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Book No...
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Discontinuous Boundary Value Problems of the First Kind for Poisson's Equation.[*]

BY GRIFFITH C. EVANS.

1. *Introduction.* Let $\Phi(e)$ be a completely additive function of point sets, representing thus a general mass distribution, defined in the open plane region T which, for simplicity, we take as bounded and simply connected. Let $\Phi(w)$ be the corresponding additive and bounded function of regular curves w in T, with regular discontinuities. We have

$$\Phi(w) = \int_T q(w, P)\, d\Phi(e_P), \tag{1}$$

where $q(w, P)$ is the density function of the region ω, bounded by w, with respect to P:

$$\begin{aligned} q(w,P) &= 1 \quad , \ P \text{ inside } w, \\ &= 0 \quad , \ P \text{ outside } w, \\ &= \psi/2\pi \ , \ P \text{ on } w, \end{aligned}$$

ψ being the angle from the forward to the backward tangent at P.[†]

The equation to be considered is what we shall call Poisson's equation:

$$\int_w D_n u\, ds = \Phi(w), \tag{2}$$

although what is commonly known as Poisson's equation is the special case of this where $\Phi(w)$ is absolutely continuous. In fact, if $\Phi(e)$ or $\Phi(w)$ is the integral of some point function $\phi(M)$ and $\phi(M)$ is suitably restricted, so as to imply the existence and integrability of the second derivatives, the equation (2) reduces to the usual form

$$\nabla^2 u = -\phi(M). \tag{3}$$

But we assume no specialization of this sort, since we allow $\Phi(e)$ to represent, as we have said, the most general possible distribution of finite positive and negative mass on T.

---

[*] Preliminary report presented to the American Mathematical Society, September, 1927.

[†] H. E. Bray and G. C. Evans, "A Class of Functions Harmonic in the Sphere," *American Journal of Mathematics*, Vol. 49 (1927), pp. 153-180. See p. 169. For brevity in reference this memoir will be cited as I.

We shall obtain solutions of the first boundary value problem and its generalization, in the sense of our earlier memoirs on Laplace's equation.[*] Somewhat analogous results may be obtained for three dimensions, but for a restricted class of regions. It may be remarked that the problems proposed for (2) are in general simpler to treat than the traditional ones for (2′), at the same time that they are wider in scope. In this sense it may be said that the situation is "correctly stated."

We need assume that (2) is required to hold merely on some special class of curves, e. g., on *almost all two-dimensional segments* s in T, that is to say, on all the rectangles formed from lines $x = a$, $y = b$ except possibly those which correspond to values of a or b respectively forming sets of linear measure zero. It will then follow that (2) will hold on more extended classes of curves. We shall be concerned especially with curves w of class Γ,[†] that is, simple regular curves such that for each curve there is a constant Γ with the property

$$\int_w \frac{|\cos n, MP|}{MP}\, ds_P < \Gamma,$$

holding irrespective of the position of M, n being the interior normal to the curve at P. And we shall say that a property holds for *almost all* curves of class Γ if it holds for all curves of the class except those which, from some given set E of superficial measure zero, contain portions, on their arcs, which are of positive exterior measure. The two definitions of "almost all" just given, for segments and curves respectively, are consistent.

The quantity $D_n u$ is also used in the sense of our previous memoirs. It is not always the derivative of u in the direction n, as usually defined, although for a fixed direction n it is identical with that derivative, if the latter is summable superficially, except on a set of measure zero. For the purposes of this paper, if the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ exist almost everywhere and are summable superficially, $D_\alpha u$ may be defined as the vector

$$D_\alpha u = (\partial u/\partial x) \cos x, \alpha + (\partial u/\partial y) \cos y, \alpha. \tag{3}$$

A more symmetrical definition is the following:

---

[*] G. C. Evans, *Logarithmic Potential—Generalized Dirichlet and Neumann Problems*, New York (1927); see Ch. V. This book will be cited as II.

[†] G. C. Evans, "Fundamental Points of Potential Theory," *Rice Institute Pamphlet*, Vol. 7 (1920), pp. 252-329, cited hereafter as III. See p. 261.

$$(3') \qquad D_\alpha u = \lim_{w \to 0} (1/\omega) \int_w u \, d\alpha',$$

where α' is the direction $\pi/2$ in advance of α and ω is the region interior to w, approaching zero as a regular family (see III, p. 274). For the functions used in this paper the two definitions would agree except possibly on a set of superficial measure zero.

2. *The Principal Solution.* We take M, a point in T, and consider what we call the *principal solution*

$$(4) \qquad U(M) = (1/2\pi) \int_T g(M,P) \, d\Phi(e_P),$$

where $g(M,P)$ is the Green's function for T with pole at P, and differs from $\log 1/MP$ by a function which is harmonic, and, for a fixed M, bounded in T. Moreover, the directional derivative or gradient of $g(M,P)$ as a function of P satisfies a relation of the form

$$\int_{\sigma_1} |\nabla g(M,P)|^2 \, d\sigma = \int_{\sigma_1} \{(\partial g/\partial x)^2 + (\partial g/\partial y)^2\} \, d\sigma$$

$$= \int_{\sigma_1} \frac{d(g,h)}{d(x,y)} \, d\sigma \leq 2\pi g_1$$

for any region σ_1 in T which does not contain a neighborhood of M, g_1 being the largest value of $g(M,P)$ for P in σ_1, and $h = h(M,P)$ being the function conjugate to $g(M,P)$. It follows then that we have the inequality:

$$(4') \qquad \int_T |\nabla g(M,P)| \, d\sigma < A,$$

where A is some finite number depending on T.

For the circle, the Green's function may of course be expressed directly. If we let Σ be the region interior to the unit circumference C with center 0, M a point of Σ and M' its inverse with respect to C, we may write

$$(5) \qquad \begin{aligned} G(M,P) &= \log(M'P \cdot OM/MP), && M, P \text{ in } \Sigma,\ M \text{ not at } 0, \\ &= 0, && M \text{ or } P \text{ on } C, \\ &= \log(1/OP), && M \text{ at } 0. \end{aligned}$$

This is a function whose symmetry with respect to M and P as well as whose identity with the Green's function follow at once from the similarity of triangles.

The integral (4) is to be interpreted as a generalized Stieltjes or Daniell integral, since the part

$$L(M) = \int_T \log (1/MP)\, d\Phi(e_P)$$

is so given (see III, p. 257). The quantities U, $\partial U/\partial x$, $\partial U/\partial y$, $D_a U$ are defined almost everywhere in T, and although they need not remain bounded or even finite are summable over any region contained in T, and over T itself. The proof of these facts is, with the help of (4′), the same as for the case of $L(M)$ which is given in the memoir just cited. Finally, we have

$$\int_\omega D_a U(M)\, d\omega_M = \int_w U(M)\, d\alpha_M', \tag{6}$$

where α' is defined as before, valid for all rectangles, and in fact for all curves of class Γ for which $\int_w \log MP\, ds_P$ is defined when M is on w.[*]

If we let E_0 be the set of points, of superficial measure zero, where the relation

$$D_a U = \int_T \frac{\partial g(M,P)}{\partial \alpha_M}\, d\Phi(e_P), \quad \alpha \text{ arbitrary,}$$

may fail to have meaning, the identity

$$\int_w D_n U(M)\, ds_M = \int_T d\Phi(e_P) \int_w \frac{\partial g(M,P)}{\partial n_M}\, ds_M$$
$$= \int_T q(w,P)\, d\Phi(e_P)$$

or

$$\int_w D_n U(M)\, ds_M = \Phi(w) \tag{7}$$

can fail to be valid only on those segments or on those curves of class Γ which are made up in part of subsets of E_0 of positive exterior linear measure (see III, p. 285, with correction II, p. 146).

Equation (6) states that $U(M)$ is a *potential function for its vector derivative* or gradient vector $D_a U$ in the sense of our previous memoirs,[†] and (7) that it satisfies (2). We have then the theorem:

---

[*] Or in fact for all curves of class Γ if a principal value is taken, when necessary, for the improper integral over w; see III, p. 264.

[†] III, p. 274. See also Evans, "Note on a Theorem of Bôcher," *American Journal of Mathematics*, Vol. 50 (1928), p. 123-126 at p. 124. This note will be cited as IV.

THEOREM 1. *The principal solution* (4) *is a potential function for its vector derivative* $D_a U$ *and satisfies the Poisson equation* (2) *for almost all curves of class* Γ *in* T*; in particular for almost all rectangles and almost all circles.*

2.1. If $\Phi(e)$ is a function of positive type, corresponding therefore to an arbitrary distribution of finite positive mass, the $U(M)$ given by (4) will obviously be ≥ 0 or positively infinite. A characteristic property of the principal solution is described in the following theorem:

THEOREM 2. *Let* M_0 *be a point of* T *and* C_ρ *a circumference of center* M_0 *and radius* ρ *small enough to lie entirely in* T*; let* $M_1 = (\rho_1, \theta_1)$ *be a generic point of* C_ρ *when* $\rho = \rho_1$. *Then, if* $\Phi(e)$ *is of positive type,*

$$U(M_0) = (1/2\pi) \int_0^{2\pi} U(M_1)\, d\theta_1 + \lim_{\rho_2=0} (1/2\pi) \int_{\rho_2}^{\rho_1} (1/\rho)\Phi(C_\rho)\, d\rho, \tag{8}$$

admitting $+\infty$ *as a possible value of* $U(M_0)$.

Hence $U(M)$ *cannot have a strict minimum at any point of* T.

It is only the first part of the theorem which requires proof. If we form the quantity $D_\rho U(M)$, we have, for almost all ρ,

$$\rho \int_{C_\rho} D_\rho U(M)\, d\theta = -\Phi(C_\rho)$$

by Theorem 1. But if we form the integral of $D_\rho U(M)$ over the annular region $\rho_2 < \rho < \rho_1$, we have for it the value (III, p. 282 and p. 288):

$$\int_{\rho_2}^{\rho_1} d\rho \int_{C_\rho} D_\rho U(M)\, d\theta = \int_0^{2\pi} d\theta \int_{\rho_2}^{\rho_1} D_\rho U(M)\, d\rho$$
$$= \int_{C_{\rho_1}} U(M_1)\, d\theta - \int_{C_{\rho_2}} U(M_2)\, d\theta,$$

and therefore

$$\int_{C_{\rho_2}} U(M_2)\, d\theta = \int_{C_{\rho_1}} U(M_1)\, d\theta + \int_{\rho_2}^{\rho_1} (1/\rho)\Phi(C_\rho)\, d\rho \tag{9}$$

It remains merely to show that the limit of the left-hand member of (9) as ρ_2 approaches zero is $2\pi U(M_0)$.

[illegible]

$$\int_{C_\rho} U(M)\, dC = (1/2\pi) \int_T d\Phi(e_P) \int_{C_\rho} g(M, P)\, d\theta_M.$$

the change in the order of integration being justifiable (see the similar integrals, III, p. 264). But

$$g(M,P)=\log(1/MP)+a(M,P)$$

where $a(M,P)$ is harmonic for M, P in T. We have

$$\int_0^{2\pi} a(M,P)\,d\theta_M=2\pi\, a(M_0,P)$$

$$\int_0^{2\pi}\log(1/MP)\,d\theta_M=2\pi\, k_\rho(M_0,P),$$

where

$$\begin{aligned} k_\rho(M_0,P) &= -\log M_0P, && M_0P\geqq\rho \\ &= -\log\rho, && M_0P<\rho \end{aligned}$$

and yields thus an increasing sequence of continuous functions whose limit as ρ tends to zero is $-\log M_0P$. Hence

$$\int_{C\rho} U(M)\,d\theta=\int_T k_\rho(M_0,P)\,d\Phi(e_P)+\int_T a(M_0,P)\,d\Phi(e_P)$$

and

$$\begin{aligned}\lim_{\rho=0}\int_{C\rho} U(M)\,d\theta &=\int_T \log(1/M_0P)\,d\Phi(e_P)+\int_T a(M_0,P)\,d\Phi(e_P)\\ &=\int_T g(M_0,P)\,d\Phi(e_P),\end{aligned}$$

by definition of the Daniell or generalized Stieltjes integral.

But the right-hand member is precisely $2\pi U(M_0)$, and the point is proved.

It will be noticed that equation (9) is valid whether or not $\Phi(e)$ is of positive type, and if $\Phi(e)$ is written, as it may be, as the difference of two functions of positive type the rest of the analysis may be carried through provided the integral in (4) is convergent. *Equation* (8) *is valid for a general distribution of mass* $\Phi(e)$ *for points* M_0 *where* (4) *is a convergent integral, and that is, almost everywhere in* T.

2.2. Another characteristic property of the principal solution (4), when $\Phi(e)$ is of positive type, is that it is *lower semi-continuous.*

THEOREM 3. *If* $\Phi(e)$ *is of positive type, and* $U(M)$, *given by* (4), *is* $>N$ *at some point* M_0 *of* T, *then there is a neighborhood about* M_0 *for which* $U(M)>N$, M *in this neighborhood.*

We admit the possible value $U(M_0) = +\infty$, in which case N would be any positive number. The theorem requires proof only if $\Phi(T) > 0$, since if $\Phi(T) = 0$, $U(M)$ vanishes identically.

We consider the function

$$V(M) = (1/2\pi) \int_T \log (1/MP) d\Phi(e_P),$$

since $U(M) - V(M)$ is continuous in T; if the property holds for $V(M)$ it will also hold for $U(M)$. Let us assume then that $V(M_0) > N'$ and show that $V(M) > N'$ for a sufficienly small neighborhood; take $\epsilon < V(M_0) - N'$.

With the same notation as for Theorem 3, we take ρ small enough so that, whether or not the integral $V(M_0)$ converges to a finite value, we have

$$\int_T k_\rho(M_0, P) d\Phi(e_P) > N' + (\epsilon/2).$$

We take then M within $C\rho_1$, $\rho_1 < \rho$, where ρ_1 is small enough to insure the validity of the inequality

$$\log (1/MP) > k_\rho(M_0, P) - \epsilon/2\Phi(T).$$

This inequality will in fact be satisfied if ρ_1 is small enough so that

$$(MP/M_0P) < \epsilon^\eta, \quad \eta = \epsilon/2\Phi(T),$$

when M is on C_{ρ_1} and P is on C_ρ.

It follows then that

$$\int_T \{\log (1/MP) - k_\rho(M_0, P)\} \, d\Phi(e_P) > -(\epsilon/2\Phi(T)) \int_T d\Phi(e),$$

or

$$V(M) > \int_T k_\rho(M_0, P) \, d\Phi(e_P) - (\epsilon/2).$$

Hence, by means of the earlier inequality on the integral of the second member,

$$V(M) > N',$$

and the point is proved.

2.3. We may draw an immediate corollary from Theorems 2 and 3 with

[illegible]

[illegible] tions in T which [illegible] potential functions of their vector deriva-

[illegible] satisfy (2) for almost all segments (or almost all circles, or [illegible] all curves of class Γ) will be such a solution of (2), with $\Phi(w) \equiv 0$. But a theorem which the author has already established (IV, p. 123) states that such functions are, except for removable discontinuities on a set of at most superficial measure zero, continuous with all their derivatives and solutions of Laplace's equation. We may state then the following proposition:

COROLLARY. *Let $u(M)$ be a potential function for its vector derivative, and a solution of* (2), *with* $\Phi(e) \geqq 0$, *for almost all segments* (*or almost all circles, or almost all curves of class* Γ) *in T. By changing the value of $u(M)$ at most in the points of a set of superficial measure zero it may be made lower semi-continuous and to satisfy* (8); *it will not then have a strict minimum at any point of T.*

3. *The class and boundary values of $U(M)$ for the circle.* As far as boundary values are concerned, it is perhaps easier to handle the problem first for the unit circle Σ, and obtain the results for other simply connected regions by conformal transformation. In this section, we show that $U(M)$ for Σ is of the class (ii), and therefore of the class (i) of the author's treatment of Laplace's equation (see II, Ch. III), and that it takes on zero boundary values for almost all points on the circumference C as M approaches C along a radius.

Let M have polar coordinates r, θ with respect to the center O of the circle, and form the integral

$$\begin{aligned}\int_{\theta'}^{\theta''} U(r,\theta)\,d\theta &= (1/2\pi)\int_{\theta'}^{\theta''} d\theta \int_{\Sigma} g(r,\theta;P)\,d\Phi(e_P)\\ &= (1/2\pi)\int_{\Sigma} d\Phi(e_P)\int_{\theta'}^{\theta''} g(r,\theta;P)\,d\theta,\end{aligned}$$

writing $g(r,\theta;P)$ now for the Green's function (5) for C. The quantity

$$\int_{\theta'}^{\theta''} g(r,\theta;P)\,d\theta$$

is a continuous function of P, bounded independently of r, and the limit of it is zero as r tends to 1, for any P in Σ. Hence by a fundamental property of the generalized integral:

$$\lim_{r=1}\int_{\theta'}^{\theta''} U(r,\theta)\,d\theta = 0. \tag{10}$$

We need however a still stronger result.

Consider a finite number n of non-overlapping intervals (θ_i', θ_i'') at values of r, $r = r_i$, all $> r_0$ where r_0 is some value of r which will eventually be allowed to approach 1, and write

$$2\pi I = \sum_1^n{}_i \; 2\pi \int_{\theta_i'}^{\theta_i''} U(r_i, \theta)\,d\theta = \sum_1^n{}_i \int_{\Sigma} d\Phi(e_P) \int_{\theta_i'}^{\theta_i''} g(r_i, \theta; P)\,d\theta.$$

We shall prove that

$$\lim_{r_0=1} \quad I = 0. \tag{11}$$

Again, but temporarily, take $\Phi(e)$ as of positive type so that $U(r, \theta) \geqq 0$. Let $r_0 = 1 - x$ and draw a second circumference C_ρ with center O and smaller radius, $\rho = 1 - (x)^{1/2}$; denote by Σ_ρ the open circular region bounded by C_ρ. Consider the integrals I_ρ and $I_{1-\rho}$ extended respectively over Σ_ρ and $\Sigma - \Sigma_\rho$, writing

$$I = I_\rho + I_{1-\rho}.$$

In Σ_ρ we have, by (5),

$$g(r, \theta; P) = g(M, P) = \log\,(OM \cdot M'P)/MP$$
$$\leq \log \frac{x^{1/2} + x - x^{3/2}}{x^{1/2} - x}.$$

for the upper bound of $g(M, P)$ when $OP < \rho$ and $OM > r_0$ will be the value of $g(M, P)$ when P, M, M' are collinear and P is on C_ρ while M is on C_{r_0}. But this quantity tends to 0 with x. Hence, given ϵ, we can take r_0 near enough to 1 so that

$$g(M, P) < 2\pi\epsilon \quad \begin{cases} r > r_0 \\ OP < \rho \end{cases}$$

Accordingly

$$I_\rho < \epsilon\,\Phi(\Sigma_\rho) \leq \epsilon\,\Phi(\Sigma)$$

and approaches zero as r_0 approaches 1.

On the other hand, if P lies in $\Sigma - \Sigma_\rho$, the integral

$$J = \Sigma \int_{\theta_i'}^{\theta_i''} g(M, P)\,d\theta_M$$

is bounded, independently of r_0, P as r_0 approaches 1. For

$$g(M, P) \leq \log 2 - \log MP \leq \log 2 - \log M_1P,$$

[illegible] M_1P is the numerical length of the perpendicular to OM from P. In other words, M_1 is a projection of M along OM on the circumference of diameter OP.

Hence

$$J \leq 2\pi \log 2 + \int_0^{2\pi} | \log M_1P \mid d\theta_M,$$

which quantity will be seen to be bounded. In fact

$$M_1P = OP \mid \sin OM, OP \mid = OP \mid \sin \psi \mid$$

where ψ is the angle OP, OM and $d\psi = d\theta$, and accordingly

$$\int_0^{2\pi} | \log M_1P \mid d\theta_M \leq 2\pi \mid \log OP \mid - 2 \int_0^{\pi} \log \sin \psi \, d\psi$$

$$\leq 2\pi \mid \log OP \mid - 4 \int_0^{\pi/2} \log \sin \psi \, d\psi$$

$$\leq 2\pi \mid \log OP \mid - 4 \int_0^{\pi/2} \log (\psi/2) \, d\psi$$

which is bounded as r_0 approaches 1 for P in $\Sigma - \Sigma_\rho$.

Hence there is a constant K, independent of r_0 as r_0 tends to 1 such that

$$I_{1-\rho} \leq K \int_{\Sigma-\Sigma_\rho} d\Phi(e_P)$$

$$\leq K \left(\Phi(\Sigma) - \Phi(\Sigma_\rho)\right).$$

But this also approaches zero as r_0 approaches 1, since Σ is an open set.

Accordingly equation (11) is verified, if $\Phi(e)$ is of positive type. But since any completely additive function of e is the difference of two functions of positive type, equation (11) holds for any $\Phi(e)$. It holds, moreover, even if the number of intervals is infinite.

3.1. We shall now use these results in order to show that $\lim\limits_{r=1} U(r, \theta) = 0$ for almost all θ, restricting ourselves first to functions $\Phi(e)$ of positive type.* Suppose that the statement is not true, and that there is therefore a set κ of values of θ and positive numbers m and ϵ such that

$$\text{meas } \kappa > m > 0$$

$$\overline{\lim_{r=1}} \, (U(r, \theta) > \epsilon > 0, \qquad \theta \text{ in } \kappa.$$

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* This is approach to zero "in the narrow sense." The reader may devise an example to show that approach to zero "in the wide sense" (see II, Ch. II) may be impossible at every point of C. Such an example has been constructed by C. H. Dix.

We shall show that this is impossible by slightly extending a result already proved for continuous functions (See III, p. 317).

We take r_0 sufficiently near 1 so that the integral I of (11) is small, $< \epsilon m$. Let E_{r_0} be the set of points (r, θ), $r_0 < r < 1$, for which $U(r, \theta) > \epsilon$; the projection of E_{r_0} on C includes κ and is of measure $> m$. Moreover E_{r_0} is an open set, by Theorem 3, and hence is the sum of a denumerable infinity of curvilinear rectangles, bounded by arcs and radii. On the boundary of each rectangle $U(r, \theta) > \epsilon$, and accordingly, again by Theorem 3, each rectangle is contained in one whose projection on C is an open set, but for every point of which $U(r, \theta) > \epsilon$. The resulting rectangles constitute a family $\mathcal{E}$, the members of which may, of course, overlap.

On the other hand we can take a closed set κ' in κ whose measure is as near that of κ as we please, and thus so that meas. $\kappa' > m$. Each point of κ' will be interior to some arc which is the projection of some one or more of the rectangles of $\mathcal{E}$. Since κ' is a closed set there will be a finite number of these rectangles which have the same property, and thus whose projection on C will constitute a set of measure $> m$.

We may now finally choose a finite number of arcs (θ_i', θ_i'') lying each one interior to one of these rectangles, thus corresponding to values $r > r_0$, but whose projections on C do not overlap and yet have a total measure $> m$. The integral I extended over these arcs will therefore be $> \epsilon m$. But this is a contradiction.

Hence if $\Phi(e)$ is of positive type, $\lim\limits_{r=1} U(r, \theta) = 0$, for almost all θ. But the same result therefore holds if $\Phi(e)$ is any completely additive function of point sets in Σ.

A further result may be obtained. If $\Phi(e)$ is of positive type, so that $U(r, \theta) \geq 0$, we have the two equations

$$\lim_{r-1} \int_0^\theta U(r, \theta)\, d\theta = 0$$

$$\lim_{r=1} U(r, \theta) = 0, \quad \text{almost everywhere.}$$

It follows then, from the well known theorem of de la Vallée Poussin, that [illegible] absolute continuity of $\int U(r, \theta)\, d\theta$ is uniform for any denumerable [illegible]

[illegible]

[illegible] prove directly, however, a more general property. Since $U(r, \theta)$ [illegible] is a potential functoin for its vector derivative we have, by integrating over the region $r_1 < r < r_2$, $\theta' < \theta < \theta''$, which we denote by $\sigma(r_1, r_2)$,

$$\int_{\sigma(r_1,r_2)} (1/r) D_r U \, d\omega = \int_{\theta'}^{\theta''} U(r_2, \theta)\, d\theta - \int_{\theta'}^{\theta''} U(r_1, \theta)\, d\theta,$$

so that if $\Phi(e)$ is of positive type,

$$\int_{\theta'}^{\theta''} U(r_2, \theta)\, d\theta \leq \int_{\theta'}^{\theta''} U(r_1, \theta)\, d\theta + \int_{\sigma(r_1,1)} (1/r) \mid D_r U \mid d\omega$$

$$\leq \int_{\theta'}^{\theta''} U(r_1, \theta)\, d\theta + \int_{\theta'}^{\theta''} d\theta \quad \int_{r_1}^{1} \mid D_r U \mid dr$$

since the double integral of the right-hand member may be written as an iterated integral.

But both terms of the right-hand member now define absolutely continuous functions of θ which are independent of r_2. Hence the absolute continuity of the function $\int_0^\theta U(r_2, \theta)\, d\theta$ is uniform for all r_2, $r_1 < r_2 < 1$. The same result holds accordingly even if $\Phi(e)$ is not of positive type.

The results of this section may be expressed in the following theorem:

THEOREM 4. *The principal solution* $U(r, \theta)$, *for the unit circle* C, *given by* (4), *satisfies the relations*

$$\lim_{r=1} \int_0^\theta U(r, \theta)\, d\theta = 0 \tag{12}$$

$$\lim_{r=1} U(r, \theta) = 0, \quad \textit{for almost all } \theta, \tag{13}$$

(14) $\int_0^\theta U(r, \theta)\, d\theta$ *is absolutely continuous as a function of* θ, *uniformly in* r, *for* $0 < r_1 < r < 1$,

(15) $\int_0^{2\pi} \mid U(r, \theta) \mid d\theta$ *is bounded*, $0 < r_1 < r < 1$.

Equation (14) is the condition (ii), already mentioned, and (15) is the condition (i), which of course is a consequence of (ii).

4. *Discontinuous first boundary value problems for the circle.* If two solutions of (2) are taken as the same when they differ at most on sets of points of zero superficial measure, the following theorem may be stated.

THEOREM 5. *Consider the class of functions $u(r,\theta)$ which are potential functions of their vector derivatives and satisfy Poisson's equation* (2), *for almost all segments in the unit circle* Σ *(or almost all circles in* Σ*). Let $F(\theta)$ be of limited variation, with regular discontinuities and such that $F(\theta+2\pi) = F(\theta)+F(2\pi)-F(0)$; and let $f(\theta)$ be summable in the Lebesgue sense over the closed interval* $(0, 2\pi)$.

There is one and only one $u(r,\theta)$ which satisfies (*i*) [*i. e., equation* (15)] *and the boundary condition*

$$\lim_{r=1}\int_0^\theta u(r,\theta)\,d\theta = F(\theta)-F(0). \tag{16}$$

This solution takes on the boundary values $F'(\theta)$ almost everywhere, for approach along a radius.

There is one and only one $u(r,\theta)$ which satisfies (ii) [i. e., equation (14)] *and takes on the boundary values $f(\theta)$, almost everywhere, for approach along a radius.*

These functions in both cases will satisfy (2) for almost all curves of class Γ.

In fact, as already remarked in § 2. 3, the difference $u(r,\theta)-U(r,\theta)$ will be, except for removable discontinuities on at most a set of zero superficial measure, continuous with all its derivatives and a solution of Laplace's equation in Σ. This difference will then be uniquely determined, if it is of the class (i), by the boundary values of its integral, and if it is of the class (ii), by its own boundary values taken on almost everywhere, approaching the boundary along radii, that is, in the *narrow* sense (see II, Ch. III). But since $U(r,\theta)$ satisfies both (i) and (ii), $u(r,\theta)-U(r,\theta)$ satisfies whichever of these conditions is satisfied by $u(r,\theta)$; moreover the boundary condition on $u(r,\theta)-U(r,\theta)$ is the same as that on $u(r,\theta)$, by (12) and (13) of Theorem 4.

The requirement that (2) be satisfied for almost all segments or almost all circles in Σ may be modified by substituting for these other special classes of closed curves, e. g., almost all cells of a translatable regular net of curves of class Γ (see IV, p. 126).

5. *The general bounded simply connected plane region T.* Let now $g(Q,M)$ be again the Green's function for T with pole Q, and $h(Q,M)$ its

conjugate. The conditions (i) and (ii) take a form which is invariant of a conformal transformation:

(*i*) *The integral*

$$\int_0^{2\pi} |u(M)|\, dh(Q,M) = \int_{g=x} |u(M)| \frac{\partial g(Q,M)}{\partial n_M} ds_M$$

extended over the closed curve $g(Q,M) = x$ *remains bounded as* x *tends to zero.*

(*ii*) *The integral*

$$\int_0^h u(M)dh(Q,M),$$

extended along the curve $g(Q,M) = x$ *is absolutely continuous in* h, *uniformly with respect to* x, *as* x *tends to zero.*

We consider *a class* (A) of functions $u(M)$ which are potential functions of their vector derivatives and solutions of (2) for almost all segments (or almost all circles, or other special classes of curves, as already mentioned, or almost all curves of class Γ). We let $F(P) = F(h)$ *be of limited variation on the accessible boundary points of* T (see II, p. 78), with regular discontinuities and such that $F(h+2\pi) = F(h) + F(2\pi) - F(0)$, and we let $f(P) = f(h)$ be summable in the Lebesgue sense on the boundary (see II, p. 78) with respect to $h(Q,P)$. We state the theorem:

THEOREM 6. *There is one and only one* $u(M)$ *of class* (A) *which satisfies* (*i*) *and the boundary condition*

$$\lim_{x=0} \int_0^h u(M)dh(Q,M) = F(P) - F(P_0) \tag{17}$$

for points of the boundary accessible along curves $h =$ *const. from* Q, *where* $F(P_0)$ *is the value of* $F(P)$ *which corresponds to* $h = 0$ *and the integral is extended along curves* $g(Q,M) = x$.

This solution takes on the boundary values dF/dh *almost everywhere, in the narrow sense, that is, along curves* $h =$ const., *and satisfies* (2) *for almost all curves of class* Γ.

There is one and only one $u(M)$ *of class* (A) *which satisfies* (*ii*) *and the boundary condition*

(18) $\lim\limits_{M=P} u(M) = f(P)$, *in the narrow sense, almost everywhere on the boundary of T.*

This solution satisfies (2) *for almost all curves of class* Γ.

The meanings assigned to $F(P)$ and $f(P)$, almost all, etc., depend upon a definition of the order of boundary points (see II, Ch. V). The $f(P)$ is independent of the pole Q, but the $F(P)$ depends on Q, for the same function $u(M)$, according to the equation

$$F_Q(P) - F_Q(P_0) = \int_0^h \frac{dh(Q,P')}{dh(Q',P')}\, dF_{Q'}(P').$$

Certain continuous boundary value problems occur as special cases of Theorem 6, precisely as in the case of Laplace's equation (see II, Ch. V). A function $u(M)$ which happens to remain bounded in the neighborhood of the boundary of T of course satisfies (ii).

5.1. The theorem will follow immediately from the similar theorems on Laplace's equation (see II, Ch. V), by showing merely that the principal solution $U(M)$ for T satisfies (i) and (ii) and takes on zero boundary values in the sense of (17) and (18); for the difference $u(M) - U(M)$ will then, except for removable discontinuities, be a solution of Laplace's equation. The proof will be accomplished by means of a conformal transformation of T into the interior Σ of the unit circle. In fact, considerably more of the general problem might have been handled by this method, since the quantity

$$\int_w D_n u\, ds$$

is invariant of a conformal transformation, provided the curves are such as to give the integral meaning both in T and Σ; but the treatment is briefer in the present fashion.

It will evidently be sufficient to consider $\Phi(e)$ of positive type.

Let us make then a transformation of T into Σ by means of which Q passes into O and $h = 0$ into $\theta = 0$, the function $-\theta$ being conjugate to log $1/r$. We define a function $\Psi(e')$ in Σ by writing

$$\Psi(e') = \Phi(e),$$

where e is the set in T which transforms into e' by the one-to-one correspondence. The function $\Psi(e')$ will then also be of positive type and bounded in Σ.

Let $\{e_i'\}$ denote a denumerable infinity of sets in Σ, no two of them with common points, and let $\{e_i\}$ be the aggregate of corresponding sets in T. No two of the latter have common points. The set $E = e_1 + e_2 + \cdots$ will then correspond to the set $E' = e_1' + e_2' + \cdots$; for if a point M' is in E' it will be a member of one of the sets e_i' and its corresponding point M will lie in e_i and therefore in E; also, similarly, if M lies in E its transform M' will lie in E'. It follows that $\Psi(e')$ is completely additive. In fact, with the notation just used,

$$\sum_1^n \Psi(e_i') = \sum_1^n \Phi(e_i)$$

and

$$\lim_{n=\infty} \sum_1^n \Psi(e_i') = \lim_{n=\infty} \sum_1^n \Phi(e_i) = \Phi(E).$$

But, by definition,

$$\Psi(E') = \Phi(E)$$

and therefore

$$\Psi(E') = \lim_{n=\infty} \sum_1^n \Psi(e_i').$$

Moreover, if we denote by $G(M', P')$ and $g(M, P)$ the respective Green's functions in the two regions Σ and T, we shall have

$$\int_\Sigma G(M', P') d\Psi(e'_{P'}) = \int_T g(M, P) d\Phi(e_P). \tag{19}$$

In order to prove this identity, we denote by Σ_x and T_x respectively the open regions bounded by the curves $G(M', P') = x$ and $g(M, P) = x$, and form for $n > x$ the functions $H_n'(P')$ and $H_n(P)$:

$$\begin{aligned} H_n(P) &= g(M, P) \quad , \quad g(M, P) < n, \\ &= n \quad , \quad g(M, P) \geq n, \\ H_n'(P') &= G(M', P') \quad , \quad G(M', P) < n, \\ &= n \quad , \quad G(M', P') \geq n. \end{aligned}$$

Then

$$H_n'(P') = H_n(P)$$

and

$$\int_{\Sigma_x} H_n'(P')\, d\Psi(e') = \int_{T_x} H_n(P)\, d\Phi(e)$$

since both may be written as limits of the same Riemann sum.

But if we let x approach zero, we have

$$\int_\Sigma H_n'(P')\, d\Psi(e') = \int_T H_n(P)\, d\Phi(e)$$

since

$$\int_T H_n(P)\, d\Phi(e) - \int_{T_x} H_n(P)\, d\Phi(e) = \int_{T-T_x} H_n(P)\, d\Phi(e)$$
$$\leq x\Phi(T - T_x),$$

with a similar inequality for the left-hand member. If now we let n become infinite we have the result desired, from the definition of the *general integral.*

Equation (19) states that the fundamental solutions $U'(M')$ in Σ and $U(M)$ in T are equal for corresponding points M' and M. We accordingly conclude that, in T, $U(M)$ satisfies (i) and (ii), and also the equations

$$(17') \qquad \lim_{x=0} \int_0^h U(M)\, dh(Q, M) = 0,$$

where the integral is extended along the curve $g(Q, M) = x$, and

$$(18') \qquad \lim_{M=P} U(M) = 0,$$

almost everywhere, in the narrow sense. This completes the proof of Theorem 6.

It is worthy of notice that if we transform the function of curves $\Phi(w)$ as we transform the function of point sets $\Phi(e)$, the regularity of the discontinuities is preserved. That is, if we denote by w' the curve in Σ which corresponds to the regular curve w in T, w' will be regular, and if we write $\Psi(w') = \Phi(w)$, the equation

$$\Psi(w') = \int_\Sigma q(w', P')\, d\Psi(e')$$

will hold for $\Psi(w')$ in Σ' if the corresponding equation (1) holds in T. In fact, since the transformation is conformal the density function of w at P in T becomes the density function of w' at P' in Σ:

$$q(w, P) = q(w', P'),$$

and hence, as may easily be proved,

$$\int_\Sigma q(w', P')\, d\Psi(e') = \int_T q(w, P)\, d\Phi(e).$$

[illegible]

The problem for three dimensions. There is no essential difficulty in generalizing the results of §§ 1 and 2 to three dimensions; and §§ 3 and 4 may evidently be followed in an analysis of the sphere. But we cannot say the same for § 5, since a transformation of the kind there used is not available in three dimensions. Results similar to those here expounded for the general plane region may however be obtained for space regions bounded by surfaces which are sufficiently smooth, and the integration with respect to the conjugate of the Green's function in the conditions (i) and (ii) may be replaced by integration with respect to the element of area over surfaces close to the boundary surface. Corresponding simplifications occur with sufficiently smooth boundaries in the plane. But further consideration of these situations may well be postponed until corresponding treatments of the Dirichlet problem, now in preparation, are available.

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On Irreducible Cross-cuts of Plane Simply Connected Regions.[*]

BY WALLACE ALVIN WILSON.

1. It is a well-known theorem that, if R is a simply connected plane egion whose frontier F is bounded and C is a simple arc lying in R save or the end-points, which are on F, then C cuts R into two simply connected egions and the frontier of each of these contains C. Moreover, if m is a oint of one of these regions and n one of the other, no closed proper part of separates m from n in R. These properties are obviously not confined to imple arcs and it is the purpose of this paper to extend these results to a nore general class of continua.

It is an easy guess that the generalization of the simple arc C will be a ounded continuum irreducible between two or more sets of points and at irst sight it seems that most properties obtainable would be obvious corolaries of certain theorems already obtained by C. Kuratowski[†] and the uthor[‡] regarding regular frontiers (i. e., frontiers which are the common rontiers of at least two components of their complements). However, there re two complications: first, the frontier F may not be regular; and second, may itself lie in different components of the complement of C. As a simple xample let F be the union of the segments ab and cd, where in terms of artesian co-ordinates $a = (-2, 0)$, $b = (3, 0)$, $c = (1, 0)$, and $d = (0, 1)$. et M and N be circumferences of radius 1 and centers $(-2, 0)$ and $(2, 0)$, espectively, and let C be the union of M and N and an open curve approaching M and N asymptotically, but not meeting $M + N + F$. Then C is irreucible between the set $b + c$ and the point $(-1, 0)$ and divides R, the comlement of F, into five simply connected regions, two having C as a proper art of their frontiers. But F is not regular and lies in three components f the complement of C.

The work falls naturally into three parts: §§ 2-7, containing definitions and general properties; §§ 8-10, containing some useful lemmas; and §§ 11-14, containing the principal theorems.

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[*] Presented to the American Mathematical Society, September, 1928.

[†] (A) C. Kuratowski, "Sur les coupures du plan," Fundamenta Mathematicae, Vol. 6, pp. 130-145.

[‡] (B) W. A. Wilson, "On Bounded Regular Frontiers in the Plane," *Bulletin of he American Mathematical Society*, Vol. 34, pp. 81-90.

2. *Definitions.* In the following let R be a simply connected plane region whose frontier F is bounded. If C is a sub-set of $\bar{R}$, m and n are two points of R not on C, and every continuum contained in R and joining m and n meets C, we say that C is a *cross-cut of R between m and n* and we write that C is an $S(m, n, R)$. If C is closed and no closed proper part of C is an $S(m, n, R)$, we say that C is an irreducible $S(m, n, R)$; if C is an $S(m, n, R)$, but is not itself an $S(m, n)$, it will be called *regular*. It is easily seen that, if C is a bounded closed $S(m, n, R)$, it contains an irreducible $S(m, n, R)$.

Examples. Let R be the interior of a circumference F and let C be a circumference in $\bar{R}$ tangent to F at some point. If m lies within and n without C, C is a non-regular irreducible $S(m, n, R)$.

Let R be as before and let C be the union of F and a double spiral starting at the center of R and approaching F asymptotically. Then C cuts R into two regions and is a non-regular irreducible $S(m, n, R)$. Here $C \supset F$.

Let R be as before and let C be the union of a circumference in R and two segments joining this circumference to F. Here C is a non-regular $S(m, n, R)$, and is not irreducible.

3. *Notation.* Aside from the ordinary notation of the aggregate theory, the following will be used.

The whole plane will be denoted by Z.

If F is a continuum, a component of $Z - F$ whose frontier is F is called a *principal* component of $Z - F$.

The symbol $V_\delta(a)$ denotes those points of the plane whose distances from the point a are less than δ.

If every continuum in the plane which joins the points m and n meets C, we say that C is an $S(m, n)$.

The notation "$A \subseteq B$" signifies that the set A is a part of the set B and may be identical with it.

4. THEOREM. *Let R be a simply connected region whose frontier F is bounded and let C be a bounded regular irreducible $S(m, n, R)$.*[*] *Then $F \cdot C$ is not connected.*

Proof. As neither F nor C is an $S(m, n)$, $F + C$ is not an $S(m, n)$

If $F \cdot C$ is a continuum, by a theorem essentially due to Z. Janiszewski.* But $F + C$ is an $S(m, n)$, since any arc mn not meeting F lies in R and hence meets C.

5. THEOREM. *Let R be a simply connected region whose frontier F is bounded. For the bounded closed set C to be an irreducible $S(m, n, R)$ it is necessary and sufficient that m and n lie in components R_m and R_n of $R - R \cdot C$ whose frontiers both contain C and that $C = \overline{R \cdot C}$.*

This theorem and the following are easy generalizations of similar theorems by C. Kuratowski [See reference (A), pp. 133 and 136]. The proof of the above is as follows.

Let C be an irreducible $S(m, n, R)$. It is evident that $R \cdot C$ is an $S(m, n, R)$ and that $\overline{R \cdot C}$ is a closed $S(m, n, R)$. Hence $C = \overline{R \cdot C}$. Clearly m and n lie in different components R_m and R_n of $R - R \cdot C$. Let H be the frontier of R_m and $H \cdot C \neq C$. Any continuum in R joining m and n must cut H. It does not cut F and hence must cut $H \cdot C$, since $H \subset F + C$. Thus $H \cdot C$ is an $S(m, n, R)$. Therefore $H \cdot C = C$.

Conversely, suppose that C is a part of the frontiers of both R_m and R_n and that $C = \overline{R \cdot C}$. Obviously C is an $S(m, n, R)$. Since it is bounded and closed, there is a closed sub-set D which is an irreducible $S(m, n, R)$. If $D \neq C$, there is a point x of $C - D$ which lies in R. Then there is a $\delta > 0$ such that $V_\delta(x) \subset R$ and $V_\delta(x)$ contains points of both R_m and R_n, which contradicts the hypothesis regarding D. Hence C is an irreducible $S(m, n, R)$.

6. THEOREM. *Let R be a simply connected region whose frontier F is bounded. Let C be a bounded irreducible $S(m, n, R)$. Then C is a continuum and $C^* = R \cdot C$ is connected.*

Proof. If C^* is not connected, $C^* = M + N$, where $\bar{M} \cdot N = M \cdot \bar{N} = 0$ and neither M nor N is void. Let R_m and R_n be the components of $R - R \cdot C$ containing m and n, respectively. If x is a point of M, there is a $V_\delta(x)$ containing no point of $N + F$, but containing points of both R_m and R_n. Hence $N + F$ is not an $S(m, n)$, and the same thing is true of $M + F$. But both of these sets are closed and $(N + F)(M + F) = F$. This is a contradiction by reference (C). Hence C^* is connected. As $C = \overline{C^*}$ by § 5, C is a continuum.

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* Z. Janiszewski, "Sur [illegible] and neither P nor Q is an $S(m, n)$, then $P + Q$ is not an $S(m, n)$. [illegible] Straszewicz, *Fundamenta Mathematicae*, Vol. 4, p. 129.

7. If for each pair of points $\{a_i\}$, $i = 1, 2, \cdots, n$, C is a bounded irreducible $S(a_i, a_j, R)$, it is convenient to write that C is an irreducible $S(a_i; R)$. The previous theorems are of course valid in this case. In particular, it follows that there are n components of $R - R \cdot C$ such that C is a part of the frontier of each one.

As an example of an irreducible $S(a_i; R)$, $i = 1, 2, \cdots, k$, we need only to modify the construction of the continuum H described in reference (B), pp. 88-90. Let F be the segment ab. It is apparent from the discussion there given that the construction can be made so that there are k simply connected regions analogous to $E_1 + G_4 + \cdots + G_{3n-2} + \cdots$ and that the frontier of each of these is the union of H and a segment of ab. If we take $C = H$, $R = Z - F = Z - ab$, and the points $\{a_i\}$ one in each of the k connected regions just described, it is evident by § 5 that C is an irreducible $S(a_i; R)$.

8. Before proceeding further it is desirable to consider some general properties of the union of two bounded continua whose divisor is not connected and of their complementary domains. Let H and K be the continua in question, $F = H + K$, $H^* = F - K$, $K^* = F - H$, and let α be any component of $H \cdot K$. If Z denotes the plane, the components of $Z - H$ and $Z - K$ containing no points of K^* and H^*, respectively, fall into three classes: first, components $\{R_h\}$ whose frontiers are parts of H but not of K; second, components $\{R_k\}$ whose frontiers are parts of K but not of H; third, components $\{R_{hk}\}$ whose frontiers are parts of $H \cdot K$. No component of one class has a point in common with one of another and each one is a component of $Z - F$. Also the frontier of each R_{hk} is a part of some component of $H \cdot K$.

The components of $Z - F$ fall into four classes: first, those having frontier points on both H^* and K^*, second, those having frontier points on H^* but not on K^*; third, those having frontier points on K^* but not on H^*; fourth, those having frontier points on neither H^* nor K^*. It is easily shown that the components of the second, third, and fourth classes are identical with the components $\{R_h\}$, $\{R_k\}$, and $\{R_{hk}\}$, respectively, discussed above. Furthermore, by an extension of a theorem by S. Straszewicz [*] it can be shown that, if $H \cdot K$ has n components (or an infinity), there are at least n components of $Z - F$ of the first class (or an infinity).

If we set $H_1 = H + \sum R_h + \sum R_{hk}$ and $K_1 = K + \sum R_k + \sum R_{hk}$, we obtain the following results. Each component of $H_1 \cdot K_1$ is a component α of $H \cdot K$ together with such components $\{R_{hk}\}$ as have their frontiers a part of α. If H^* is connected, so is H^*_1. If $F_1 = H_1 + K_1$, the components of $Z - F_1$ are identical with those of $Z - F$ of the first class.

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[*] See reference (B), p. 83 and S. Straszewicz, "Über die Zerschneidung der Ebene," *Fundamenta Mathematicae*, Vol. 7, p. 174.

9. LEMMA. *Let H and K be bounded continua. Let $H^* = H - H \cdot K$ be connected and $\overline{H^*} = H$. Then all the points of K lying in any component R of $Z - H$ are on a component of $K \cdot \bar{R}$.*

Proof. Let us assume the existence of two points m and n in $R \cdot K$ such that any sub-continuum M of K irreducible between m and n contains points not in $\bar{R}$. Let C be a simple arc in R whose ends are m and n. Then $C \cdot M$ is a closed set and $C \cdot M \neq C$. Hence $C - C \cdot M$ is a finite or enumerable set of open arcs whose end-points lie in M. There is at least one of them, with end-points m' and n', such that no sub-continuum M' of M irreducible between m' and n' lies in $\bar{R}$; let it together with its end-points be denoted by C'. This last statement holds, as otherwise the hypothesis regarding M would be false.

As $C' \cdot M' = m' + n'$, there are exactly two principal components S and T of $Z - (C' + M')$, by reference (B), p. 86. As $C' \subset R$, $S \cdot R \neq 0$ and $T \cdot R \neq 0$. As M' contains points not in $\bar{R}$, $S \cdot (Z - \bar{R}) \neq 0$ and $T \cdot (Z - \bar{R}) \neq 0$. Thus both S and T contain points of H, as they connect points in R and points not in $\bar{R}$. Since $H^* \cdot (C' + M') = 0$ and H^* is connected, H^* lies in one component of $Z - (C' + M')$. Suppose that $S \cdot H^* = 0$. Then $S \cdot \overline{H^*} = S \cdot H = 0$, which is a contradiction.

The theorem follows at once. A consequence is that a sub-continuum of K joining any point of $K \cdot \bar{R}$ not on the component containing $R \cdot K$ to this component must contain points not in $\bar{R}$.

10. By means of a device employed by R. L. Moore in his paper "On the Separation of the Plane by a Continuum" (*Bulletin of the American Mathematical Society*, Vol. 35, pp. 303-307), there can be obtained certain separation properties embodied in the following lemma and corollaries. The term "upper semi-continuous image" used in the lemma signifies that there is a one to one correspondence between the elements $\{X\}$ of J and the points $\{t\}$ of a circumference, in which $X = f(t)$ is upper semi-continuous. The properties of such relations are discussed in an article by Moore ("Concerning [illegible] continua [illegible] *Mathematical Society*, Vol. 27, pp. 416-428), one by L. S. Hill ("Properties of Certain Aggregate Functions," this *Journal*, Vol. 49, pp. 419-432),

and one by C. Kuratowski ("Sur les decompositions semi-continues d'espaces metriques compacts," *Fundamenta Mathematicae,* Vol. 11, pp. 170-185).

LEMMA. *Let H and K be bounded continua and $H \cdot K$ be not connected. Let $H^* = H - H \cdot K$ be connected. Let $K^* = K - H \cdot K$ lie in one component of $Z - H$. Then there is a bounded continuum J having these properties: $J \supset H \cdot K$; $Z - J$ has two components, one containing H^* and the other K^*; J is the upper semi-continuous image of a circumference; each element of J is a point not on $H + K$ or a component α of $H \cdot K$ together with such components of $Z - (H + K)$ as have their frontiers on α.*

Proof.[*] At least one component of $Z - (H + K)$ has frontier points on both H^* and K^*. By inversion there is no loss in generality in assuming this unbounded.

Let H_1 and K_1 be as defined in § 8. Then the continua H_1 and K_1 are bounded by the previous paragraph and neither cuts the plane by our hypotheses regarding H^* and K^*. Also H_1^* is connected. $H_1^* \supseteq H^*$, and $K_1^* \supseteq K^*$.

If each component of $H_1 \cdot K_1$ is regarded as an element and each point of $Z - H_1 \cdot K_1$ as an element, this set of elements is a space S topologically equivalent to the plane, by Moore's paper in the *Transactions* referred to above. Let the images in S of sets in Z be denoted by the corresponding small letters. Then the set $h_1 \cdot k_1$ is totally disconnected and so h_1 and k_1 satisfy the hypothesis of another theorem by Moore.[†] There is then a simple closed curve j in S such that $j \supset h_1 \cdot k_1$ and h_1^* and k_1^* are in different components of $S - j$. Returning to the plane, we see that H_1^* and K_1^*, and a fortiori H^* and K^*, lie in different components of $Z - J$.

As j is a simple closed curve in S, J is the upper semi-continuous image of a circumference. Its construction insures that the elements of J are as described in the statement of the lemma.

COROLLARY 1. *The points of $J - H_1 \cdot K_1$ are common frontier points of*

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[*] Since this article was submitted to the editors, there has appeared an article by C. Kuratowski ("Sur la separation d'ensembles situes sur le plan," *Fundamenta Mathematicae,* Vol. 12, pp. 214-239) which contains an interesting discussion of the subject of separation of sets by continua of the nature of J. By virtue of § 9 this lemma can be deduced as a corollary of Theorem V by Kuratowski. The proof here given has been retained as being perhaps of interest to those familiar with Moore's earlier work along the same line.

[†] R. L. Moore, "Concerning the Separation of Point Sets by Curves," *Proceedings of the National Academy of Sciences,* Vol. 11, p. 470.

the two components of $Z - J$ and they all lie in the component of $Z - H$ containing K^*.

For, if x is such a point, there is a $\delta > 0$ so small that all the points of J in $V_\delta(x)$ lie on an arc of J composed wholly of elements which are points. Since J is a simple closed curve of elements, the part of J not within this arc does not separate Z. Hence x is a frontier point of both components of $Z - J$. Now, if two points of $J - H_1 \cdot K_1$ are in different components of $Z - H$, there are two arcs L_1 and L_2 of J whose elements are points and which lie in different components of $Z - H$. Then, if I is the component of $Z - J$ containing H^*, L_1 and L_2 can be joined in the other component E of $Z - J$, without crossing H, a manifest contradiction.

COROLLARY 2. *If R is a component of $Z - (H + K)$ which has frontier points on H^* and K^*, then $J' = J \cdot R$ is the homeomorphic image of a simple open arc and $\overline{J'} - J'$ consists of two continua lying on different components of $H \cdot K$.*

Proof. If J' is not connected, there is in R a simple arc mn such that $J' \cdot (mn) = m + n$ and m and n lie on different components of J'. From the construction of J, m and n lie on simple arcs A_1 and A_2, respectively, of J, which lie in R, and $\overline{J - (A_1 + A_2)} = B_1 + B_2$, where B_1 and B_2 are continua, $B_1 \cdot B_2 = 0$, and $H \cdot K \subset B_1 + B_2$. Also both B_1 and B_2 contain points of $H \cdot K$, for otherwise one of them, say B_1, does not meet $H + K$. Then $B_1 \subset R$ and m and n would be on the same component of J'.

Let I and E be the components of $Z - J$ and $mn - (m + n) \subset E$. Then mn divides E into two simply connected regions E_1 and E_2, where the frontier of each contains points of one of the sets B_1 and B_2, but not the other. If $E \supset K^*$, we have a contradiction. For $\bar{E}_1 + B_1 + A_1 + A_2$ and $\bar{E}_2 + B_2 + A_1 + A_2$ are continua both of which contain points of the continuum K, their common part is $mn + A_1 + A_2$, and $K \cdot (mn + A_1 + A_2) = 0$. Likewise, we have a contradiction if $E \supset H^*$. Thus J' is connected.

Since J' is connected and each of its elements is a point and since J is the upper semi-continuous image of a circumference, J' is the homeomorphic image of a simple arc C. This arc is open. To prove this set $B = J - J'$. Neither B nor J' is an $S(I, E)$. If C were closed, J' would be a continuum and $B \cdot J' = 0$, which is impossible. If C were closed at one end, B J' [illegible]

The two continua making up $\overline{J'} - J'$ lie on two elements M and N of D. If $M = N$, we have the same contradiction as when $\overline{J'} - J'$ is a continuum. But $\overline{J'} - J'$ contains no inner points of M or N. Hence $\overline{J'} - J'$ lies on two components of $H \cdot K$.

11. THEOREM. *Let R be a simply connected region whose frontier K is bounded. Let H be a bounded regular irreducible $S(a_i; R)$, $i = 1, 2, \cdots, n$. Then $H \cdot K$ is the sum of n or more closed sets $\{\alpha_i\}$ between each pair of which H is an irreducible continuum.*

Proof. Since H is regular, all the components $\{R_i\}$ of $Z - (H + K)$ containing points $\{a_i\}$ lie in one component S_1 of $Z - H$. By § 9, if $K_1^* = K \cdot S_1$, there is a component K_1 of $K \cdot \bar{S}_1$ containing K_1^* and $K_1^* = K_1 - H \cdot K_1$. Now let $M = K - K_1^*$ and set $H_1 = H + M$. Since K is a continuum, so is H_1. It is evident that $H_1 + K = H + K$, S_1 is a component of $Z - H_1$, and $K_1^* = K - H_1 \cdot K$. If we set $H_1^* = H_1 - H_1 \cdot K$, $H_1^* = H^*$. By §§ 5 and 6, H^* is connected and $\overline{H^*} = H$. Moreover, neither $H + M$ nor K is an $S(a_i, a_j)$, while $H + K$ is. Hence, by reference (C), $H_1 \cdot K$ is not connected. Thus H_1 and K satisfy all hypotheses of the lemma of § 10 and there is a continuum J with the properties there given.

By § 10, Corollary 1, all the elements of J which are points of $J - H_1 \cdot K$ lie in S_1. Since each R_i has frontier points on both H^* and K^*, as H is regular, and since $R_i \subset S_1$, R_i has frontier points on both H^* and K^*_1. Hence each R_i contains an arc A_i of J whose elements are points and we can write $J = \sum_1^n A_i + \sum_1^n N_i$, where each N_i is a continuum and $N_i \cdot \overline{(J - N_i)}$ consists of an end-point of A_i and one of A_p ($p = i + 1$, mod. n). Now $H \cdot K \subset \sum N_i$ and each N_i contains points of $H \cdot K$, as otherwise $H + K$ would not be an $S(a_i, a_{i+1})$.

Now set $B_1 = N_1$ and $B_2 = \sum_3^n A_i + \sum_2^n N_i$. Let I and E be the components of $Z - J$ containing H^* and K_1^*, respectively. Let A_1^* and A_2^* denote the arcs A_1 and A_2 without their end-points. If H' is a proper closed part of H, there is a point x of H^* not in H'. For some $\delta > 0$ there is a $V_\delta(x)$ in I such that $H' \cdot V_\delta(x) = 0$. Let m be a point in $R_1 \cdot V_\delta(x)$ and n one in $R_2 \cdot V_\delta(x)$. As n lies in I, A_2^* is an open arc, and $K \cdot I = 0$, there is a simple arc nz, such that z lies in A_2^*, $nz - z \subset I$, and $(nz) \cdot (B_1 + A_1 + B_2 + K) = 0$. Since $n + z \subset R_2$, $H + K$ is not an $S(n, z)$. As $(H + K) \cdot (B_1 + A_1 + B_2 + K) = K$, a continuum, $H + K + B_1 + A_1 + B_2$ is not an $S(n, z)$. Then there is an arc nz not meeting any of these sets such that $(nz) \cdot A_2^* = z$ and $nz - z \subset I$.

By § 10, Corollary 1, $A_2 + B_1 + B_2$ is not an $S(H^*, K_1^*)$ and so m can be joined to K_1^* by a simple arc not cutting $A_2 + B_1 + B_2$. Starting from m, let u be the first point where this meets K_1^* and let w lie on mu and be in some $V_\eta(u)$ so small that $J \cdot V_\eta(u) = 0$. As w is in E, $(mw) \cdot A_1^* \neq 0$. Let mw meet A_1^* for the last time at m'. Then $m'w - m' \subset E$ and $m'w \subset R_1$, since $A_1 \subset R_1$ and $(m'w) \cdot (H + K) = 0$. That is, w lies in R_1. Thus $A_2 + B_1 + B_2 + K$ is not an $S(m, w)$; nor is $H + K$, since $m + w \subset R_1$. Hence $A_2 + B_1 + B_2 + H + K$ is not an $S(m, w)$ and there is an arc mw not cutting any of these sets. Starting from m, let y be the first point where it meets A_1^*. Then $my \subset I + y$ and $(my) \cdot H' = 0$.

Let mn be an arc in $V_\delta(x)$. Then $(mn) \cdot H' = 0$, and $yz = my + mn + nz$ is a simple arc in $I + y + z$ not meeting H' and meeting J only in the points y and z of A_1^* and A_2^*, respectively.

Let H'' be a sub-continuum of H irreducible between B_1 and B_2. If yz is any simple arc in $I + y + z$, where y lies in A_1^* and z in A_2^*, it divides I into two simply connected regions such that the frontier of each contains points of one of the sets B_1 and B_2, but not the other. As H'' is irreducible between B_1 and B_2, $H'' - H'' \cdot (B_1 + B_2)$ is connected and contains points of both these regions. Hence it meets yz, as $H''(A_1 + A_2) = 0$. This result in connection with the preceding paragraph is contradictory unless $H'' = H$, i. e., unless H is irreducible between B_1 and B_2.

Now let $\alpha_i = (H \cdot K) \cdot N_i$; then $\alpha_i \cdot \alpha_j = 0$ if $i \neq j$ and $H \cdot K = \sum_1^n \alpha_i$. Thus H is irreducible between α_i and $\sum_2^n \alpha_i$ and, a fortiori, between α_1 and each of the remaining sets $\{\alpha_i\}$. As B_1 was chosen as any one of the sets N_i, this gives the theorem.

12. THEOREM. *Let R be a simply connected region whose frontier K is bounded. Let H be a bounded regular irreducible $S(a_i; R)$ where the set $\{a_i\}$ is enumerably infinite. Then for every integer n, $H \cdot K$ is the sum of n closed sets between each pair of which H is an irreducible continuum. Also $H \cdot K$ can be expressed as the sum of an infinity of closed sets between each pair of which H is an irreducible continuum.*

Proof. The first statement is a mere corollary of § 11. To prove the second proceed as follows. Let the component of $Z - (H + K)$ containing a_i be R_i. Using the third paragraph of the proof of § 11 divides $H \cdot K$ into closed sets β_{11} and β_{12}, between which H is irreducible. If we now use the sets A_1, A_2, and A_3 in a similar manner, it is evident that $H \cdot K$ falls into three sets β_{21}, β_{22}, and β_{23}, of which one is identical with one of the sets β_{11} and β_{12}, say β_{11}, and the sum of the other two is identical with β_{12}, and that H is irreducible between each pair of these. This process may be continued indefinitely. It obviously generates a dyadic decomposition of $H \cdot K$ into an infinity of closed sets $\{\alpha\}$, no two of which have common points. Each α is the divisor of a sequence $\{\beta_{i,n_i}\}$. If α and α' are any two of these sets and i is sufficiently great, $\alpha \subset \beta_{i,n_i}$ and $\alpha' \subset \beta_{i,n'_i}$, where $n_i \neq n'_i$. Hence H is irreducible between α and α'.

13. THEOREM. *Let $F = H + K$ be the union of two bounded continua. Let $H \cdot K$ be the sum of a finite number n of closed sets $\{\alpha_i\}$ between each pair of which H is irreducible. Then there are at least n components of $Z - F$ whose frontiers contain H as a proper part.*

Proof. Since K is bounded, it contains a sub-continuum K' irreducible between α_1 and $\sum_2^n \alpha_i$. Then $K' - H \cdot K'$ is connected and lies in some component S_1 of $Z - H$. By § 9, if $K^* = K - H \cdot K$, $K^* \cdot S_1$ lies on a component K_1 of $K \cdot \bar{S}_1$ and $K^* \cdot S_1 = K_1 - H \cdot K_1$. Then K_1 contains points of n_1 sets $\{\alpha_i\}$ where $n_1 \geqq 2$. If $n_1 \neq n$, there is in like manner a component S_2 of $Z - H$ such that $K^* \cdot S_2$ lies on a component K_2 of $K \cdot \bar{S}_2$, $K^* \cdot S_2 = K_2 - H \cdot K_2$, and K_2 has points of two or more sets $\{\alpha_i\}$ including α_{n_1+1}. Continuing this process, we obtain k components $\{S_j\}$ of $Z - H$ such that for each j, $K^* \cdot S_j$ lies on a component K_j of $K \cdot \bar{S}_j$, $K^* \cdot S_j = K_j - H \cdot K_j$, and K_j contains points of n_j sets $\{\alpha_i\}$, where $n_j \geqq 2$ and $\sum n_j \geqq n$.

Now consider any one of these, say K_1. Let $K_1{}^* = K^* \cdot S_1$. Let $\alpha'_i = K_1 \cdot \alpha_i$ if this set is not void. There are n_1 sets $\{\alpha'_i\}$ and $H \cdot K_1 = \sum \alpha'_i$. Let $H^* = H - H \cdot K_1$. Then H and K_1 satisfy the hypotheses of the lemma of § 10 and there is a continuum J with the properties there described.

To each component of an α'_i corresponds an element of J made up of this component and those components of $Z - (H + K_1)$ bounded by it. If β_i denotes the set of elements of J corresponding to components of α'_i, it is easily seen that β_i is closed. Since J is the upper semi-continuous image of a circumference C, β_i corresponds to a closed set γ_i on C and $C - \sum \gamma_i$ is a set of open arcs. It is easily seen that of these at least n_1 arcs $\{l_i\}$ have their end-points on different sets γ_i. The image L_i of l_i is an open arc of elements of J which are points and L_i has improper limiting points on two different sets β_i and hence on two sets α'_i. As $L_i \cdot \sum \beta_i = 0$, $L_i \cdot (H + K_1) = 0$ and L_i lies in some component R_i of $Z - (H + K_1)$. This is also a component of $Z - (H + K)$. As $J - L_i$ does not separate H^* from $K_1{}^*$, it is evident that R_i has frontier points on both H^* and $K_1{}^*$. Now L_i is a component of $J - \sum \beta_i$. If $J_i = J \cdot R_i$, $L_i \subseteq J_i$. But, by

§ 10, Corollary 2, J_i is connected and meets no β_i. Hence $J_i = L_i$. Thus the existence of n_1 such arcs $\{L_i\}$ proves the existence of n_1 distinct components $\{R_i\}$ of $Z - (H + K_1)$ which have frontier points on both H^* and K_1^*.

Consider any one of these, as R_1. Let $\bar{L}_1$ meet α'_1 and α'_2; then $\bar{L}_1$ is irreducible between α'_1 and α'_2. So is H and $H \cdot \bar{L}_1 \subseteq \alpha'_1 + \alpha'_2$. Then by reference (B), p. 84 there are at least two principal components M and N of $Z - (H + \bar{L}_1)$. Since K_1^* lies in one component of $Z - J$ and H^* in the other, K_1^* lies in one component of $Z - (H + J)$ and, *a fortiori*, in but one component of $Z - (H + \bar{L}_1)$. Suppose that $M \cdot K_1^* = 0$. Then $M \cdot (H + K_1) = 0$ and $M \subset R_1$, as M has limiting points on L_1, which lies in R_1. As $H \cdot R_1 = 0$ and H is a part of the frontier of M, it is therefore a part of the frontier of R_1. We have therefore shown that there are at least n_1 components of $Z - (H + K)$ lying in S_1 and having H as a proper part of their respective frontiers.

As this holds for every S_j, there are at least $\sum n_j \geqq n$ components of $Z - (H + K)$ whose frontiers contain H and points of K^*.

COROLLARY. *Let $F = H + K$ be the union of two bounded continua. For every integer n let $H \cdot K$ be the sum of n closed sets between each pair of which H is irreducible. Then there is an enumerably infinite set of components of $Z - F$ whose frontiers contain H as a proper part.*

Remarks. The conclusion of the above theorem may be stated as follows by using § 5: *Then H is an irreducible $S(a_i; R)$ where R is the component of $Z - K$ containing $H - H \cdot K$ and the points $\{a_i\}$ lie in R and number at least n.* To see that the converse of the above theorem is false, let H be a circumference and K a straight line segment joining a point within H to one without.

14. The actual existence of continua H and K satisfying the hypothesis of §§ 11-13 can be readily established by modifying Ex. 2 on p. 88 of reference (B), as indicated in § 7. On the other hand, Ex. 1, just preceding this, shows that $H \cdot K$ may be the sum of an infinity of closed sets between each pair of which H is irreducible and yet H need not be an irreducible $S(m, n, R)$ for any two points in R. Finally, it should be noted that H may be a regular irreducible $S(a_i; R)$, $i = 1, 2, \cdots, n$ and be an irreducible $S(a_i; R)$. [illegible]

[illegible] components.

Various theorems may be deduced from §§ 11-13 by modifying the hypotheses. Two examples are given, of which the first is obtained from § 11, and the second from § 13.

THEOREM. *Let R be a simply connected region whose frontier K is bounded. Let H be a bounded continuum dividing R into two or more regions and be a part of the frontier of each of these. Let $H = \overline{H \cdot R}$ and let H itself separate no two points of R. Then H is irreducible between any pair of components of $H \cdot K$.*

To prove this let m and n be any two points of R, lying in components R_m and R_n, respectively, of $R - R \cdot H$. Then H is a regular irreducible $S(m, n, R)$. By §§ 11 and 12 we know that $H \cdot K$ is the sum of two closed sets between which H is irreducible. Let A and B be components of one of these and H be reducible between them. Then there is a sub-continuum h of H irreducible between A and B, and $H - h$ is connected. But by § 8 there are at least two components S and T of $Z - (h + K)$ whose frontiers contain points of both $h - h \cdot K$ and $K - h \cdot K$; let S be the one containing points of $R \cdot H$ not in h. Then T is a component of $R - R \cdot H$ whose frontier does not contain H, contrary to hypothesis.

THEOREM. *Let $F = H + K$ be the union of two bounded continua. Let $H \cdot K$ have a finite number of components and H be irreducible between each pair of them. Let H not cut the plane. Then H is a part of the frontier of every component of $Z - F$ whose frontier is not a part of K.*

For, if there were a component S of $Z - F$ whose frontier contained points of $H - H \cdot K$, but not the whole of H, there would be a proper sub-set h of H such that $h + K$ separated a point m of S from a point n of some component of $Z - F$ whose frontier contains H. But m and n lie in the same component of $Z - K$, since $H - H \cdot K$ is connected. Then K is not an $S(m, n)$, nor is h, since H does not cut the plane. There is then a contradiction by reference (C) unless h contains a continuum joining two components of $H \cdot K$, which in turn is impossible, since H is irreducible between these components.

NEW HAVEN, CONN.

Number of Abelian Subgroups in Every Prime Power Group.

BY G. A. MILLER.

It is well known that whenever a group of order p^m, p being a prime number, contains an abelian subgroup of order p^3 then the number of its abelian subgroups of this order is of the form $1 + kp$.* This is obviously a special case of the following theorem: *If the order of a group is divisible by* $p^{\alpha(\alpha-1)/2+1}$ *then the number of its abelian subgroups of order* p^α *is always of the form* $1 + kp$. This theorem results almost immediately from the fact that every invariant abelian subgroup whose order does not exceed $p^{\alpha-1}$ which appears in such a group must be contained in an invariant abelian subgroup of larger order. Hence it results that if we count every invariant abelian subgroup of order p^α as many times as the number of its subgroups of order $p^{\alpha-1}$ which are invariant under the entire group and note that each of the latter subgroups appears in a number of the former subgroups which is of the form $1 + kp$, and that this is also the form of the number of the latter subgroups which appear in one of the former, it results that if the number of the latter groups is of the form $1 + kp$ this must also be the form of the number of the former subgroups. Hence the given theorem has been established since the number of the abelian subgroups of order p^3 is of the given form.

From the preceding paragraph it results that when α is given we can always make m sufficiently large so that in every group whose order is divisible by p^m the number of abelian subgroups of order p^α is of the form $1 + kp$. Moreover, it is well known that if a non-abelian group of order p^m contains an abelian subgroup of index p then the number of its subgroups of this index is always exactly $1 + p$. In this case, the number of the abelian subgroups of index p^2 must always be of the form $1 + kp$ since every abelian subgroup of this index must be contained in an abelian subgroup of index p. On the contrary, when a group of order p^m, $m > 7$, does not involve an abelian subgroup of index p but involves abelian subgroups of index p^2 the number of the latter subgroups is not necessarily of the form $1 + kp$. In fact, it is obviously possible to construct such groups which contain exactly two abelian subgroups of index p^2 and have a central of index p^4, for every value of p and every value of $m > 7$. To construct such a group we may start with an abelian group of order p^{m-2} and of type $(1, 1, 1, \cdots)$, and extend this group by an operator s_1 of order p which has p^2 conjugates under this extended group. The group thus obtained may be extended by an operator s_2 which is commutative with s_1 and gives rise to a commutator subgroup of order p^2 which is found in the central of the preceding group and has only the identity in common with the commutator subgroup to which s_1 gives rise. The resulting group satisfies the given conditions.

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* Miller, Blichfeldt, Dickson, *Finite Groups* (1916), p. 128.

From the preceding paragraph it results that the number of the abelian subgroups of index p^2 which appear in a group of order p^m is not necessarily either 0 or of the form $1 + kp$. It is, however, easy to prove that whenever a group of order p^m contains at least one abelian subgroup of index p^2 then it must contain an invariant abelian subgroup of this index. To prove this theorem we note that if H_1 is such an abelian subgroup which is non-invariant then it must have a conjugate H_2 such that each of the two subgroups H_1 and H_2 transforms the other into itself in view of the general theorem that every set of conjugate subgroups, or conjugate operators, of a prime power group involves at least p conjugates such that each of them transforms each of the remaining $p - 1$ into itself. If H_1 and H_2 would generate the entire group then each of these subgroups would be invariant. If they generate a non-abelian subgroup of index p then the number of its abelian subgroups of order p^{m-2} must be of the form $1 + kp$. As this subgroup of index p is invariant under the entire group we have established the following theorem: *If a group of order p^m contains at least one abelian subgroup of index p^2 then it must contain an invariant abelian subgroup of this index.*

It was noted above that whenever $m > 7$ it is possible to construct a group of order p^m which contains exactly 2 abelian subgroups of index p^2. In particular, it is possible in this case to construct a group in which the number of abeilan subgroups of index p^2 is not of the form $1 + kp$. We proceed to prove that this is not possible whenever $m \leqq 7$. For this purpose we shall first prove that if a group G of order p^7 contains an abelian subgroup H_1 of order p^5 then the number of its abelian subgroups of this order is always of the form $1 + kp$. It is evidently only necessary to prove that the number of its invariant abelian subgroups of order p^5 is of this form since the number of its non-invariant abelian subgroups of any order must be divisible by p. In view of the theorem at the end of the preceding paragraph it may be assumed that H_1 is an invariant subgroup of G.

If all the other abelian subgroups of order p^5 contained in G are non-invariant our theorem requires no proof. Hence it may be assumed that G contains two invariant abelian subgroups H_1 and H_2 whose common order is p^5. These subgroups either generate G or they generate a subgroup of order p^6. If, in the latter case, this subgroup is abelian but G is non-abelian then G contains either only one abelian subgroup of order p^6 and hence the number of its abelian subgroups of order p^5 is of the form $1 + kp$, or G contains exactly $p + 1$ abelian subgroups of order p^6 and hence has a central of order p^5. In this case the number of its abelian subgroups of order p^5 is clearly again of the form $1 + kp$. Hence we may assume for the purpose of proving the theorem under consideration that H_1 and H_2 generate a non-abelian subgroup of order p^6 which involves exactly $p + 1$ abelian subgroups of order p^5. These $p + 1$ subgroups are invariant under G since at least two of them have this property.

When G contains an abelian subgroup of order p^6 the number of its abelian subgroups of order p^5 is obviously of the form $1 + kp$. Hence we shall assume in what follows that no abelian subgroup of order p^6 appears in G but that it contains abelian subgroups of order p^5. If G contains an invariant abelian subgroup of this order in addition to the $p + 1$ considered in the preceding paragraph, and if this involves the cross-cut of these $p + 1$ subgroups then this cross-cut is the central of G and it appears in each of the abelian subgroups of order p^5 which appear in G. The number of these subgroups must therefore again be of the form $1 + kp$. On the other hand, if this additional invariant abelian subgroup does not involve the given cross-cut there will be $p + 1$ abelian subgroups of order p^5 which have for their cross-cut another subgroup of order p^4, and p of these do not appear in the group generated by H_1 and H_2. The central of G in this case is of order p^3. As similar remarks apply to the additional sets of $p + 1$ abelian invariant subgroups of order p^5, in case such sets exist, it results that the number of the invariant abelian subgroups of order p^5 contained in G is of the form $1 + kp$ except possibly in the case when H_1 and H_2 generate G and there are no two invariant abelian subgroups of order p^5 in G whose cross-cut is of order p^4.

It may also be assumed that G contains an invariant abelian subgroup of order p^4 which is not found in an invariant abelian subgroup of the order p^5 since the number of the former is of the form $1 + kp$ according to the general theorem stated in the first paragraph of this article. Hence it results that the central of G is of order p^3 and that every abelian subgroup of order p^4 which is found in G involves this central and that no two of these subgroups have any common operators besides those found in the central of G. Moreover, the central of G and its commutator subgroup must be identical. From this it follows that G involves exactly $p + 1$ invariant abelian subgroups and the following theorem has been established: *If a group of order p^7 contains an abelian subgroup of order p^5 then the number of its abelian subgroups of this order is of the form* $1 + kp$.

We shall now prove that if a group of order p^m contains an abelian subgroup of order p^4 then the number of its abelian subgroups of this order is always of the form $1 + kp$. From the general theorem stated in the first paragraph it results that this theorem is true whenever $m > 6$. Hence it remains only to prove it when $m = 6$, since it is obviously true when $m = 5$. We may again confine our attention to the invariant abelian subgroups of order p^4. Hence we assume that G is a non-abelian group of order p^6 which contains at least two invariant abelian subgroups H_1, H_2 of order p^4. If every abelian invariant subgroup of order p^3 contained in G were found in an invariant abelian subgroup of order p^4 our theorem would clearly be true since the number of the former is known to be of the form $1 + kp$. Hence we may assume in what follows that G contains an invariant abelian subgroup K of order p^3 which does not appear in H_1.

If H_1 and K generate G their cross-cut is of order p and hence the commutator subgroup of G is of order p. This is impossible since it was assumed that K does not appear in an abelian invariant subgroup of order p^4. Hence it results that H_1 and K generate a group of order p^5 whose central is of order p^2 and whose commutator subgroup coincides with this central. This group of order p^5 contains no abelian subgroup of order p^4 besides H_1. The remaining operators of G cannot be commutative with all the operators of the central of this subgroup since K is not contained in an invariant abelian subgroup of order p^4. As every invariant subgroup of index p^2 in any group must involve the commutator subgroup of this group it results that G contains only one invariant abelian subgroup of order p^4. Hence the following theorem has been established: *If a group of order p^m contains an abelian subgroup of order p^4 then the number of its abelian subgroups of this order is always of the form* $1 + kp$.

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Finite Groups in which Conjugate Operations are Commutative.

BY C. HOPKINS.

In an article entitled *On Groups in which every two Conjugate Operations are Permutable,* Burnside[*] developed certain general properties of a group G which is subject to the sole restriction that its conjugate operations be commutative. In particular, he proved that if G is of finite order, it is the direct product of its Sylow subgroups,[†] and is either an abelian or a metabelian group if no Sylow subgroup is of order 3^m. In the same year Fite published his dissertation *On Metabelian Groups,*[‡] in which he defined the class of a group G as the number of terms in the sequence $G', G'', \cdots, 1$, where G' is the group of cogredient isomorphisms of G and each succeeding $G^{[i]}$ is the group of cogredient isomorphism of the preceding one. From this definition it is evident that abelian and metabelian groups form the categories of groups of class 1 and 2, respectively. Moreover, these two categories are included in the category of groups defined by the stipulation that every two conjugate operations be commutative. Hence it appears that a group in which conjugate operations are commutative may be conveniently classified and discussed according to its class. This procedure was adopted by Fite in a later article,[||] in which he asserted that the class of a group of finite order in which every two conjugate operations are commutative can not exceed 5.

In this paper we shall prove that the class of a finite group G in which every two conjugate operations are commutative can not exceed 3; we shall examine in some detail the case where G is of class 3 and has exactly three independent generating operations, and we shall exhibit a method of constructing all groups of this sort.

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[*] Burnside, *Proceedings of the London Mathematical Society*, Vol. 35 (1903), pp. 28-37.

[†] *Loc. cit.*, p. 34.

[‡] [illegible] of the American Mathematical Society, Vol. [illegible] ([illegible]), [illegible]

[illegible] *loc. cit.*, p. 348.

[||] [illegible] *Mathematische Annalen*, Vol. 67 (1909), pp. 498-510.

1. *General Results.* In what follows the symbol G shall denote a group of order 3^m in which every two conjugate operations are commutative. The assumption that G is of order 3^m involves no loss of generality, since it has been demonstrated by Burnside that if G is of finite order it is the direct product of its Sylow subgroups, and is of class 1 or 2 if no Sylow subgroup is of order 3^m. In what follows we are interested only in groups whose class exceeds 2. If G is of class k it cannot have fewer than k independent generating operations. This follows from the fact that such a group must contain an invariant commutator $P_{1\,2\,\ldots\,k}$ which is generated by the k independent operations $P_1, P_2, \cdots, P_k$.[*]

If a finite group G of order p^m and of class k whose conjugate operations are commutative has more than k independent generating operations, it must contain a subgroup of class k which has exactly k independent generating operations. To prove this we note that the commutator $P_{1\,2\,\ldots\,k}$ is generated by the k operations $P_1, P_2, \cdots, P_k$, no one of which can be expressed in terms of the remaining $k-1$. [If one of the P's could be expressed as the product of powers of the remaining $k-1$, then $P_{1\,2\,\ldots\,k}$ could be represented as a commutator whose k subscripts would involve only $k-1$ distinct symbols. But such a commutator must be the identity].[†] Consequently the subgroup $H = \{P_1, P_2, \cdots, P_k\}$ has exactly k independent generating operations and is of class k, since its kth, but not its $(k-1)$st, commutator subgroup [‡] is the identity. It is clear that H and G cannot coincide, since G was assumed to have more than k independent generating operations.[§]

It is a well-known fact that there exist groups of class k whose conjugate operations are commutative for values of k ranging from 1 to 3.[||] We shall now show that no groups exist for $k > 3$ in which conjugate operations are commutative. To establish this result it will be sufficient to consider the case $k = 4$, since a group of class k must contain a subgroup of class $k-1$. It was proved above that a group of class k with more than k independent generating operations must contain a subgroup of class k which has exactly k independent generating operations. Consequently we assume that the group

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[*] Burnside, *loc. cit.*, p. 29; this follows also from the definition of *class*.

[†] Burnside, *loc. cit.*, p. 30.

[‡] *Transactions of the American Mathematical Society*, Vol. 7 (1906), p. 61.

[§] Note that the number of independent generating operations of a group of order p^n is an invariant of the group. *Vide* Miller, *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 21.

[||] For $k = 1$ and $k = 2$ we have the categories of abelian and metabelian groups, respectively. For $k = 3$ we have the example exhibited by Fite, *loc. cit.*, p. 503.

$G_{(4)}$, whose conjugate operations are commutative, is of class 4 and has the four independent generating operations (S_1, S_2, S_3, S_4). The operations $S_{ij}, S_{ijk}, S_{ijkl}$ will be defined by the equations*

$$S_i^{-1}S_jS_i = S_jS_{ij};\ \ S_i^{-1}S_{jk}S_i = S_{jk}S_{ijk};$$
$$S_i^{-1}S_{jkl}S_i = S_{jkl}S_{ijkl}.$$

Burnside † has proved that the operation $S_{ab\ldots n}$ is changed into its inverse or remains unchanged according as an odd or an even permutation is effected on its subscripts. Now

$$S_{(12)(34)} = \overset{-1}{S}_{34}\overset{-1}{S}_{12}S_{34}S_{12}$$

is the inverse of $S_{(34)(12)}$. To show that $S_{(12)(34)} = \overset{-1}{S}_{1234}$ we proceed as follows:

$$\begin{aligned} S_{34}S_{(12)(34)} &= \overset{-1}{S}_{12}\,S_{34}S_{12} = \overset{-1}{S}_1\overset{-1}{S}_2S_1S_2\cdot S_{34}\cdot\overset{-1}{S}_2\overset{-1}{S}_1S_2S_1 \\ &= \overset{-1}{S}_1\overset{-1}{S}_2S_1\cdot S_{34}\overset{-1}{S}_{234}\cdot\overset{-1}{S}_1S_2S_1 = \overset{-1}{S}_1\overset{-1}{S}_2\cdot S_{34}\overset{-1}{S}_{134}\overset{-1}{S}_{234}S_{1234}\cdot S_2S_1 \\ &= \overset{-1}{S}_1\cdot S_{34}S_{234}\overset{-1}{S}_{134}\overset{-1}{S}_{2134}\overset{-1}{S}_{234}S_{1234}\cdot S_1 = \overset{-1}{S}_1\cdot S_{34}\overset{-1}{S}_{134}\overset{-1}{S}_{2134}S_{1234}S_1 \\ &= S_{34}S_{134}\overset{-1}{S}_{134}\overset{-1}{S}_{2134}S_{1234} = S_{34}\overset{2}{S}_{1234} = S_{34}\overset{-1}{S}_{1234}. \end{aligned}$$

Consequently ‡ $S_{(12)(34)} = \overset{-1}{S}_{1234}$. In the same manner we may show that $S_{(34)(12)} = \overset{-1}{S}_{3412}$. Since $S_{(12)(34)} = \overset{-1}{S}_{(34)(12)}$, we must have $\overset{-1}{S}_{3412} = S_{1234}$. But S_{3412} may be derived from S_{1234} by an even permutation on its subscripts; therefore $\overset{-1}{S}_{1234} = S_{1234}$, which is impossible, since S_{1234} is of order 3.§ From this contradiction we have the theorem: *There exists no finite group G of class greater than 3 in which conjugate operations are commutative.*

2. *Groups of Class* 3. In Art. 1 we proved that the group G of class k has exactly k independent generating operations or else contains a subgroup with this property. Consequently we shall investigate only those groups of

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* Burnside defines the symbol P_{ab} by the equation $P_b^{-1}\ P_a\ P_b = P_a\ P_{ab}$. It is evident that this notation differs only superficially from that given above.

† *Loc. cit.*, p. 31.

‡ In the reduction above use was made of the following: $S_{12} = S_2^{-1}\ S_1^{-1}\ S_2\ S_1$; the operations $S_{ij\ldots n}$ and $S_{ab\ldots l}$ are commutative if their subscripts have a letter [illegible] its inverse according as an even or an odd permutation is effected on the letters in its subscript. The last three results are due to Burnside.

§ Burnside, *loc. cit.*, p. 31.

class k which have k independent generating operations. We are not interested in groups of class 1 or 2, which are contained in the categories of abelian and metabelian groups, respectively, and we have proved that G cannot be of class greater than 3. We assume, therefore, that G is generated by S_1, S_2, and S_3 of orders 3^{m_1}, 3^{m_2}, 3^{m_3}, respectively.[*] In the way of notation let the symbols c_{ij}, c_{kij} be defined by the equations $s_i^{-1}s_js_i = s_jc_{ij}$, $s_k^{-1}c_{ij}s_k = c_{ij}c_{kij}$.

Now from the results of Burnside[†] we have the following relation: (a) s_j is commutative with c_{ij} and c_{ki}; (b) the c's are relatively commutative; (c) $c^3_{ijk} = 1$; (d) $c_{ijk} = c_{kij} = \overset{-1}{c_{jik}}$ etc., that is, an even permutation on the symbols of the subscript of any c leaves it unchanged, while an odd permutation changes c into its inverse.

These relations evidently form a set of necessary conditions that G be of class 3. That they are also sufficient will be seen in the following paragraph. But we also obtain a set of sufficient conditions if we replace (c) by the weaker condition that c_{ijk} be invariant under the group $\{s_1, s_2, s_3\}$.

We assume that there exists a group $\Gamma = \{s_1, s_2, s_3\}$ which has the following properties: (a) s_1, s_2, s_3 are of orders 3^{m_1}, 3^{m_2}, 3^{m_3}, respectively; (b) s_i is commutative with every commutator of Γ whose subscript contains i; (c) the commutators of Γ are relatively commutative; (d) $c_{ijk} = c_{kij} = c_{jki} = \overset{-1}{c_{jik}} = \overset{-1}{c_{kji}} = \overset{-1}{c_{ikj}}$; (e) c_{ijk} is invariant in Γ. The notation will be made clearer by reference to the following equations:

$$\begin{aligned} s_1^{-1}s_2s_1 &= s_2c_{12} & \qquad s_1^{-1}c_{23}s_1 &= c_{23}k \\ s_2^{-1}s_3s_2 &= s_3c_{23} & \qquad s_2^{-1}c_{31}s_2 &= c_{31}k \\ s_3^{-1}s_1s_3 &= s_1c_{31} & \qquad s_3^{-1}c_{12}s_3 &= c_{12}k \end{aligned}$$

where $k = c_{123}$.

It is not difficult to see that any operation in Γ can be written in the form $s_1^{x_1} s_2^{x_2} s_3^{x_3} c_{12}^{y_3} c_{23}^{y_1} c_{31}^{y_2} k^{z}$.[‡]

Let $S = s_1^{a_1} s_2^{a_2} s_3^{a_3} c_{12}^{\alpha_3} c_{23}^{\alpha_1} c_{31}^{\alpha_2} k^{\gamma}$ and $T = s_1^{b_1} s_2^{b_2} s_3^{b_3} c_{12}^{\beta_3} c_{23}^{\beta_1} c_{31}^{\beta_2} k^{\delta}$ be any two operations of Γ and let $S^{-1}TS = TC_{st}$ and $T^{-1}ST = SC_{ts}$. If we can show that $S^{-1}C_{st}S = C_{st}$ and $T^{-1}C_{st}T = C_{st}$, then it must follow that any two conjugate operations of Γ are commutative.[*] One may easily verify the equation

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[*] Since Burnside proved that G is metabelian if its order is p^m where $p \neq 3$, and, further, that G is always the direct product of its Sylow subjects, we shall consider only the case where the order of G is a power of 3.

[†] *Loc. cit.*, pp. 29-31.

[‡] Burnside, *loc. cit.*, p. 32.

$$S^{-1}TS = Tc_{12}^{-a_2b_1+a_1b_2}\ c_{23}^{-a_3b_2+a_2b_3}\ c_{31}^{-a_1b_3+a_3b_1}k^{\lambda}$$

where $\lambda = -a_1b_1 - a_2b_2 - a_3b_3 + a_1\beta_1 + a_2\beta_2 + a_3\beta_3 - a_1a_2b_3 - a_2a_3b_1 + a_3b_1b_2 + a_1a_3b_2 - a_2b_1b_3$.

Also, $S^{-1}C_{st}S = C_{st}k^{-a_1a_3b_2+a_1a_2b_3}k^{-a_1a_2b_3+a_2a_3b_1}k^{-a_2a_3b_1+a_1a_3b_2} = C_{st}$. Therefore S is commutative with C_{st}.

Moreover, $T^{-1}S_{st}T = C_{st}k^{-a_3b_1b_2+a_2b_1b_3}\ k^{-a_1b_2b_3+a_3b_1b_2}\ k^{-a_2b_1b_2+a_1b_2b_3} = C_{st}$. Accordingly T is commutative with C_{st}.

From this result that conjugate operations of Γ are commutative it is easy to show that the order of any operation of Γ is a power of 3. Accordingly, if conditions $(a), \cdots, (e)$ are fulfilled, the group $\{s_1, s_2, s_3\}$ is a group of order 3^m of class 3 whose conjugate operations are commutative. Thus the sufficiency of these conditions is demonstrated. From the necessary conditions obtained by Brunside for such groups it follows that the invariant operation k is of order 3.

3. *A Method of Constructing a Group of Class* 3. That groups of order 3^m and class 3 whose conjugate operations are commutative do not constitute a null-class was demonstrated by the example given by Fite.[†] We shall now develop a general method of constructing groups of this character which have three independent generating operations.

Let s_2 be a substitution of order 3^{m_2} on the letters $a_1, \cdots, a_{3a}$. Let c_{12} be any substitution of order $3^{m_{12}}$, $m_{12} \leqq m_2$, which is commutative with s_2. The group $\{s_2, c_{12}\}$ admits an automorphism in which s_2 corresponds to s_2c_{12} while c_{12} corresponds to itself. We write $\{s_2, c_{12}\}$ as a regular group and denote by s_1' a substitution of order $3^{m_{12}}$ in the holomorph of $\{s_2, c_{12}\}$ which effects this automorphism. Let s_1'' be any substitution of order $3^{m_1''}$ on a set of letters distinct from the letters of $\{s_2, c_{12}\}$. The substitution s_1'' is therefore commutative with s_1' and with every substitution of $\{s_2, c_{12}\}$. Now $s_1 = s_1'\ s_1''$ is a substitution of order 3^{m_1}, where m_1 is equal to the larger of the two exponents m_{12} and m_1''. One readily sees that the group $\{s_2, c_{12}, s_1\} = \{s_1, s_2\}$ is a metabelian group whose order is a power of 3.

We can find a substitution k of order 3 which is commutative with every operation of the metabelian group $\{s_1, s_2\}$. The group $\{s_1, s_2, k\}$ admits an automorphism in which $s_1 \sim s_1$; $k \sim k$; $c_{12} \sim c_{12}$; $s_2 \sim s_2k$. We write

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[*] [illegible] that $C_{ts} = C_{st}^{-1}$; hence the two equations above involve the two additional equations $S^{-1}C_{ts}S = C_{ts}$ and $T^{-1}C_{ts}T = C_{ts}$.

[†] Fite, *loc. cit.*, p. 503.

$\{s_1, s_2, k\}$ as a regular group and denote by c_{31}' a substitution of order 3 in its holomorph which effects this automorphism. Let c_{31}'' be any substitution of order $3^{m_{31}''}$ on a set of letters distinct from the letters of $\{s_1, s_2, k\}$, where m_{31}'' does not exceed m_1. It is clear that c_{31}'' is commutative with c_{31}' and with every substitution of $\{s_1, s_2, k\}$. The order of $c_{31} = c_{31}' \, c_{31}''$ is $3^{m_{31}}$, where m_{31} is the larger of the two exponents m_{31}'' and 1. It is obvious that $\{s_1, s_2, k, c_{31}\} = \{s_1, s_2, c_{31}\}$ is a group whose order is a power of 3.

The group $\{s_1, s_2, c_{31}\}$ admits an automorphism which is defined by the following correspondences:

$$s_1 \sim s_1k^{-1}\,;\ s_2 \sim s_2\,;\ c_{12} \sim c_{12}\,;\ c_{31} \sim c_{31},\ k \sim k.$$

Let c_{23}' be a substitution of order 3 in the holomorph of the regular group $\{s_1, s_2, c_{31}\}$ which brings about this automorphism. Let c_{23}'' be a substitution of order $3^{m_{23}''}$ on a set of letters distinct from the letters of the regular group $\{s_1, s_2, c_{31}\}$, where $3^{m_{23}''}$ is not to exceed the order of s_2. Then $c_{23} = c_{23}' \, c_{23}''$, of order $3^{m_{23}}$, transforms the substitutions of $\{s_1, s_2, c_{31}\}$ according to c_{23}', since c_{23}'' is commutative with every substitution of $\{s_1, s_2, c_{31}\}$ as well as with c_{23}'. It is clear that $\{s_1, s_2, c_{31}\}$ and c_{23} together generate a group whose order is a power of 3.

One may easily verify the fact that the following correspondences define an antomorphism of $\{s_1, s_2, c_{31}, c_{23}\}$:

$$s_1 \sim s_1c_{31}\,;\ s_2 \sim s_2c_{23}^{-1}\,;\ c_{12} \sim c_{12}k\,;\ c_{31} \sim c_{31}\,;\ c_{23} \sim c_{23}\,;\ k \sim k.$$

$[s_1^{-1}\, s_2s_1 = s_2c_{12}\,;\ c_{31}^{-1}\, s_1^{-1}\, s_2c_{23}^{-1}\, s_1c_{31} = c_{31}^{-1}\, s_2c_{12}c_{23}^{-1}\, k^{-1}\, c_{31} = s_2k^{-1}\, c_{12}c_{23}^{-1}\, k^{-1}$ $= s_2c_{12}c_{23}^{-1}\, k$, since $k_3 = 1$. This proves that the operation corresponding to $s_1^{-1}\, s_2s_1$ is the same as the operation corresponding to s_2c_{12}. It is not difficult to show that this is true for the remaining conjugate operations.]

Let s_3' be any substitution of lowest order in the holomorph of the regular group $\{s_1, s_2, c_{31}, c_{23}\}$ which transforms the substitutions of this group according to the automorphism above. The order of s_3', which we denote by $3^{m_3'}$, is evidently equal to the larger of the two numbers $3^{m_{31}}$ and $3^{m_{23}}$, where $3^{m_{31}}$ and $3^{m_{23}}$ are the orders of c_{31} and c_{23}, respectively. Let s_3'' be any substitution of any order $3^{m_3''}$ on a set of letters different from those of the regular group $\{s_1, s_2, c_{31}, c_{23}\}$. Then $s_3 = s_3' \, s_3''$ is of order 3^{m_3} where m_3 is the larger of the two exponents m_3' and m_3''. Now s_3 and $\{s_1, s_2, c_{31}, c_{23}\}$ generate a group of order 3^m which has the three independent generating operations s_1, s_2, s_3. From the mode of construction it is clear that its operations satisfy the sufficient conditions which were developed in Art. 2. Consequently $G = \{s_1, s_2, s_3\}$ is a group of order 3^m and class 3 whose conjugate operations are commutative.

We shall now give two examples of such groups, each of order 3^7.

Example 1: In the notation of the preceding development we take $m_2 = 1$, $m_2 = 1$, $s_1'' = 1$. Then $\{s_1, s_2\}$ is a metabelian group of order 3^3 whose operations are all of order 3. Next, we take $c_{31}'' = 1$, and obtain the group $\{s_1, s_2, k, c_{31}\}$ of order 3^5, whose operations are all of order 3. Then we take $c_{23}'' = 1$, from which it follows that the group $\{s_1, s_2, k, c_{23}\}$ is of order 3^6 and contains only operations of order 3. Finally, we take $s_3'' = 1$ and obtain the group $\{s_1, s_2, s_3\}$ of order 3^7 whose operations are all of order 3. This is the group given by Fite.*

Example 2: Let s_2 and s_0 be two commutative operations of order 9 which have no powers in common except the identity. There exists an antomorphism of $\{s_2, s_0\}$ in which s_0 corresponds to itself while s_2 corresponds to $s_2 s_0$. Let s_1 be an operation of order 9 in the holomorph of $\{s_2, s_0\}$ which effects this automorphism. The group $\{s_2, s_0, s_1\} \equiv \{s_2, s_1\}$ is a metabelian group of order 3^6 which admits an automorphism defined by the correspondences $s_1 \sim s_1 s_1^{-3}$, $s_2 \sim s_2 \cdot s_2^{-3}$, $s_0 \sim s_0 \cdot s^3_0$. The holomorph of $\{s_1 s_2\}$ contains an operation s_3 of order 3 which brings about this automorphism. It is clear that s_1, s_2, and s_3 together generate a group of order 3^7 in which the sufficient conditions of Art. 2 are satisfied. This group is of a more special type than the preceding example, in view of the fact that certain of the commutators are powers of generating operations. In the notation of the preceding example, we have the relations $c_{23} = s_2^3$, $c_{31} = s_1^{-3}$, $k = c_{12}^3$.

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[*] *Loc. cit.*, p. 503.

Covariant Conditions for Multiple Roots of a Binary Form.

By L. T. Moore and J. I. Tracey.

If a binary form $(\alpha x)^n = 0$ has the roots $\alpha_1, \alpha_2, \cdots, \alpha_n$, any symmetric function of the differences of the roots and the differences between x and one or more of the roots is a covariant, provided each root enters the same number of times into the expression. By the proper use of this method invariants and covariants may be formed which will vanish on the hypothesis of any system of equalities between the roots. Cayley [*] has given a method of finding the conditions for the existence of any system of equalities among the roots of an equation, and has given the complete solution for the biquadratic and the quintic. For the general equation of order n the condition for a double root is well known, and the covariants which vanish for two double roots or for a triple root have been found.[†] This paper gives a method whereby a covariant may be formed which vanishes when an n-ic has a p-fold root, a p_1-fold root, a p_2-fold root, $\cdots$, a p_j-fold root simultaneously. It also enables one to determine the other systems of equality among the roots for which the same covariant will vanish.

1. *The condition for the existence of a p-fold root.* Let the order of the equation be m. The discriminant of the m-ic, which contains $\frac{1}{2}\, m \cdot m - 1$ factors of the type $(\alpha_1 - \alpha_2)^2$, may be represented by means of a diagram consisting of m points and having each of the m points connected by lines with the remaining $m - 1$ points. Then each factor of the above type corresponds to one of the lines in the diagram.

Next, divide the m points into two sets of q and $m - q$ points each, where $2q \leqq m$. Connect each of the $m - q$ points with all the remaining points of that set, and each of the q points with the remaining $q - 1$ points, but no point of either set is to be connected with any point of the other set. The product of terms corresponding to all the lines in the diagram will involve each of the $m - q$ roots to the degree $2(m - q - 1)$ and each of the q roots to the degree $2(q - 1)$. If this expression is multiplied by q factors of the type $(x - \alpha_i)^{2(m-2q)}$, where each α_i corresponds to one of the q set of roots, the product is a typical term of a summation which is a covariant of degree $2(m - q - 1)$ and of order $2q(m - 2q)$ which will be designated in the usual form $C_{2(m-q-1),\ 2q(m-2q)}$.

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[*] *Collected Mathematical Papers*, Vol. 2, p. 465.

[†] Tracey, *American Journal of Mathematics*, Vol. 36, p. 31.

Obviously every term of this covariant will vanish if the m-ic has a triple root or root of greater multiplicity. It will not vanish for any number of double roots less than or equal to q, but will vanish if there are more than q double roots. Hence it cannot be made to vanish for any possible number of double roots provided $m = 2q$ or $m = 2q + 1$.

If the m points can be arranged into sets and connected in such a way that for any p points chosen at least two of them will be connected, and further, that some $p - 1$ of the m points may be chosen which are unconnected, then an invariant or covariant formed by the above method will vanish if the m-ic has a p-fold root but will not vanish if the multiplicity of the root is less than p. Divide the m points into $p - 1$ sets so that no two sets differ by more than one in the number of points which they contain; then connect each point of every set with the remaining points in that set but with no other point. The covariant or invariant which corresponds to this arrangement will vanish for a p-fold root or roots of greater multiplicity, but will not vanish for any combination of simultaneous multiple roots provided that each has a multiplicity of less than p. This statement is readily seen to be true, for if the multiplicity of the roots be $s_1, s_2, \cdots, s_g < p$ and $s_1 + s_2 + s_3 + \cdots + s_g \gtreqless m$ these roots can be distributed among the $p - 1$ sets so that no set contains two roots which belong to the same s_i.

If m is a multiple of $p - 1$ there will be the same number of roots in each set and the resulting form is an invariant of degree $2(r - 1)$ where $r = m/(p - 1)$. If m is not a multiple of $p - 1$ the form will be a covariant since factors $(x - \alpha_i)$ must be introduced to make each root occur to the same degree. Let $(m + k)/(p - 1) = r$ where k is an integer between 0 and $p - 2$ inclusive, then in the diagram there will be $p - k - 1$ sets which contain r points and k sets which contain $r - 1$ points. The covariant obtained will be of degree $2(r - 1)$ and of order $2k(r - 1)$. The following table gives the forms which vanish when the m-ic has the indicated multiple root:—

Double root,		$I_{2(m-1)}$.
Triple root,	m even	I_{m-2}.
	m odd	$C_{(m-1),(m-1)}$.
Fourfold root,	$m = 3r$,	$I_{2(r-1)}$.
	〃 $= 3r - 1$	C [illegible]
	$m = 3r - 2$,	$C_{2(r-1),4(r-1)}$.
p-fold root,	$m = (p - 1)r - k$,	$C_{2(r-1),2k(r-1)}$.

There are $p-1$ possible cases according to the integral value of k less than $p-1$ which makes $(m+k)/(p-1)$ an integer r.

A different distribution of the roots among the $p-1$ sets will give rise to other covariants than those indicated in the table. All covariants, however, which are represented by such unequal distributions will vanish for some combination of simultaneous multiple roots even though none of the roots are of multiplicity equal to p.

2. *The condition that an n-ic have a p-fold and a p_1-fold root.* If $p_1 \geqq p$, the covariant which vanishes when the n-ic has a p-fold and a p_1-fold root may be represented in a similar manner by a diagram of n points. Divide these into p_1 sets so that p_1-p+1 sets contain a single point each, and distribute the remaining $n-p_1+p-1$ points among the remaining $p-1$ sets exactly as the m points were distributed among the $p-1$ sets in the preceding section. It is evident from this arrangement that p_1 unconnected points can be selected, after which, not more than $p-1$ points can then be selected which are unconnected. The corresponding covariant will vanish for a p_1-fold and a p-fold root; it will vanish for a p_1+1-fold root alone, but will not vanish for any number of simultaneous multiple roots of multiplicity less than p. It will vanish, however, if the n-ic has g multiple roots of multiplicity $s_1, s_2, \cdots, s_g$, respectively, provided each s_i satisfies the relation $p_1 > s_i > p$ and provided furthermore that $s_1+s_2+s_3+\cdots+s_g > (g-1)(p-1)+p_1$. This statement readily follows from the fact that of any s_i unconnected points chosen $s_i-(p-1)$ must be chosen from the p_1-p+1 set which contain single points.

Let $m=n-p_1+p-1$, then the degree of the covariant desired is seen to be the same, in terms of m, as that of the covariant which vanishes when an m-ic has a p-fold root, and the order is the order of that covariant increased by $2(p_1-p+1)(r-1)$ where r is the greatest number of roots in any set. Thus if $m/(p-1)$ is an integer r the required form is $C_{2(r-1),2(r-1)(n-m)}$; or in general, if $(m+k)/(p-1)$ is an integer r where k is some integer between 0 and $p-2$ inclusive then the covariant desired is a

$$C_{2(r-1),2(r-1)(n-m+k)}.$$

3. *The condition for a p-fold, a p_1-fold, and a p_2-fold root.* An extension of the methods of the two previous sections leads to the following results. Let $p_2 > p_1 > p$, the diagram consists of p_2-p_1 sets which contain single points, of p_1-p+1 sets which contain two points, and $p-1$ sets which contain the remaining m points of either r or $r-1$ points in each set. The

covariant represented by this diagram will vanish for a $p_2 + 1$-fold root alone; for a q_1-fold root and a q_2-fold root simultaneously provided $q_1 + q_2 > p_1 + p_2$; for a q-fold a q_1-fold and a q_2-fold root simultaneously provided $q + q_1 + q_2 > p + p_1 + p_2$; and for g simultaneous roots of multiplicity $s_1, s_2, \cdots, s_g$, provided each $s_i > p$, and provided also that $s_1 + s_2 + s_3 + \cdots + s_g > (g - 2)(p - 1) + p_1 + p_2$. It will not vanish for any number of simultaneous multiple roots of multiplicity less than p.

If, as before, we call $(m + k)/(p - 1)$ an integer r the covariant which vanishes for a p-fold, a p_1-fold, and a p_2-fold root is a

$$(1) \qquad C_{2(r-1),\, 2[(r-1)(p_2-p_1+k)+2(r-2)(p_1-p+1)]}.$$

Finally we can say that the covariant which vanishes when an n-ic has $j + 1$ multiple roots of multiplicity $p, p_1, \cdots p_j$ respectively, where each $p_{i+1} > p_i > p$, is a

$$(2) \qquad C_{2(r-1),\, 2[(p_j-p_{j-1}+k)(r-1)+2(p_{j-1}-p_{j-2})(r-2)+\ldots +(j-1)(p_2-p_1)(r-j+1)+j(p_1-p+1)(r-j)]}.$$

(1) and (2) may be written in the alternate forms:

$$(1') \qquad C_{2(r-1),\, 2[(r-1)(n-m+k)-2(p_1-p+1)]}.$$

and

$$(2') \qquad C_{2(r-1),\, 2[(r-1)(n-m+k)-2(p_{j-1}+2p_{j-2}+\ldots+(j-1)p_1-\frac{j(j-1)}{2}(p-1))]}$$

respectively.

where $m = n - (p_1 + p_2 + \cdots + p_j) + j(p - 1)$, and k is the lowest integer which makes $(m + k)/(p - 1)$ an integer r.

4. *Another method of constructing Covariants.* There is another method of forming a covariant which will vanish when an n-ic has a p_1-fold, a p_2-fold, $\cdots$, a p_j-fold root simultaneously, where $p_1 + p_2 + \cdots + p_j = m$ and which will not vanish for any other combination of multiple roots of multiplicity $q_1, q_2, \cdots, q_k$ respectively provided $q_1 + q_2 + \cdots + q_k < m$. A typical term of the summation which gives the covariant in terms of the roots of the n-ic may be represented by a diagram as follows:

Divide the n points into two sets, the first containing $m - 1$ points and the second containing $n - m + 1$ points. Connect each point of the first set with every point of the second set. Connect also every point of the second set with every other point of that set, but no two points of the first set are to be connected. Calling the points in the first set α_i and in the second set

β_i, there will be three types of factors in each term of the covariant, namely $(\alpha_i - \beta_j)$, $(\beta_i - \beta_k)^2$ and $(x - \alpha_i)^{n-2}$. There will be $m - 1$ factors of the latter type in each term, and as each root enters to the degree $2n - m - 1$ the covariant obtained by this process is a

$$C_{2n-m-1,\,(m-1)\,(n-2)}.$$

We observe that the factors of the type $(\alpha - \beta)$ only need to enter linearly for the existence of this covariant. The particular advantage of this method over the former is when the number of simultaneous multiple roots is large and the multiplicity of each relatively small. Thus for example, if this covariant is formed for the maximum number of double roots, say k, then it will vanish only for some combination of multiple roots whose combined multiplicity is $2k$. However, if the covariant which vanishes for k double roots is formed by the first method it will also vanish for a triple root.

On Certain Finitely Solvable Equations Between Arithmetical Functions.

By E. T. Bell.

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1. *Introduction and Summary.* One of the fascinations of arithmetic is that almost any one of its general theorems, no matter how old, will give something new and unexpected when properly treated. The origin of the present note is a conjecture by Descartes, put forth 290 years ago, and proved in 1911 by Dubouis. The connection between this and what follows will be pointed out in § 4. For the moment we may remark that the number of equations between arithmetical functions for which we can prove either that they have no solution or only a finite number of solutions, and in the latter case state the solutions, is extremely small. In general it is difficult to say anything significant about such an equation; a classic instance is that of the equation defining the perfect numbers. If however an arithmetical problem containing at least one statement involving 'all' or 'any' can be solved in more than one way, a comparison of two distinct modes of solution will yield the complete theory of an equation of the type described. The functions to be considered are as follows:

$F(n)$, ≡ the number of odd classes of binary quadratic (Gauss) forms for the negative determinant $-n$; $F_1(n)$, ≡ the number of even classes; $E(n)$, $\equiv F(n) - F_1(n)$. In these definitions all the usual conventions are to hold, as in H. J. S. Smith's *Report*, or as in the classnumber relations in Dickson's *History*.

$\epsilon(n)$, ≡ 1 or 0 according as n is or is not the square of an integer > 0.

$N_r(n)$, ≡ the number of representations of n as a sum of r squares with integer roots $\gtreqless 0$, the order of the squares in a particular representation being relevant.

$Z_r(n)$, ≡ the similarly defined function with the restriction that the roots of the squares are $\gtrless 0$.

$\sigma(n)$, ≡ the sum of all the divisors of n.

$\tau(n)$, ≡ the sum of the odd divisors of n.

$\xi(n)$, ≡ the excess of the number of $4k+1$ $(k \geqq 0)$ divisors of n over the number of $4k+3$ divisors.

(n, r), ≡ the coefficient of x^r in $(1+x)^n$.

[illegible] the known theorems

$$N_1(n) = 2\epsilon(n), \quad N_2(n) = 4\xi(n), \quad N_3(n) = 12E(n),$$
$$N_4(n) = [2 + (-1)^n]\tau(n).$$

Among others we shall prove the following. Unity is not counted as a prime.

I. *If* $m \equiv 3$ *mod* 8, *the only solutions of* $4F(m) = \sigma(m)$ *are* $m = 3, 11$.

II. *If* $m \equiv 1$ *mod* 4, *the only solutions of*

$$6F(m) = \sigma(m) + 3\xi(m) - \epsilon(m)$$

are $m = 1, 5, 9, 17, 29, 41$.

III. *The only odd squares* m^2 *such that*

$$6F(m^2) = \sigma(m^2) + 3\xi(m^2) - 1$$

are $m^2 = 1, 9$.

IV. *The only numbers* $m \equiv 1$, *mod* 4, *that are not squares, such that*

$$6F(m) = \sigma(m) + 3\xi(m)$$

are $m = 5, 17, 29, 41$. *Hence the only primes* $p \equiv 1$ *mod* 4 *such that* $6F(p) = p + 7$ *are* $p = 5, 17, 29, 41$.

V. *The only primes* $q \equiv 3$ *mod* 4 *such that* $4F(q) = q + 1$ *are* $q = 3, 11$.

VI. *The only* $m \equiv 3$ *mod* 4 *such that* $2F(2m) = \sigma(m)$ *are* $m = 3, 7$.

VII. *The only primes* q *of the form* $4k + 3$ *such that* $2F(2q) = q + 1$ *are* $q = 3, 7$.

VIII. *The only* $m \equiv 1$ *mod* 4 *such that* $2F(2m) = \sigma(m) + \xi(m)$ *is* $m = 1$.

IX. *If* p *is a prime of the form* $4k + 1$, $2F(2p) = p + 3$ *has no solution.*

X. *If* $m \equiv 1$ *mod* 4, $6F(m) = 3\sigma(m) + 3\xi(m) - \epsilon(m)$ *has no solution.*

Some of these are of course immediate corollaries of others. Nevertheless they are simple enough to merit separate statement. All follow from Descartes' conjecture which was proved by Dubouis, combined with the theorem

$$\text{XI.}\quad Z_r(n) = \sum_{j=0}^{r-1} (-1)^j \, (r, j) \, N_{r-j}(n),$$

which we shall prove.

2. *An algebraic lemma.* Let x be the umbra of the one-rowed matrix $(x_0, x_1, x_2, \cdots)$, and similarly for y and $(y_0, y_1, y_2, \cdots)$. Then, as always in the symbolic or umbral calculus, $x^n \equiv x_n$, and $x = y$ means only $x_n = y_n$ $(n = 0, 1, 2, \cdots)$. Each of $x = y + 1$ and $y = x - 1$ implies the other. That is,

$$x_n = (y+1)^n \equiv \sum_{j=0}^{n} (n,j) y_{n-j} \qquad (n = 0, 1, \cdots)$$

mplies and is implied by

$$y_n = (x-1)^n \equiv \sum_{j=0}^{n} (-1)^j (n,j) x_{n-j} \qquad (n = 0, 1, \cdots).$$

If in particular $y_0 = 0$, we see that

$$\sum_{j=0}^{n-1} (n, n-j) y_{n-j} = 0 \qquad (n = 1, 2, \cdots, r)$$

mplies

$$y_r = \sum_{j=0}^{r-1} (-1)^j (r,j) x_{r-j}.$$

Incidentally it may be noted that the last gives immediately the expansion of the determinant

$$\begin{vmatrix} x_r & (r, r-1) & (r, r-2) & \cdots & (r, 1) \\ x_{r-1} & (r-1, r-1) & (r-1, r-2) & \cdots & (r-1, 1) \\ x_{r-2} & 0 & (r-2, r-2) & \cdots & (r-2, 1) \\ x_{r-3} & 0 & 0 & \cdots & (r-3, 1) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1 & 0 & 0 & \cdots & (1, 1) \end{vmatrix}$$

s the stated value for y_r.

3. *Proof of* § 1, XI. Let n denote an arbitrary constant integer > 0. Write N for the umbra of $(N_r(1), N_r(2), \cdots, N_r(n), \cdots)$, and similarly or Z and $Z_r(n)$ $(n = 1, 2, \cdots)$. Then if $|q| < 1$, and $S \equiv 2\sum_{n=1}^{\infty} q^{n^2}$, he coefficient of q^n in the expansion of S^r is $Z_r(n)$, and that of q^n in the xpansion of $(1+S)^r$ is $N_r(n)$. Hence, from the identity.

$$(1+S)^r \equiv 1 + (r,1)S + (r,2)S^2 + \cdots + (r,r)S^r,$$

valid for all integers $r \geqq 0$, we infer, upon comparing coefficients of q^n, that $N_r(n) = (r,1)Z_1(n) + (r,2)Z_2(n) + \cdots + (r,r)Z_r(n)$. Now every integer is a sum of integer squares in at least one way. Hence $Z_0(n) = N_0(n) = 0$, and therefore we may write the last as

$$N_r(n) = \sum_{j=0}^{r} (r,j) Z_{r-j}(n),$$

which is valid for $r = 0, 1, 2, \cdots$. Hence $N = 1 + Z$; whence, $Z = N - 1$, and by § 2 we have § 1, XI.

4. *Descartes' conjecture.*[1] In 1638 Descartes stated in letters to Mer-

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[1] See Dickson, *History of the Theory of Numbers*, Vol. 2, p. 276; *ibid.*, p. 302, [illegible]

senne that he believed the only numbers n not a sum of 4 squares > 0 to be

$$1,\ 3,\ 5,\ 9,\ 11,\ 17,\ 29,\ 41,\ 4^h\lambda \qquad (\lambda = 2,\ 6,\ 14),\ \ h \geqq 0.$$

This conjecture was proved by Dubouis in 1911.

From § 1, XI, with $r = 4$, we get

$$Z_4(n) = N_4(n) - 4N_3(n) + 6N_2(n) - 4N_1(n),$$

the left of which vanishes only for the above values of n. Referring to § 1 for $N_j(n)$ $(j = 1, \cdots, 4)$, we see that

$$[2 + (-1)^n]\,\tau(n) + 3\xi(n) = 6E(n) + \epsilon(n)$$

is solvable only for the values of n stated by Descartes.

Incidentally we note that $4^h(8k + 7)$ *is the sum of* 4 *squares with roots* $\gtrless 0$ *in* 8 *or* 24 *times* $\sigma(8k + 7)$ *ways, according as* $h = 0$ *or* $h > 0$. For, the stated number is not a square or a sum of 2 or 3 squares, and the theorem follows from $Z_4(n) = N_4(n)$, $n \equiv 4^h(8k + 7)$.

5. *Proofs of* § 1, *I-X*. From § 4 we see that the only odd integers $n > 0$ satisfying

$$\sigma(n) + 3\xi(n) = 6E(n) + \epsilon(n)$$

are $n = 1, 3, 5, 9, 11, 17, 29, 41$. We recall that $\xi(n)$ vanishes for $n = 2^h(4k + 3)$; evidently $\epsilon(n) = 0$ for the same n. Again, $E(8k + 7) = 0$, $3E(8k + 3) = 2F(8k + 3)$. Hence § 1, I follows; V is a corollary. Since $E(4k + 1) = F(4k + 1)$, § 1, II is proved, and III, IV are corollaries.

From the even values of n given as in § 4 by Descartes, we have the following:

XII. *If* $n = 2^{2h+k}\mu$, $h \geqq 0$, $k > 0$, *and if* $\mu \equiv 3$ *mod* 4, *the only solutions of*

$$2E(2^{2h+k}\mu) = \sigma(\mu)$$

are $h \geqq 0$, $k = 1$, $\mu = 3$ *and* $h \geqq 0$, $k = 1$, $= 7$.

XIII. *With h, k as in XII, and* $\mu \equiv 1$ *mod* 4, *the only solutions of*

$$3\sigma(\mu) + 3\xi(\mu) = 6E(2^{2h+k}\mu) + \epsilon(2^{2h+k}\mu)$$

are $h \geqq 0$, $k = 1$, $\mu = 1$.

XIV. *If* $p \equiv 1$ *mod* 4 *is prime,* $2E(2^{2h+k}p) = p + 3$ *has no solution.*

Since $E(4n) = E(n)$, it is sufficient to consider XII-XIV with $h = 0$. From XII we thus get § 1, VI, and VII is a corollary. Similarly XIII gives § 1, VIII, to which IX is a corollary.

In the equation in XIII take $k = 2$. Since $\epsilon(2^2\mu) = \epsilon(\mu)$, we get § 1, X.

On the Number of Representations of Integers by Certain Ternary Quadratic Forms.

By J. V. Uspensky.

In a recent paper * Professor L. E. Dickson has considered the possibility of representing integers by certain positive ternary quadratic forms. For the forms considered by Professor Dickson as well as for some other particular forms it is possible, however, to give the exact number of representations. The number of cases when the exact number of representations can be given seems to be limited, wherefore the discussion of such cases may present certain interest. In the following we consider two groups of closely related forms. The first group includes the following forms:

(I) $x^2 + 2y^2 + 3z^2$
(II) $x^2 + 2y^2 + 6z^2$
(III) $2x^2 + 3y^2 + 6z^2$
(IV) $x^2 + 3y^2 + 6z^2$
(V) $x^2 + y^2 + 3z^2$
(VI) $x^2 + 3y^2 + 3z^2$
(VII) $x^2 + y^2 + 6z^2$
(VIII) $x^2 + 6y^2 + 6z^2$,

whose coefficients do not contain other prime factors except 2 and 3. Similarly, the second group, consisting of 6 forms:

(IX) $x^2 + y^2 + 5z^2$
(X) $x^2 + 5y^2 + 5z^2$
(XI) $x^2 + 2y^2 + 5z^2$
(XII) $x^2 + 5y^2 + 10z^2$
(XIII) $x^2 + 2y^2 + 10z^2$
(XIV) $2x^2 + 5y^2 + 10z^2$

contains only forms whose coefficients involve prime factors 2 and 5.

1. *Forms* (I), (II), (III), (IV). All these forms are closely related to the form $x^2 + y^2 + z^2$. The number of representations of integers by this form was first determined by Gauss in the fifth section of Disquisitiones as an application of his general theory of ternary quadratic forms. In a more elementary way the same result may be derived from Kronecker's classnumber relations as shown by Kronecker himself. Finally, there exists a very simple and lucid deduction, based on the arithmetic of quaternions and due to Mr. Venkov.†

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* Dickson, "Integers Represented by Positive Ternary Quadratic Forms," [illegible] Vol. 33 (1927), pp. 63–70.

† Venkov, "On the Arithmetic of Quaternions," *Bulletin of the Russian Academy of Sciences*, 1922.

Gauss' expression for the number of representations by the form $x^2+y^2+z^2$ involves two numerical functions $G(n)$ and $F(n)$ defined as follows: $G(n)$ is the number of all classes of positive binary quadratic forms corresponding to the determinant $-n$, this number being diminished by $1/2$ when n is a perfect square, and by $2/3$ when n is triple of a perfect square; similarly, $F(n)$ denotes the number of classes of all those forms, corresponding to the determinant $-n$ which can represent odd numbers with the exception that this number should be diminished by $1/2$ when n is a square of an *odd* number. It is also customary to set

$$G(0) = -(1/12), \qquad F(0) = 0.$$

This being so, we have in all cases

$$(1) \qquad N(n = x^2 + y^2 + z^2) = 12[2F(n) - G(n)].$$

Starting from this fundamental result it is not difficult to show that

$$(2) \qquad N(n = x^2 + y^2 + 2z^2) = 4(2 + (-1)^n)[2F(2n) - G(2n)]$$

and

$$(3) \qquad N(n = x^2 + 2y^2 + 2z^2) = 4(2 - (-1)^{(n-1)/2})[2F(n) - G(n)]$$

for an *odd* n, while

$$(4) \qquad N(n = x^2 + 2y^2 + 2z^2) = 4(2 + (-1)^{n/2})[2F(n) - G(n)]$$

for an *even* n. Now, making use of the relation

$$(5) \qquad N(3k = x^2 + 2y^2) + N(k/3 = x^2 + 2y^2) = 2N(k = x^2 + y^2)$$

holding true for every k [*] we find

$$2N(n = x^2 + 2y^2 + 6z^2) = N(3n = x^2 + 2y^2 + 18z^2) + N(n/3 = x^2 + 2y^2 + 2z^2).$$

Again,

$$N(3n = x^2 + 2y^2 + 18z^2) = \tfrac{1}{2}[N(3n = x^2 + 2y^2 + 2z^2) - N(n/3 = x^2 + 2y^2 + 2z^2)],$$

so that

$$(6) \qquad N(n = x^2 + 2y^2 + 6z^2) = \tfrac{1}{4}N(3n = x^2 + 2y^2 + 2z^2) + \tfrac{3}{4}N(n/3 = x^2 + 2y^2 + 2z^2),$$

for every n. This way the number of representations by the form $x^2+2y^2+6z^2$

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[*] In the following to avoid separate consideration of different cases we suppose that all the numerical functions we have to consider vanish when their argument is a fractionary number.

may be determined by making use of (4), but it is hardly necessary to develop the final formulas.

For any number n we obviously have

$$N(2n = x^2 + 2y^2 + 6z^2) = N(n = x^2 + 2y^2 + 3z^2),$$

whence and from (6) it follows that

$$(7) \qquad N(n = x^2 + 2y^2 + 3z^2) \\ = \tfrac{1}{4}N(6n = x^2 + 2y^2 + 2z^2) + \tfrac{3}{4}N(2n/3 = x^2 + 2y^2 + 2z^2),$$

where the right hand member may be expressed by means of (4).

The same relation (5) leads easily to the following equation:

$$N(3n = x^2 + 2y^2 + 6z^2) + N(n = 2x^2 + 3y^2 + 6z^2) \\ = 2N(n = x^2 + 2y^2 + 2z^2)$$

which, combined with (6), gives

$$N(n = 2x^2 + 3y^2 + 6z^2) = -\tfrac{1}{4}N(9n = x^2 + 2y^2 + 2z^2) \\ + (5/4)N(n = x^2 + 2y^2 + 2z^2).$$

Now, taking into account (4) as well as the general formulas

$$G(9m) = [4 + (m/3)]\, G(m) - 3G(m/9)$$
$$F(9m) = [4 + (m/3)]\, F(m) - 3F(m/9),$$

we finally find:

$$(8) \qquad N(n = 2x^2 + 3y^2 + 6z^2) = \{[1 - (n/3)]/4\}\, N(n = x^2 + 2y^2 + 2z^2),$$

provided, n is not divisible by 3.

Finally, to determine the number of representations by the form $x^2 + 3y^2 + 6z^2$ we notice that

$$N(2n = 2x^2 + 3y^2 + 6z^2) = N(n = x^2 + 3y^2 + 6z^2)$$

and therefore, supposing n non-divisible by 3,

$$(9) \qquad N(n = x^2 + 3y^2 + 6z^2) \\ = \{[1 + (n/3)]/4\}\, N(2n = x^2 + 2y^2 + 2z^2).$$

2. *Forms* (V), (VI), (VII), (VIII). These four forms are again intimately related. With respect to forms

$$x^2 + y^2 + 3z^2 \text{ and } x^2 + 3y^2 + 3z^2$$

we confine ourselves to find the number of representations of integers which are not divisible by 3. To this end we resort to certain classnumber relations established by elementary means in our paper "On Gierster's class-number relations."

Denoting by n any integer, we have in case of an odd n

$$\sum \mathrm{F}(3n - 9h^2) = (1/2)\phi(3n) - \phi(n) + (3/2)\psi(n/3)$$
$$\sum G(3n - 9h^2) = \phi(3n) - (8/3)\phi(n) + 3\psi(n/3)$$

while for an *even* n

$$\sum F(3n - 9h^2) = X(n) - 3\phi(n/12) + 3\psi(n/12)$$
$$\sum G(3n - 9h^2) = 2\phi(n/2) - 6\phi(n/12 + \psi(n/12)$$
$$\sum G(3n - 9h^2) = 2\phi(n/2) - 6\phi(n/12) + \psi(n/12).$$

Here, as usual, $\phi(n)$ represents the sum of all divisors of n, while $X(n)$ and $\psi(n)$ are defined by

$$X(n) = \sum d, \quad n = d\delta, \quad \delta \textit{ odd}$$
$$\psi(n) = \sum (d' - d), \quad n = dd\,, \quad d < d.$$

At the same time

$$\sum F[(n/3) - h^2] = (1/2)\phi(n/3) + (1/2)\psi(n/3)$$
$$\sum G[(n/3) - h^2] = (1/3)\phi(n/3) + \psi(n/3)$$

for an odd n, and

$$\sum F[(n/3) - h^2] = X(n/3) - \phi(n/12) + \psi(n/12)$$
$$\sum G[(n/3) - h^2] = 2\phi(n/2) - 2\phi(n/12) + 2\psi(n/12)$$

for an even n. Putting

$$\omega(n) = 3G(3n) - 2F(3n) - 3[3G(n/3) - 2F(n/3)],$$

we derive from the preceding equations

$$2\sum \omega(n - 3h^2) = (-1)^n\, 4\sum (-1)^d d,$$

the righthand member sum being extended over *all* the divisors of n, which are not divisible by 3.

On the other hand, we have

$$N(n = x^2 + y^2 + 3z^2 + 3t^2) = (-1)^n\, 4\sum(-1)^d d,$$

so that, putting for brevity

$$N(n = x^2 + y^2 + 3z^2) = X(n),$$

we have for every n

$$2\sum \omega(n - 3h^2) = \sum X(n - 3h^2),$$

and this necessitates

$$X(n) = 2\omega(n),$$

that is,

$$N(n = x^2 + y^2 + 3z^2) = 2\omega(n).$$

In case n is not divisible by 3 this reduces to

$$N(n = x^2 + y^2 + 3z^2) = 2[3G(3n) - 2F(3n)]. \tag{10}$$

Now, if we substitute $3n$ instead of n and suppose $n \equiv 1 \pmod 3$, we get readily

$$N(n = x^2 + 3y^2 + 3z^2) = 4[3G(n) - 2F(n)], \tag{11}$$

while for $n \equiv 2 \pmod 3$ we obviously have

$$N(n = x^2 + 3y^2 + 3z^2) = 0.$$

To find the number of representations by the remaining forms $x^2 + y^2 + 6z^2$ and $x^2 + 6y^2 + 6z^2$ we make use of the general identity (VIII) of our above mentioned paper. We assume in it

$$F(x, y, z) = 0 \text{ for an even } x$$
$$F(x, y, z) = (-1)^{(x-1/2)} \text{ for an odd } x,$$

and after a simple discussion we arrive at the following useful relations:

$$\sum_{(a)} (-1)^x = \sum_{(b)} (-1)^x - [(-1)/2]^{n/2} \{N[3n = h^2 + 2k^2 + 2l^2] - N[(n/3) = h^2 + 2k^2 + 2l^2]\} \tag{12}$$

$$\sum_{(a)} (-1)^x = \sum_{(b)} (-1)^x + [(-1)/2]^{(n-1)/2} \{N[3n = h^2 + 2k^2 + 2l^2] - N[(n/3) = h^2 + 2k^2 + 2l^2]\} \tag{13}$$

holding true for even and odd n respectively, the extent of summations being as follows:

$$\text{(a)} \quad n = 3x^2 + y^2 + z^2; \qquad \text{(b)} \quad (n/3) = x^2 + y^2 + z^2.$$

We first suppose n non divisible by 3; denoting by R and S respectively the number of solutions of the equations

$$n = 12x^2 + y^2 + z^2$$
$$n = 3x^2 + y^2 + z^2, \quad x \text{ odd},$$

we have,

$$R - S = -[(-1)/2]^{n/2} N(3n = h^2 + 2k^2 + 2l^2) \text{ for an even } n$$
$$R - S = [(-1)/2]^{(n-1)/2} N(3n = h^2 + 2k^2 + 2l^2) \text{ for an odd } n.$$

Now the righthand member can be found from (3) and (4), while the sum $R+S$ is given by (10). We can, therefore, find R and S separately and the final expression for R is

$$R=2(-1)^{n/2}\,[3G(3n)-4F(3n)] \text{ for an even } n$$
$$R=2(-1)^{(n-1)/2}\,[3G(3n)-4F(3n)] \text{ for an odd } n.$$

On the other hand, it is obvious that

$$R(2n)=N(n=x^2+y^2+6z^2)$$

and, using the preceding expression for R,

$$(14) \qquad N(n=x^2+y^2+6z^2)=2(-1)^n\,[3G(6n)-4F(6n)]$$

for any number non-divisible by 3.

In a similar way, denoting by P and Q respectively the number of solutions of the equations

$$n=4x^2+3y^2+3z^2$$
$$n=x^2+3y^2+3z^2,\ x \text{ odd},$$

and replacing n by $3n$ [$n\equiv 1 \pmod{31}$ in (12) and (13)], we find

$$Q=(-1)^{(n-1)/2}\,4[4F(n)-3G(n)]$$

for an odd n and

$$P=(-1)^{n/2}\,4[3G(n)-4F(n)]$$

for an even n. But obviously

$$Q=N(n=x^2+6y^2+6z^2),$$

if n is odd and

$$P=N(n=x^2+6y^2+6z^2)$$

if n is even. That is, finally,

$$(15) \qquad N(n=x^2+6y^2+6z^2)=(-1)^{(n+1)/2}\,4[3G(n)-4F(n)]$$
$$(16) \qquad N(n=x^2+6y^2+6z^2)=(-1)^{n/2}\,4[3G(n)-4F(n)]$$

for an *odd* and *even* n respectively.

At the same time it is clear from the preceding analysis that the general expression for the number of representations by two forms $2x^2+2y^2+3z^2$ and $2x^2+3y^2+3z^2$ can be found, but it does not seem worth while to write down the final expressions.

3. *Forms* (*IX*) *and* (*X*). Both these forms are closely related to the form $x^2+y^2+z^2$. We start from the almost obvious relation

$$(17)\qquad N(5n = x^2 + y^2) + N(n/5 = x^2 + y^2) = 2N(n = x^2 + y^2)$$

and, replacing n by $n - 5z^2$, summing up all the equations thus obtained by making z assume all integral values so that $n - 5z^2$ remains $\geqq 0$, we get

$$2N(n = x^2 + y^2 + 5z^2) = N(5n = x^2 + y^2 + 25z^2) \\ + N(n/5 = x^2 + y^2 + z^2),$$

whence it follows,

$$(18)\qquad N(n = x^2 + y^2 + 5z^2) = (1/6)N(5n = x^2 + y^2 + z^2) \\ + (5/6)N(n/5 = x^2 + y^2 + z^2),$$

which gives the number of representations by (IX).

Suppose now $n \equiv \pm 1 \pmod 5$. From the same relation (17), denoting by Q the number of solutions of the equation

$$n = x^2 + y^2 + z^2,$$

where

$$x^2 \equiv \pm 1 \pmod 5,$$

signs corresponding to those in $n \equiv \pm 1 \pmod 5$, we derive

$$Q + N(n = x^2 + 25y^2 + 25z^2) = 2N(n = x^2 + 5y^2 + 5z^2).$$

But it is easy to see that

$$Q + N(n = x^2 + 25y^2 + 25z^2) = (2/3)N(n = x^2 + y^2 + z^2),$$

whence it follows that

$$N(n = x^2 + 5y^2 + 5z^2) = (1/3)N(n = x^2 + y^2 + z^2)$$

for every $n \equiv \pm 1 \pmod 5$.

4. *Forms* (XI), (XII), (XIII), (XIV). We are going to determine first the number of solutions of the equation

$$8n = i^2 + 2j^2 + 5k^2$$

in odd numbers i, j, k. For this purpose we take for our starting point the equations (20) and (21) established in our paper "On Gierster's classnumber relations" already quoted. These equations may be written as follows:

$$\sum F(5n - 25h^2) = (1/4)N(2n = x^2 + 2y^2 + 5z^2 + 10t^2) \\ + (1/16)\,N(5n = x^2 + y^2 + z^2 + 25t^2) + (5/2)\psi(n/5)$$

for an *odd* n and

$$\sum F(5n - 25h^2) = (1/4)N(2n = x^2 + 2y^2 + 5z^2 + 10t^2) \\ + (1/16)N(5n = x^2 + y^2 + z^2 + 25t^2) \\ + (5/2)\phi(n/10) + 5\psi(n/20)$$

for an *even* n. These relations, being combined with the known expressions for the sum

$$\sum F[n/5-h^2],$$

give

$$\begin{aligned}\sum \{F(5n-25h^2)-5F[(n-5h^2)/5]\} &= (1/4)N(2n=x^2+2y^2+5z^2+10t^2)\\ &\quad -(5/16)N(n/5=x^2+y^2+z^2+t^2)\\ &\quad +(1/16)N(5n=x^2+y^2+z^2+25t^2),\end{aligned}$$

and this holds true whether n be odd or even.

On the other hand, we have found in the same paper (see equation (36)):

$$\sum(-1)^z=(5/4)N[n/5=x^2+y^2+z^2] -(1/4)N(5n=x^2+y^2+z^2), \tag{20}$$

sum being extended over all solutions of the equation

$$8n=x^2+2y^2+5z^2,$$

whence we find easily

$$\sum Q(n-5h^2)=N(2n=x^2+2y^2+5z^2+10t^2) -(5/4)N(n/5=x^2+y^2+z^2+t^2)+(1/4)N(5n=x^2+y^2+z^2+25t^2).$$

Here $Q(n)$ stands for the number of solutions of the equation

$$8n=i^2+2j^2+5k^2$$

with *odd* i, j, k. Comparing this result with one previously found, we have

$$\sum Q(n-5h^2)=4\sum\{F(5n-25h^2)-5F[(n-5h^2)/5]\}$$

for every n, and hence we conclude

$$Q(n)=4[F(5n)-5F(n/5)]. \tag{21}$$

The value of $Q(n)$ being thus determined, we find from the same equation (20)

$$N(2n=x^2+2y^2+5z^2)=3G(5n)-2F(5n) +10F(n/5)-15G(n/5), \tag{22}$$

and this holds true for every n. Now, supposing first n non divisible by 5, we can write down the preceding equation as follows

$$N(n=x^2+2y^2+5z^2)=3G(10n)-4F(10n)$$

for every *even* number n non divisible by 5.

To find the number of representations of an *odd* number n we observe that

$$N(n = x^2 + 2y^2 + 5z^2) = N(4n = x^2 + 2y^2 + 5z^2)$$

The righthand member here is determined by the preceding equation, and, some obvious simplifications made, we get

$$(23) \qquad N(n = x^2 + 2y^2 + 5z^2) = (-1)^n\,[3G(10n) - 4F(10n)],$$

provided n is not divisible by 5.

Next, we replace n by $5n$ in (22) which gives

$$N(2n = x^2 + 5y^2 + 10z^2) = 3G(25n) - 2F(25n) + 10F(n) - 15G(n).$$

Here the righthand member can be simplified using the general relations

$$\begin{aligned} F(25n) &= [6 - (n/5)]\,F(n) - 5F(n/25) \\ G(25n) &= [6 - (n/5)]\,G(n) - 5G(n/25). \end{aligned}$$

Confining ourselves to the most interesting case $2n \equiv \pm 1 \pmod 5$, we find

$$N(2n = x^2 + 5y^2 + 10z^2) = 2[3G(n) - 2F(n)],$$

or, which is the same,

$$N(n = x^2 + 5y^2 + 10z^2) = 2[3G(2n) - 4F(2n)]$$

for every *even* $n \equiv \pm 1 \pmod 5$. For an *odd* n we evidently have

$$N(n = x^2 + 5y^2 + 10z^2) = N(4n = x^2 + 5y^2 + 10z^2),$$

whence and from the preceding we derive the following general expression

$$N(n = x^2 + 5y^2 + 10z^2) = (-1)^n\,2[3G(2n) - 4F({}^2n)]$$

provided $n \equiv \pm 1 \pmod 5$.

In the same paper already quoted we established the following equations holding true for $n \equiv \pm 2 \pmod 5$:

$$N(2n = x^2 + 5y^2 + 10z^2) + 2(-1)^{(n-1)/2}\,N(n = 2x^2 + 5y^2 + 10z^2) = (1/2)N(n = x^2 + y^2 + z^2)$$

$$N(2n = x^2 + 5y^2 + 10z^2) - 2(-1)^{n/2}\,N(n = 2x^2 + 5y^2 + 10z^2) = (1/2)N(n = x^2 + y^2 + z^2)$$

for an *odd* and *even* n respectively. These equations may serve to obtain the number of representations by the form $2x^2 + 5y^2 + 10z^2$. We find

$$(24) \qquad N(n = 2x^2 + 5y^2 + 10z^2) = (-1)^{n/2}\,2[3G(n) - 4F(n)]$$

$$(25) \qquad N(n = 2x^2 + 5y^2 + 10z^2) = (-1)^{(n-1)/2}\,2[3G(n) - 4F(n)]$$

for an even and odd $n \equiv \pm 2 \pmod 5$ respectively.

It remains to consider the form $x^2 + 2y^2 + 10z^2$. Supposing n *even*, we have the obvious relation

$$N(n = x^2 + 2y^2 + 10z^2) = N(n/2 = x^2 + 2y^2 + 5z^2),$$

whence and from (23) it follows,

$$N(n = x^2 + 2y^2 + 10z^2) = (-1)^{n/2}\,[3G(5n) - 4F(5n)], \tag{27}$$

provided n is not divisible by 5.

To find the number of representations by the same form in case of an *odd* n we make use of the relation

$$\Sigma\,(-1)^z = -\,(1/4)\,\Sigma\,(-1)^{i+j},$$

where sums are extended over solutions of the equations

$$\begin{aligned} 2n &= x^2 + 2y^2 + 5z^2 \\ 5n &= i^2 + j^2 + k^2 \end{aligned}$$

respectively and n denotes any number non-divisible by 5. This relation follows immediately from one established in § 11 of our paper quoted above. Let us denote by P and Q respectively the number of solutions of the equation

$$2n = x^2 + 2y^2 + 5z^2$$

with *even* and *odd* z. The preceding relation gives for $n \equiv 1 \pmod 4$

$$P - Q = 2F(5n) - G(5n),$$

while

$$P + Q = 3G(5n) - 2F(5n),$$

whence

$$P = N(n = x^2 + 2y^2 + 10z^2) = G(5n).$$

For $n \equiv 3 \pmod 4$ it follows from the same equation,

$$P - Q = 3G(5n) - 6F(5n)$$

and

$$P + Q = 3G(5n) - 2F(5n),$$

whence

$$P = N(n = x^2 + 2y^2 + 10z^2) = 3G(5n) - 4F(5n).$$

Both expressions thus obtained may be summarized in a single equation:

$$N(n = x^2 + 2y^2 + 10z^2) = (-1)^{(n+1)/2}\,[3G(5n) - 4F(5n)], \tag{28}$$

provided n is an *odd* number non-divisible by 5.

Representation of Integers in the Form. $x^2 + 2y^2 + 3z^2 + 6w^2$.

By L. W. Griffiths.

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1. *Introduction.* Jacobi [*] first proved that every positive integer N is represented in the form

$$(1) \qquad x^2 + 2y^2 + 3z^2 + 6w^2.$$

Liouville [†] gave the number of representations when N is even, a power of 3, or a multiple of 7, as a linear function of the sum of the divisors prime to 6 of N; he gave upper and lower limits only for other odd N. The problem for N odd is recognized now as an extremely difficult and important one. In this paper the results of Liouville for N odd are proved, and better limits than his are found. Secondly, although the methods of this paper seem to be ineffective in determining the number of representations when N is an odd prime different from 3, the problem for N odd and composite is reduced to that for N odd and prime.

In particular, the essential notations and results of this paper are as follows. Write $N = 3^b M$ where $b \geqq 0$ and M is prime to 6; define $f(M)$ as the sum of the distinct positive divisors of M, and $g(N)$ and $\beta(N)$ by $g(N) = \beta(N) f(M) = (3^{b+1} - 2) f(M)$. It is easily shown that the number of representations of N by (1) is even, in notation $2T(N)$. Then $T(N) \equiv g(N) \pmod 2$, and $T(N)$ satisfies Liouville's inequality $g(N)/3 \leqq T(N) \leqq g(N)$. Also $T(N) \leqq g(N) - 2t(N)$, where $t(N) = E(N) + k(N)$, $E(N)$ is the greatest integer in $(2N/3)^{1/2}$, and $k(N)$ is 0 or 1 according as $E(N)$ is even or odd. Hence a better inequality than Liouville's is $g(N)/3 \leqq T(N) \leqq g(N) - 2t(N)$. The problem is reduced to that for P^n, where P is an odd prime and n is a positive integer, since $D(N)$ defined to be $2T(N) - g(N)$ is a factorable function, that is $D(PQ) = D(P)D(Q)$ when P and Q are relatively prime. The problem is reduced to that of odd primes by $T(P^n) = T(P^{n-2}) + K(P)$, where $K(P)$ is a polynomial in $f(P)$ and $T(P)$. By this reduction of the problem to that for P^n, a third inequality, better than the second, is obtained for N composite but not a power of a prime. By the same reduction, $2T(N) = g(N)$ if N is divisible by 7, while for every $A > 0$ there are infinitely many integers N such that $|2T(N) - g(N)| > A$.

[illegible]

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[*] [illegible], *History of the Theory of Numbers*, Vol. 3, 229.

[†] [illegible] de *Mathématiques*, Vol. 9, 2 série (1864).

[‡] [illegible] Vol. 50 (1928), 303-314.

obtained the preceding results, of Jacobi and of Liouville for N even, in a discussion of generalized quaternion algebras. There the following notations and relations were established. The integral quantities of the algebra with parameters $(-2, -3)$ are the quantities $q = \kappa + \lambda i + \mu H + \nu L$ in which $\kappa, \lambda, \mu, \nu$ are integers; the conjugate of q is $\bar{q} = \kappa + \mu - \lambda i - \mu H - \nu L$ and the norm $q\bar{q}$ is the integer

$$(2) \qquad \kappa^2 + \kappa\mu + \mu^2 + 2\lambda^2 + 2\lambda\nu + 2\nu^2,$$

that is the value of (1) with

$$(3) \qquad x = \kappa + \mu/2, \qquad y = \lambda + \nu/2, \qquad z = \mu/2, \qquad w = \nu/2.$$

Every integer $N \geqq 0$ is represented by (1) since there is in the algebra a quantity q having μ and ν even and norm N. From $1 = x^2 + 2y^2 + 3z^2 + 6w^2$ and (3), the units are ± 1, $\pm H$, $\pm (1 - H)$. Since the norm of a product is the product of the norms, all the quantities of norm N are associated in sets of six each, the quantities in any one set being the products qu where q is any specific one in the set and u is in turn each of the six units. q is proper (definition) if the greatest common divisor of $\kappa, \lambda, \mu, \nu$ is 1. Hence if q is a proper quantity having norm $N = P_1 P_2 \ . \ . \ . \ P_h$ in which the P's are primes arbitrarily arranged, then there are quantities $p_1, p_2, \cdots, p_h$ of norms $P_1, P_2, \cdots, P_h$ respectively such that $q = p_1 p_2 \cdots p_h$; *and* each of the p's is successively unique up to a unit right factor if not more than one of the P's is 3. If the norm of q is not divisible by 3 then q is proper if and only if no factor p in the product is a right associate of the conjugate of its predecessor. Write $N = 2^a 3^b M$ ($a \geqq 0$, $b \geqq 0$, M prime to 6). Then the number of distinct quantities of norm N is $6g(N)$, that is the number of representations of N in (2) is $6g(N)$ and the number of sets of right associated quantities of norm N is $g(N)$. According as $a > 1$ or $a = 1$ six or two of the six right associates in each set have μ and ν even. Hence, as stated by Liouville, *the number of representations of N in* (1) *is* $6g(N)$ *if* $a > 1, 2g(N)$ *if* $a = 1$.

3. *Number of Representations for N odd reduced to the case of N odd and prime.* Since $x^2 = (-x)^2$ this number of representations is even, in notation $2T(N)$. It is necessary to enumerate, for $N = 2^a 3^b M$ as above but with $a = 0$, those of the $6g(N)$ representations which have μ and ν even. Now $N \equiv 1 \pmod 2$ implies that κ and μ are not both even. Hence $\kappa, \lambda, \mu, \nu$ reduced modulo 2 determine one of the following quadruples

$$(4)\qquad \begin{array}{cccc} 1\,0\,0\,0 & & 0\,0\,1\,0 & & 1\,0\,1\,0 \\ 1\,1\,0\,0 & & 0\,0\,1\,1 & & 1\,1\,1\,1 \\ 1\,1\,0\,1 & & 0\,1\,1\,0 & & 1\,0\,1\,1 \\ 1\,0\,0\,1 & & 0\,1\,1\,1 & & 1\,1\,1\,0. \end{array}$$

Table (4) is basic in the sequel. It is easily verified that if q is in the first column then qH and $q(1-H)$ are in the second and third respectively of the same row; hence each of the $g(N)$ sets in § 2 corresponds to a row of the table. If q is in the first column Hq is in the second column of the row determined by the permutation (1)(324), and $(1-H)q$ in the third column of the row determined by (1)(342). Define $2R(N)$ and $2S(N)$ as the numbers of representations of the first column, first and second rows respectively. Then

$$(5)\qquad T=R+S, \qquad g=R+3S, \qquad D=R-S.$$

The last equation defines $D(N)$; the first holds since only the first and second rows of the table correspond to μ and ν both even. The second holds since (2) is symmetrical in κ and μ, for then a representation in the second row determines one in the third and this in turn one in the fourth.

THEOREM 1. *If* $N=\prod_1^n M_h$, *where n is a positive integer and every two of the M_h are relatively prime, then*

$$2T(N)=\prod_1^n g(M_h)+\prod_1^n [2T(M_h)-g(M_h)].$$

Case $n=2$. The products pq are the distinct quantities of norm M_1M_2 if p runs through the distinct (proper and not proper) quantities of norm M_1 and q similarly for M_2. To determine how many of these products pq have μ and ν even, that is are in the first column, first or second row, of (4), it is sufficient to take p and q each in the first column of (4). Then the following table gives the row of pq in (4), the row of p in (4) being read off down the left and that of q across the top:

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

[illegible] product pq, p may be chosen in the first column of (4) in $R(M_1)$ ways from the first row (written R_1) and in S_1 ways from each of the other three rows: and similarly for q. Hence by the diagonal of the table

$$(6) \qquad R_{12} = R(M_1M_2) = R_1R_2 + S_1S_2 + S_1S_2 + S_1S_2 = R_1R_2 + 3S_1S_2,$$

and similarly

$$(7) \qquad S_{12} = S(M_1M_2) = R_1S_2 + S_1R_2 + S_1S_2 + S_1S_2 = R_1S_2 + S_1R_2 + 2S_1S_2.$$

But by (5), (6) and (7)

$$D_{12} = R_{12} - S_{12} = D_1D_2, \quad 2T_{12} = g_1g_2 + D_1D_2.$$

Case $n > 2$. By the familiar properties of $f(N)$, since the M's are relatively prime, $g(M_2 \cdots M_n) = g_2 \cdots g_n$. Furthermore, as for D_{12} above, $D(M_2 \cdots M_n) = D_2 \cdots D_n$.

Hence $2T(M_1 \cdots M_n) = g_1 \cdots g_n + D_1 \cdots D_n$.

THEOREM 2. $2T(3^b) = \beta(3^b) + (-1)^b$,

$R(3^b) = [\beta(3^b) + 3(-1)^b]/4, \quad S(3^b) = [\beta(3^b) - (-1)^b]/4.$

By the reference in § 2, the distinct quantities of norm 3^b are known, and the non-proper among them are segregated and counted. Then (6) and (7) give $R(3^b) = R(3^{b-2}) + (3^b + 3^{b-1})/2, \qquad S(3^b) = S(3^{b-2}) + 2 \cdot 3^{b-1}$, from which the theorem follows.

Theorem 2 and theorem 1 with $M_1 = 3^b$ give Liouville's result,

$$2T(N) = g(N) + (-1)^b[2T(M) - f(M)], \quad a = 0, \quad b > 0.$$

THEOREM 3. *If* $n \geqq 2$ *and* P *is a prime* $\neq 2, 3$ *then:* $T(P^n) = T(P^{n-2}) + K(P)$, *where* $K(P)$ *is a polynomial in* $f(P)$ *and* $T(P)$*;* $T(P^n)$ *is a polynomial in* $f(P)$ *and* $T(P)$.

$2T(P^n)$ is less than the value from theorem 1 of $2T(M_1 \cdots M_n)$ with $M_1 = \cdots = M_n = P$. Define $2T_n'$ as the latter; and $2c_n$, $2d_n$, $2e_n$ as the numbers respectively of non-proper, proper, and distinct non-proper quantities enumerated by $2T_n'$. Then

$$(8) \qquad \begin{aligned} &T'_n = c_n + d_n, \\ &T(P^n) = d_n + e_n = T_n' - c_n + e_n, \\ &d_0 = 1, \; d_1 = T(P), \quad c_0 = c_1 = e_0 = e_1 = 0, \\ &\qquad\qquad\qquad\qquad c_2 = f, \; e_2 = 1, \quad d_2 = T(P^2) - 1. \end{aligned}$$

A quantity which is not proper is of the form $p\bar{p}q$, where the norm of q is P^{n-2} (this includes quantities factored, when enumerated in T_n', as $qp\bar{p}r$ having the norm of qr equal to P^{n-2}). In c_n there are fT'_{n-2} products $p\bar{p}q$ of norm P^n in which the first and second factors certainly make the product non-proper; for $h = 1, 2, \cdots n-2$ there are $d_h(f-1)T'_{n-2-h}$ non-proper products of norm P^n in which $p_1 \cdots p_h$ is proper. Hence

$$c_n = fT'_{n-2} + (f-1)\sum_h d_h T'_{n-2-h}, \quad h = 1, 2, \cdots, n-2, \qquad n \geqq 2.$$

Similarly

$$e_n = d_{n-2} + d_{n-4} + \cdots + d_k = T(P^{n-2}), \quad k = [1-(-1)^n]/2, \quad n \geqq 2.$$

Hence by (8)

$$T(P^n) - T(P^{n-2}) = T'_n - fT'_{n-2} + (1-f)\,TT'_{n-3}$$
$$+ (1-f)\sum_h [T(P^h) - T(P^{h-2})]\,T'_{n-2-h},$$
$$(h = 2, 3, \cdots, n-2) \qquad (n \geqq 2).$$

Since $T(P^2) = T'_2 - f + 1$, $T(P^3) = T'_3 + 2(1-f)T$, we have $T(P^n) = T(P^{n-2}) + K(P)$, where $K(P)$ is a polynomial in f and T. Hence $T(P^n)$ is a polynomial in f and T for every positive integer n.

4. *Conditions on* $T(N)$ *for* N *odd.* *By* (5), $g - T = 2S$ and $3T - g = 2R$. Hence

THEOREM 4. *If* N *is odd,* $T(N) \equiv g(N)$ (*mod* 2) *and* $g(N)/3 \leqq T(N) \leqq g(N)$.

Liouville stated this inequality for N odd and prime to 3. He instanced $N = 1$ as attaining the upper limit and $N = 5$ the lower. We prove next that if $N > 1$ the upper limit is lowered and that if N is not certain powers of certain primes the lower limit is raised.

THEOREM 5. *If* N *is odd* $T(N) \leqq g(N) - 2[E(N) + k(N)]$, *where* $E(N)$ *is the greatest integer in* $(2N/3)^{1/2}$ *and* $k(N) = 0$ *or* 1 *according as* $E(N)$ *is even or odd.*

This follows from $T = g - 2S$ and Lemmas 1 and 2.

LEMMA 1. *If* N *is odd* $S(N) \geqq 2h$, *where* h *is the number of odd solutions of* $W^2 < 2N/3$.

If W and N are odd and $W^2 < 2N/3$, then $4N - 6W^2 > 0$, $4N - 6W^2 \not\equiv 0 \pmod 4$, $4N - 6W^2 \not\equiv 10 \pmod{16}$. Hence * there are integers X, Y, Z (at least two of which are different from 0) such that $4N - 6W^2 = X^2 + 2Y^2 + 3Z^2$ and $4N = X^2 + 2Y^2 + 3Z^2 + 6W^2$. Hence X, Y, Z, W and $-X$, Y, $-Z$, W are different. Now X, Y, Z, W determines $q = \sigma + \xi i + \eta H + \zeta L$ having norm N by (3), thus $\sigma = (X - Z)/2$, $\xi = (Y - W)/2$, $\eta = Z$, $\zeta = W$; and $-X$, Y, $-Z$, W determines $q_1 = -\sigma + \xi i - \eta H + \zeta L$. Hence $q_1 \not\equiv q \pmod 2$, q and q_1 each contribute 1 to S. Finally $S \geqq 2h$.

LEMMA 2. *If* N *is odd and* h *is the number of odd solutions of*

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* Dickson, *Bulletin of the American Mathematical Society*, Jan., 1927.

$W^2 < 2N/3$ *then* $2h = E(N) + k$, *where* $k = 0$ *or* 1 *according as* $E(N)$ *is even or odd.*

Let h' be the number of all solutions different from 0 of $W^2 < 2N/3$. Then $2h = h' + k$, where k is 0 or 1 according as h' is even or odd. Also $h'^2 < 2N/3 < (h' + 1)^2$. Hence $h' = E(N)$.

THEOREM 6. *If* $N = \prod_1^n M_h$, *where* n *is a positive integer and every two of the* M_h *are relatively prime, then* $g(N)/3 \leqq T(N) \leqq g(N)/2 + \frac{1}{2} \prod_1^n [g(M_h) - 2t(M_h)] < g(N) - 2t(N)$, *where* $t(N) = E(N) + k(N)$.

The first two inequalities follow from theorems 1, 4 and 5. The last inequality is proved by induction.

THEOREM 7. $T(N) > g(N)/3$ *if* N *is odd and different from* P^n, *where* n *is odd and* P *is a prime of the form* $6m + 5$ *or* $n \equiv 2 \pmod 3$ *and* P *is a prime of the form* $6m + 1$; *but not conversely.*

If N is a power of 3, theorem 2 gives the result. If N is a power of a prime not excluded in the theorem, $g(N) \not\equiv 0 \pmod 3$. Then $T > g/3$ by theorem 4 since T is an integer. If N is not a power of a prime, write $N = N_1 N_2$ where $N_1 > 1$, $N_2 > 1$, and N_1 is relatively prime to N_2. The result follows by theorem 1 and the inequality $-g_h/3 \leqq 2T_h - g_h \leqq g_h$ for $h = 1, 2$ which is equivalent to that in theorem 4. The converse is not true since $T(11) = 8$, $g(11) = 12$.

Theorems 6 and 7 suggest that asymptotically there may be some simple function of $g(N)$ which gives the value of $T(N)$. Since $2T(7) = 8 = g(7)$, $2T(N) = g(N)$ if N is divisible by 7. On the other hand, for every $A \geqq 0$ there are infinitely many integers N such that $| 2T(N) - g(N) | > A$. For if P is a prime of the form $4m + 1$, $g(P) = 4m + 2$ and $2T(P) \neq g(P)$ since $T \equiv g \pmod 2$. Let n be any positive integer such that $2^n > A$, and $P_1, \cdots, P_n$ any n distinct primes of the form $4m + 1$. Then $| 2T(P_h) - g(P_h) | > 1$ for $h = 1, \cdots, n$, and by theorem 1

$$| 2T(N) - g(N) | \geqq 2^n > A \text{ if } N = \prod_1^n P_h.$$

For positive odd integers $N \leqq 75$, for $N = 3^4$, and for $N = 5^3$, the solutions of $N = x^2 + 2y^2 + 3z^2 + 6w^2$ have been found. The values of $T(N)$ thus obtained check the preceding theorems. The upper and lower limits of theorems 4, 5 and 6 coincide if $N = 5$; the upper limit is attained for $N = 1, 3, 7, 11, 17$, the lower for $N = 23$.

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Normal Ternary Continued Fraction Expansions for Cubic Irrationalities.

By P. H. Daus.

I. Introduction.

1. *Ternary Continued Fraction.* In a previous article* the expression *Normal Ternary Continued Fraction* was defined and expansions for the cube roots of integers discussed. It is the purpose of this article to discuss expansions for cubic irrationalities defined by the equation $x^3+px^2+qx+r=0$.

It will be convenient to collect here some of the definitions and results of the paper referred to above that will be needed later.

Definition. *If u_1, v_1, w_1 be any three numbers, we define a ternary continued fraction expansion for them by the equations*

$$(1) \qquad u_{n+1}=v_n-p_nu_n\,;\quad v_{n+1}=w_n-q_nu_n\,;\quad w_{n+1}=u_n.$$

These equations may be written by combining them,

$$(2) \qquad \begin{aligned} u_n &= && w_{n+1}, \\ v_n &= u_{n+1} && +\,p_n\,w_{n+1}, \\ w_n &= \quad v_{n+1} && +\,q_n\,w_{n+1}. \end{aligned}$$

The set of numbers (u_n, v_n, w_n) is called the nth complete quotient set, and (p_n, q_n) is called the nth partial quotient set. The numbers (A_n, B_n, C_n), defined by the recursion formulas (3) below form the nth convergent set to the ternary continued fraction

$$(v_1/u_1,\, w_1/u_1)=(p_1,\, q_1;\; p_2,\, q_2;\; \cdots;\; p_n,\, q_n;\; \cdots).$$

$$(3) \qquad \begin{aligned} A_n &= q_nA_{n-1}+p_nA_{n-2}+A_{n-3}, \\ B_n &= q_nB_{n-1}+p_nB_{n-2}+B_{n-3}, \\ C_n &= q_nC_{n-1}+p_nC_{n-2}+C_{n-3}, \end{aligned}$$

with the initial conditions

$$\begin{vmatrix} A_{-2} & A_{-1} & A_0 \\ B_{-2} & B_{-1} & B_0 \\ C_{-2} & C_{-1} & C_0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

2. *Relations between Complete Quotient Sets.*

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* *American Journal of Mathematics*, Vol. 44 (1922), pp. 279-296. In equations (A) and (3) of this paper, p and q should be interchanged.

THEOREM I. *If u_1, v_1, w_1, any three numbers, be expanded into a ternary continued fraction, then*

$$(4)\qquad \begin{aligned} u_1 &= A_{n-2}u_{n+1} + A_{n-1}v_{n+1} + A_n w_{n+1},\\ v_1 &= B_{n-2}u_{n+1} + B_{n-1}v_{n+1} + B_n w_{n+1},\\ w_1 &= C_{n-2}u_{n+1} + C_{n-1}v_{n+1} + C_n w_{n+1},\end{aligned}$$

and

$$(5)\qquad \begin{aligned} u_{n+1} &= c_{n-2}w_1 + b_{n-2}v_1 + a_{n-2}u_1,\\ v_{n+1} &= c_{n-1}w_1 + b_{n-1}v_1 + a_{n-1}u_1,\\ w_{n+1} &= c_n\ \ w_1 + b_n\ \ v_1 + a_n\ \ u_1,\end{aligned}$$

where a_n, b_{n-1}, $\cdots$ are the cofactors of the corresponding elements of the determinant of (4).

3. *Regular Expansions.* The defining equations (1) are independent of the choice of p_n, q_n. In order to extend the continued fraction algorithm it is desirable to have the absolute values of the A's, B's and C's each form an increasing sequence of integers. Such an expansion will be called *regular,* and we will confine our attention to such expansions.

If we call $v_n/u_n = \sigma_{1,n}$ and $w_n/u_n = \sigma_{2,n}$, then the defining equations (1) may be written

$$(6)\qquad \sigma_{1,n} = p_n + 1/\sigma_{2,n+1}; \quad \sigma_{2,n} = q_n + \sigma_{1,n+1}/\sigma_{2,n+1}.$$

Jacobi[*] chose p_n and q_n as the greatest integers in $\sigma_{1,n}$ and $\sigma_{2,n}$, which led him into difficulties. We notice that the equation defining $\sigma_{1,n}$ is the defining equation for an ordinary continued fraction, but that for $\sigma_{2,n}$ is not. Because of this, it was found desirable to remove this limitation on q_n, and select q_n as indicated below, the purpose in view being to make the expansion regular. If $\sigma_{1,n}$ is positive, we select p_n equal to the greatest integer in $\sigma_{1,n}$. (If $\sigma_{1,n}$ is an integer p_n may be equal to or one less than $\sigma_{1,n}$.) If $\sigma_{1,n}$ is negative, then $| p_n |$ is selected equal to or one more than the greatest integer in $| \sigma_{1,n} |$. The essential point of this selection is to make $| u_{n+1} | < | u_n |$. In either case q_n is selected arbitrarily, usually approximately equal to $\sigma_{2,n}$, subject to the limitation that the expansion be regular. We give as an illustration two examples, showing the reduction and rebuilding of (A_n, B_n, C_n).

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[*] Jacobi, C. G. J. *Gesammalte Werke,* Vol. VI, pp. 385-426.

Example 1.

u	v	w	p, q;	A	B	C
				1	0	0
				0	1	0
				0	0	1
17	44	13	2, 0;	1	2	0
10	13	17	1, 1;	1	3	1
3	7	10	2, 1;	3	7	2
1	7	3	6, 1;	10	27	8
1	2	1	2, 1;	17	44	13
0	0	1				

It will be noted that q_3 and q_4 are less than the greatest integer in the corresponding σ's. The selection in this particular example was guided by the next theorem.

Example 2.

u	v	w	p, q;	A	B	C
				1	0	0
				0	1	0
				0	0	1
9	-3	-53	$0, -6$;	1	0	-6
-3	1	9	$0, -3$;	-3	1	-18
1	0	-3	$0, -3$;	9	-3	-53
0	0	1				

This example illustrates a case when the σ's are negative.

4. *Skew-palindromic Expansions.* An important case of a regular expansion (illustrated by the examples above) is one which may be considered as an extension of an ordinary continued fraction in which the partial quotients are palindromic, that is, read the same backward as forward. In our case, however, we define a *skew-palindromic* expansion as one in which the q's are palindromic and the p's omitting the first, are also palindromic.

THEOREM II. *If u_1, v_1, w_1, three integral rational numbers, relatively prime, be expanded into a ternary continued fraction, then the necessary and sufficient condition that the sequence $u_1, u_2, \cdots, u_n$, be the sequence A_ν,* [illegible] *A_1, ($A_\nu = u_1$), where (A_ν, B_ν, C_ν) is the last convergent set, is that if we omit the first partial quotient set, the balance of the expansion be skew-palindromic.*

For such an expansion, it is evident from the manner in which the p's are selected that $|\, w\text{'s}\, |$ form a decreasing sequence and consequently the $|\, A\text{'s}\, |$ an increasing sequence of integers. From the properties of convergents it will later be shown that for the type of expansion we are to discuss, that $|\, B\text{'s}\, |$ and $|\, C\text{'s}\, |$ also form an increasing sequence of integers. In the case the p's and q's are always positive, this is evident from the form of the recursion formulas (3).

II. NORMAL TERNARY CUBIC FORMS.

5. *Normal Cubic.* The discussion of ternary continued fractions is closely allied to the problem of determining the units in a given domain, defined by a root θ, of the irreducible equation

$$(7) \qquad x^3 + px^2 + qx + r = 0.$$

That is, we seek integers x, y, z such that the norm of $(x + \theta y + \theta^2 z)$ will be unity. If we indicate the three roots of (7) by $\theta, \theta', \theta''$, then

$$(8) \qquad N(x + \theta y + \theta^2 z) = (x + \theta y + \theta^2 z)(x + \theta' y + \theta'^2 z)(x + \theta'' y + \theta''^2 z).$$

This expression, when multiplied out and simplified by means of the well-known relations between the roots, will be called the *normal ternary cubic* and denoted by $N(x, y, z)$. It becomes

$$(9) \qquad N(x, y, z) = x^3 - px^2y + qxy^2 - ry^3 + (p^2 - 2q)x^2z + (q^2 - 2pr)xz^2 + pry^2z - qryz^2 + r^2z^3 + (3r - pq)xyz.$$

In the special case of the reduced cubic $p = 0$ and

$$(9') \qquad N(x, y, z) = x^3 + qxy^2 - ry^3 - 2qx^2z + q^2xz^2 - qryz^2 + r^2z^3 + 3rxyz.$$

In the further special case $p = 0$, $r = -D$, so that $x^3 = D$, we have the form of the so-called, misnamed, Pellian cubic,

$$(9'') \qquad N(x, y, z) = x^3 + Dy^3 + D^2z^3 - 3Dxyz.$$

For convenience in computation, it is highly desirable to have equation (9) in determinant form. To that end, we consider the automorphs of $N(x, y, z)$.*

Let x, y, z be a solution of $N(x, y, z) = 1$, and let $(x_1 + \theta y_1 + \theta^2 z_1)$ and $(x_2 + \theta y_2 + \theta^2 z_2)$ be two numbers, such that

$$(10) \qquad x_2 + \theta y_2 + \theta^2 z_2 = (x + \theta y + \theta^2 z)(x_1 + \theta y_1 + \theta^2 z_1).$$

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* See H. W. Tanner, "Notes on a Ternary Cubic," *Proceedings of the London Mathematical Society,* Vol. 10 (1895), p. 187.

Multiplying out and equating coefficients of like irrationalities and reducing by means of (7), we have

$$\text{(11)}\qquad \begin{aligned} x_2 &= xx_1 && - rzy_1 && + (-ry + rpz)z_1,\\ y_2 &= yx_1 && + (x - qz)y_1 && + (-rz - qy + pqz)z_1,\\ z_2 &= zx_1 && + (y - pz)y_1 && + (x - qz - py + p^2z)z_1. \end{aligned}$$

Now equation (10) is equivalent to

$$\text{(10')}\qquad N(x_2, y_2, z_2) = N(x, y, z)N(x_1, y_1, z_1).$$

Interpreting (11) as a transformation on the form $N(x, y, z)$, we have

$$N(x, y, z)\begin{bmatrix} x & -rz & -r(y-pz)\\ y & x-qz & -rz-q(y-pz)\\ z & y-pz & x-qz-p(y-pz) \end{bmatrix} N(x, y, z),$$

where x, y, z is a solution of $N(x, y, z) = 1$. Applying this transformation to the right hand member of $N(x, y, z) = 1$ itself, we see

$$\text{(12)}\qquad N(x, y, z) = \begin{vmatrix} x & -rz & -r(y-pz)\\ y & x-qz & -rz-q(y-pz)\\ z & y-pz & x-qz-p(y-pz) \end{vmatrix}.$$

The corresponding special cases become

$$\text{(12')}\qquad N(x, y, z) = \begin{vmatrix} x & -rz & -ry\\ y & x-qz & -rz-qy\\ z & y & x-qz \end{vmatrix},$$

and

$$\text{(12'')}\qquad N(x, y, z) = \begin{vmatrix} x & Dz & Dy\\ y & x & Dz\\ z & y & x \end{vmatrix}.$$

6. *Normal Cubic Surface.* Let us now consider some elementary geometrical properties of the surface $N(x, y, z) = m$.[*] Let us denote the expression $x + \theta y + \theta^2 z$ by U and the corresponding expressions in θ' and θ'' by U' and U''. Our surface is then

$$\text{(13)}\qquad U\,U'\,U'' = m.$$

It is apparent that the surface has three binodes in the plane at infinity in the direction given by the intersection of the asymptotic planes $U = 0$, $U' = 0$, $U'' = 0$, taken in pairs. There are two cases to consider. First, when equation [illegible] has but one real root and secondly when it has three real roots.

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[*] See H. W. Tanner, *loc. cit.* For further details see W. E. H. Berwick, *Proceedings of the London Mathematical Society*, (2) Vol. 12 (1913), p. 393-429.

In the first case, only one of the asymptotic planes is real, say $U = 0$. The real line of intersection of the imaginary planes becomes an asymptotic line. If we cut the surface, $N(x, y, z) = m$, by the plane $U = k$, we get a section given by $U' U'' = m/k$. We readily recognize that the section is an ellipse which decreases continuously in size as the cutting plane moves further away from the origin, gradually closing in on the line $U' = 0$, $U'' = 0$. The surface has a spur in the direction of this line, which grows larger as we progress from infinity towards the origin, and finally spreads out over the plane $U = 0$, the entire surface lying on one side of this plane. The surface may be visualized by a deformation of the canonical form $(x^2 + y^2)z = m$, where $z = 0$ plays the rôle of $U = 0$, and the z-axis that of the asymptotic line.

In the second case, when all the roots are real, the three asymptotic planes are real and divide space into eight regions. The surface consists of four sheets, no two of which lie in adjacent regions. Every section by a plane through an intersection corresponds to the canonical form $xy^2 = c$, while the section parallel to one of the planes is a hyperbola. This surface can be visualized by a deformation of the canonical form $xyz = m$.

7. *Direction Cosines of Asymptotic Line.* The direction cosines of the line of intersection of the planes $U' = 0$, $U'' = 0$ are proportional to the determinants formed from the matrix

$$\left\| \begin{matrix} 1 & \theta' & \theta'^2 \\ 1 & \theta'' & \theta''^2 \end{matrix} \right\| .$$

This gives, where λ, μ, ν are the direction cosines,

$$(14) \qquad \begin{aligned} \lambda : \mu : \nu &= \theta' \theta''^2 - \theta'' \theta'^2 : -(\theta''^2 - \theta'^2) : \theta'' - \theta', \\ \lambda : \mu : \nu &= N(\theta)/\theta : p + \theta : 1 = q + p\theta + \theta^2 : p + \theta : 1. \end{aligned}$$

If we call the intersection of the planes an asymptotic line, we see that its direction cosines are given by (14) and the direction cosines $\lambda' : \mu' : \nu'$ and $\lambda'' : \mu'' : \nu''$ of the other two asymptotic lines are given by replacing θ in (14) by θ' and θ'', respectively.

It is apparent that in the first case we should expect only one expansion and in the second case three expansions. The results of this last paragraph suggest that we select as u_1, v_1, w_1, the numbers 1, $p + \theta$, $q + p\theta + \theta^2$. It is readily shown that under the conditions we have set for the selection of p_n and q_n, the partial quotients after p_1, q_1, p_2 are exactly the same as if we started with $1, \theta, \theta^2$. This can be seen by comparing the following expansions. If the expansion for $1, \theta, \theta^2$ is

$$(15) \qquad (p_1, q_1;\ p_2, q_2;\ p_3, q_3;\ \cdots),$$

then the expansion for 1, $p + \theta$, $q + p\theta + \theta^2$ is

$$(15') \qquad (p_1 + p, q + pp_1 + q_1;\ p + p_2, q_2;\ p_3, q_3;\ \cdots),$$

as may be readily verified by writing out the complete quotient sets. In fact, we might have started with 1, $l + \theta$, $m + n\theta + \theta^2$, (l, m, n, integers) and obtained the same expansion except for p_1, q_1 and p_2 The selection of 1, $p + \theta$, $q + p\theta + \theta^2$ has the advantage that the convergents (A_n, B_n, C_n) provide us directly with the solution of $N(x, y, z) = 1$.

III. PERIODIC EXPANSIONS.

8. *Periodic Expansions.* After these preliminary considerations, we come to the following theorem, which is fundamental in the consideration of periodic expansions.

THEOREM III. *If in a ternary continued fraction, the expansion becomes periodic after a finite number of terms, then* $\sigma_{1,1}$ *and* $\sigma_{2,1}$ *are roots of cubic equations with rational coefficients.**

As explained in the last section, we will confine our attention to the case where $\sigma_{1,1} = p + \theta$; $\sigma_{2,1} = q + p\theta + \theta^2$, θ being a real root of the irreducible cubic

$$(7) \qquad x^3 + px^2 + qx + r = 0.$$

It will be convenient to use as well as the complete quotient set (u_n, v_n, w_n), composed of linear functions of θ and θ^2, the rationalized complete quotient set $(\bar{u}_n, \bar{v}_n, \bar{w}_n)$, defined as follows.

$\bar{u}_n = \alpha_n =$ Norm of $u_n = u_n u_n' u_n''$, u_n' and u_n'' being the conjugates of u_n.

$$(16). \qquad \begin{aligned} \bar{v}_n &= \alpha_n' + \beta_n'\theta + \gamma_n'\theta^2 = v_n u_n' u_n''. \\ \bar{w}_n &= \alpha_n'' + \beta_n''\theta + \gamma_n''\theta^2 = w_n u_n' u_n''. \end{aligned}$$

$\bar{u}$, $\bar{v}$, $\bar{w}$ may have a common factor, which is not to be removed as in Jacobi's work. In this connection compare the illustrative example below.

We compute the rationalized quotient sets in order as follows. Suppose $\bar{u}_n, \bar{v}_n, \bar{w}_n$ equal to $u_n u_n' u_n''$, $v_n u_n' u_n''$, $w_n u_n' u_n''$ respectively have been found. Then before rationalization

$$\begin{aligned} u_{n+1} : v_{n+1} : w_{n+1} &= (v_n - p_n u_n) u_n' u_n'' : (w_n - q_n u_n) u_n' u_n'' : u_n u_n' u_n'' \\ &= u_{n+1} u_n' u_n'' : v_{n+1} u_n' u_n'' : w_{n+1} u_n' u_n''. \end{aligned}$$

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* For proof, see D. N. Lehmer, *Proceedings of the National Academy of Sciences*, Vol. 4 (1918), pp. 360-364, or my previous article.

Now to rationalize $u_{n+1}u_n'u_n''$, we multiply by the conjugates of each factor, namely the product $u'_{n+1}u''_{n+1}u_nu_n''u_nu_n'$. This gives, since $\bar{u}_n = \alpha_n$,

$$u_{n+1} : v_{n+1} : w_{n+1} = u_{n+1}u'_{n+1}u''_{n+1}\alpha^2 : v_{n+1}u'_{n+1}u''_{n+1}\alpha^2 : w_{n+1}u'_{n+1}u''_{n+1}\alpha^2.$$

We notice that we obtain $\bar{u}_{n+1}$, $\bar{v}_{n+1}$, $\bar{w}_{n+1}$, by dividing out α^2. A numerical example will perhaps make this clear.

9. *Numerical Example.* In the expansion of the real root of $x^3 + x - 5 = 0$, we find that $\bar{u}_2 = 3$, $\bar{v}_2 = 1 + 2\theta - \theta^2$, $\bar{w}_2 = 2 + \theta + \theta^2$, $p_2 = 0$ and select $q_2 = 1$. Consequently $u_3 : v_3 : w_3 = 1 + 2\theta - \theta^2 : -1 + \theta + \theta^2 : 3$. The product of the conjugates of $1 + 2\theta - \theta^2$, viz. $(1 + 2\theta' - \theta'^2)(1 + 2\theta'' - \theta''^2)$, when reduced by means of the elementary symmetric functions of the roots is $3(6 + \theta + 2\theta^2)$, so that

$$\begin{aligned} u_3 : v_3 : w_3 &= 3\,(6 + \theta + \theta^2)\,(1 + 2\theta - \theta^2) : 3(6 + \theta + 2\theta^2)\,(-1 + \theta + \theta^2) \\ &\quad : 9(6 + \theta + 2\theta^2) \\ &= 9 \cdot 7 : 9(3 + 4\theta + \theta^2) : 9(6 + \theta + 2\theta^2). \end{aligned}$$

So that

$$\bar{u}_3 = 7;\ \bar{v}_3 = 3 + 4\theta + \theta^2;\ \bar{w}_3 = 6 + \theta + 2\theta^2.$$

Below appears a convenient arrangement of the expansion, showing the convergents and the value of $N(x, y, z) = m$, using equation (12′) with $x = C_n$, $y = B_n$, $z = A_n$.

Expansion for $x^3 + x - 5 = 0$

				A	B	C	
				1	0	0	
				0	1	0	
$\bar{u} = \alpha$	$\bar{v}$	$\bar{w}$	$p, q;$	0	0	1	m
1	θ	$1 + \theta^2$	1, 3;	1	1	3	5
3	$1 + 2\theta - \theta^2$	$2 + \theta + \theta^2$	0, 1;	1	2	3	9
7	$3 + 4\theta + \theta^2$	$6 + \theta + 2\theta^2$	1, 1;	2	3	7	3
9	$6 + 3\theta$	$3 + 3\theta + 3\theta^2$	1, 1;	4	6	13	1
1	$2 + \theta$	$2 + 2\theta + 2\theta^2$	3, 4;	23	35	76	9
3	$6 + 3\theta$	$2 + 2\theta + 2\theta^2$	3, 1;	37	56	122	7
9	$8 + 4\theta + \theta^2$	$6 + 3\theta + 3\theta^2$	1, 1;	64	97	211	3
5	$5 + 2\theta - \theta^2$	$\theta + 2\theta^2$	1, 1;	124	188	409	1
1	θ	$3 + \theta + \theta^2$	1, 6;		. . .		
3	$4 + 2\theta - \theta^2$	$2 + \theta + \theta^2$	1, 1;				
7	$3 + 4\theta + \theta^2$	$6 + \theta + 2\theta^2$					
	. . .						

It may be well to point out several characteristics of this expansion, which will be the subject of discussion in the following pages. The expansion is periodic, and has one non-recurring q and two non-recurring p's. If we omit the first partial quotient set and the last two sets of the period, the balance is skew-palindromic. The values of m form the same sequence as the values of α, in the reverse order. The convergent set, two before the close of the period gives us a unit, but in this case it is not a fundamental unit. Indeed it is easily verified that $(409 + 188\theta + 124\theta^2) = (13 + 6\theta + 4\theta^2)^2$. There are certain other numerical relations which involve the last two partial quotient sets and the non-recurring elements, but these will be pointed out later.

10. *Periodicity of m_k.* We will now establish a theorem which we need in the consideration of the values of the normal cubic obtained from the convergent sets.

THEOREM IV. *If $1,\ p + \theta,\ q + p\theta + \theta^2$, be expanded into a ternary continued fraction, which ultimately becomes periodic, and if m_k is the value of $N(C_k, B_k, A_k)$, then the sequence m_k also becomes periodic.*

To prove this we will establish certain lemmas, which we also need in other connections.

LEMMA I. *If $1,\ p + \theta,\ q + p\theta + \theta^2$, be expanded into a ternary continued fraction, then the value of m_k is*

$$(17)\qquad m_k = \frac{(\beta' - p\gamma')^3 + p(\beta' - p\gamma')^2\gamma' + q(\beta' - p\gamma')\gamma'^2 + r\gamma'^3}{(\beta'\gamma'' - \beta''\gamma')^2}\,\alpha_{k+1}.$$

(The subscript for β', γ', β'', γ'' is $k+1$, but has been omitted for convenience).

From the lemma it will follow, that if the expansion ultimately becomes periodic, then the values of m_k also become periodic.

We will write instead of $A_k, A_{k-1}, \cdots, B_k, \cdots, A_0, A_1, \cdots, B_0, \cdots$ and omit the subscript $k+1$ on $\alpha, \beta, \cdots$. Using this notation, we may obtain by using Theorem I, equations (4), the following six equations*

$$(18)\qquad \begin{aligned} (C_0 - qA_0)\alpha'' + (C_1 - qA_1)\alpha' + (C_2 - qA_2)\alpha &= -rB_0\gamma'' - rB_1\gamma', \\ (B_0 - pA_0)\alpha'' + (B_1 - pA_1)\alpha' + (B_2 - pA_2)\alpha &= -rA_0\gamma'' - rA_1\gamma'. \\ A_0\alpha'' + A_1\alpha' + A_2\alpha &= C_0\gamma'' + C_1\gamma'. \end{aligned}$$

---

* For the method see the original paper, *loc. cit.*, pp. 286-289.

$$
\begin{aligned}
&A_0\beta'' + A_1\beta' = && B_0\gamma'' + && B_1\gamma',\\
(19)\quad &B_0\beta'' + B_1\beta' = && (C_0 - qA_0 + pB_0)\gamma'' + && (C_1 - qA_1 + pB_1)\gamma',\\
&C_0\beta'' + C_1\beta' = && (-rA_0 + pC_0)\gamma'' + && (-rA_1 + pC_1)\gamma'.
\end{aligned}
$$

By eliminating α', α'' and by means of certain algebraic manipulations, we obtain the desired equation (17).

11. *Reciprocal Units.* Equation (17) can be put into a simpler form due to Lemma II, which also gives us information concerning the reciprocal of a given unit.

LEMMA II. *If* $1, p+\theta, q+p\theta+\theta^2$, *be expanded into a ternary continued fraction, then* $\alpha = \beta'\gamma'' - \gamma'\beta''$ *for all values of the subscript.*

Equation (17) takes the simpler form

$$(17') \qquad \alpha_{k+1}m_k = \delta^3_{k+1} + p\delta^2_{k+1}\gamma'_{k+1} + q\delta_{k+1}\gamma'^2_{k+1} + r\gamma'^3_{k+1}, \quad (\delta = \beta' - p\gamma').$$

The proof of the lemma follows. From equations (5), adopting the abbreviated notation,

$$
(20)\qquad
\begin{aligned}
u_{k+1} &= a_2 + b_2(p+\theta) + c_2(q+p\theta+\theta^2),\\
v_{k+1} &= a_1 + b_1(p+\theta) + c_1(q+p\theta+\theta^2),\\
w_{k+1} &= a_0 + b_0(p+\theta) + c_0(q+p\theta+\theta^2).
\end{aligned}
$$

Let $u'_{k+1}u''_{k+1} = L + M\theta + N\theta^2$, so that

$$
(21)\qquad
\begin{aligned}
\alpha &= [(a_2+pb_2+qc_2) + (b_2+pc_2)\theta + c_2\theta^2]\ (L+M\theta+N\theta^2),\\
\alpha'+\beta'\theta+\gamma'\theta^2 &= [(a_1+pb_1+qc_1) + (b_1+pc_1)\theta + c_1\theta^2]\ (L+M\theta+N\theta^2),\\
\alpha''+\beta''\theta+\gamma''\theta^2 &= [(a_0+pb_0+qc_0) + (b_0+pc_0)\theta + c_0\theta^2]\ (L+M\theta+N\theta^2).
\end{aligned}
$$

Multiplying the last two of these equations out and comparing coefficients, we have

$$
(22)\qquad
\begin{aligned}
\beta' &= L(b_1+pc_1) + M(a_1+pb_1) - N(b_1q+c_1r),\\
\beta'' &= L(b_0+pc_0) + M(a_0+pb_0) - N(b_0q+c_0r),\\
\gamma' &= Lc_1 + Mb_1 + Na_1,\\
\gamma'' &= Lc_0 + Mb_0 + Na_0.
\end{aligned}
$$

Now form the product $\beta'\gamma''-\gamma'\beta''$. It can be written in the form

$$
\begin{aligned}
\beta'\gamma''-\gamma'\beta'' = {} & (b_1c_0 - b_0c_1)(L^2 - qLN + rMN)\\
& + (a_0c_1 - a_1c_0)(-rN^2 - LM + pLN)\\
& + (a_1b_0 - a_0b_1)(M^2 - pMN - LN + qN^2)\\
\equiv {} & R(b_1c_0 - b_0c_1) + S(a_0c_1 - a_1c_0) + T(a_1b_0 - a_0b_1).
\end{aligned}
$$

In determinant form

$$(23)\qquad \beta'\gamma''-\gamma'\beta''=\begin{vmatrix} R & S & T \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{vmatrix}.$$

The values of L, M, N can be found from the equation which defines them, and the elementary symmertic functions. They are

$$(24)\qquad \begin{aligned} L &= a_2^2+pa_2b_2+rb_2c_2+qb_2^2, \\ M &= -(a_2b_2+pa_2c_2+rc_2^2+qb_2c_2), \\ N &= b_2^2-a_2c_2. \end{aligned}$$

Now α may be expressed in determinant form by putting

$$x=a_2+pb_2+qc_2,\quad y=b_2+pc_2,\quad z=c_2,$$

in the general form of $N(x+\theta y+\theta^2 z)$, equation (12).
This gives

$$(25)\qquad \alpha=\begin{vmatrix} a_2+pb_2+qc_2 & -rc_2 & -rb_2 \\ b_2+pc_2 & a_2+pb_2 & -rc_2-qb_2 \\ c_2 & b_2 & a_2 \end{vmatrix}.$$

In this form, we observe that L, M, N are the cofactors of the elements of the first row of (25). If now we replace each element of (25) by its cofactor, expressed in terms of L, M, N, we get, using a well-known theorem on determinants,

$$(26)\quad \alpha^2=\begin{vmatrix} L & M & N \\ -rN & L-qN & M-pN \\ -r(M-pN) & (pq-r)N-qM & L+(p^2-q)N-pN \end{vmatrix}.$$

Also, the cofactor of any element of (26) is α times the corresponding element of (25). Consequently, forming the cofactors of the elements of the third row of (26), we obtain

$$(27)\qquad \begin{aligned} M^2-pMN-LN+qN^2 &\equiv T=c_2\alpha, \\ -rN^2-LM+pLN &\equiv S=b_2\alpha, \\ L^2-qNL+rMN &\equiv R=a_2\alpha. \end{aligned}$$

It follows by substituting these values of R, S, T in (23), that

$$\beta'\gamma''-\beta''\gamma'=\alpha\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{vmatrix}=\alpha\cdot 1=\alpha.$$

This completes the proof of the lemma, establishes equation (17′), and incidentally gives the following method for determining the reciprocal unit.

Suppose now that $N(x, y, z) = 1$, and

$$(x + \theta y + \theta^2 z)(L + M\theta + N\theta^2) = 1, \quad \text{so that}$$

$N(L, M, N)$ is also 1, and L, M, N is the reciprocal solution corresponding to x, y, z. Write $N(x, y, z)$ in determinant form.

$$\begin{vmatrix} x & -rz & -r(y-pz) \\ y & x-qz & -rz-q(y-pz) \\ z & y-pz & (x-qz)-p(y-pz) \end{vmatrix} = 1.$$

Then L, M, N are the cofactors of the elements of the first row. This is a convenient device often used in numerical calculations.

12. *Convergence.* The periodicity of the sequence m_k, and the geometry of the surface $N(x, y, z) = m$ afford a proof that B_k/A_k and C_k/A_k actually converge towards $p + \theta$ and $q + p\theta + \theta^2$ respectively, if the expansion is regular.

Consider the line which is the intersection of two of the asymptotic planes of the surface $N(x, y, z) = m$.

$$x/(q + p\theta + \theta^2) = y/(p + \theta) = z/1.$$

Let (C_k, B_k, A_k) be any point on the surface $N(x, y, z) = m$, where m is one of the values assumed by m_k. The distance d_k, from this point to the line is given by

$$(28) \quad d_k^2 = (C_k^2 + B_k^2 + A_k^2)\left[1 - \frac{(C_k w_1 + B_k v_1 + A_k)^2}{(C_k^2 + B_k^2 + A_k^2)(w_1^2 + v_1^2 + 1)}\right],$$

where $w_1 = q + p\theta + \theta^2$ and $v_1 = p + \theta$. This formula can be written in the form

$$(29) \quad (w_1^2 + v_1^2 + 1) d_k^2$$
$$= (B_k - A_k v_1)^2 + (C_k - A_k w_1)^2 + (C_k v_1 - B_k w_1)^2.$$

Now as (C_k, B_k, A_k) take on successive values which are one period apart and all solutions of $N(x, y, z) = m$, the value of $|z| = |A_k|$ continually increases by the assumption of a regular expansion. The points on the surface approach continually nearer to the asymptotic line, so that $d_k \to 0$. Since v_1 and w_1 are finite, it follows that the $|B\text{'s}|$ and $|C\text{'s}|$ each form an increasing sequence if the $|A\text{'s}|$ do, such that $B_k - A_k v_1$, $C_k - A_k w_1$, $Cv_1 - B_k w_1$, all approach zero as a limit. This result we state as

THEOREM V. *If* $1,\ p+\theta,\ q+p\theta+\theta^2$, *be expanded into a regular continued fraction, which ultimately becomes periodic, then*

$$\lim_{k\to\infty} B_k/A_k = p+\theta, \qquad \textit{and} \qquad \lim_{k\to\infty} C_k/A_k = q+p\theta+\theta^2.$$

IV. NORMAL TERNARY CONTINUED FRACTIONS.

13. *Determination of Unit.* We will use theorem IV and the corresponding equation (17′) to prove the following theorem.

THEOREM VI. *If* $1,\ p+\theta,\ q+p\theta+\theta^2$, *be expanded into a periodic continued fraction, such that the q's after the first are periodic and the p's after the second are periodic, and if the period contain n sets of partial quotients, then* (C_n, B_n, A_n) *is a solution of* $N(x,y,z)=1$.

Under the conditions of the hypothesis,

$$(30) \qquad u_{k+2} : v_{k+2} : w_{k+2} = u_{k+n+2} : v_{k+n+2} : w_{k+n+2}, \qquad (k=1,2,\cdots,\infty).$$

However, in general

$$(31) \qquad w_{k+2} = u_{k+1} \quad \text{and} \quad w_{k+n+2} = u_{k+n+1}.$$

By successive combinations of these two sets of equations we find

$$(32) \qquad \begin{aligned} u_{n+k+2} &= \lambda u_{k+2}, & (k&=0,1,\cdots,\infty),\\ v_{n+k+2} &= \lambda v_{k+2}, & (k&=1,2,\cdots,\infty),\\ w_{n+k+2} &= \lambda w_{k+2}, & (k&=1,2,\cdots,\infty), \end{aligned}$$

where

$$\lambda = u_{n+2}/u_2.$$

By referring to equations (4) of theorem I, and making the necessary changes in subscripts, we have

$$A_{n+2}w_{n+3} + A_{n+1}v_{n+3} + A_n u_{n+3} = 1,$$

or

$$\lambda(A_{n+2}w_3 + A_{n+1}v_3 + A_n u_3) = 1.$$

Therefore λ is a unit in the domain $K(\theta)$, that is, $N(\lambda)=1$. Furthermore, from (32), $N(u_{k+2}) = \alpha_{k+2}$ forms a periodic sequence beginning with $k=0$, or $\alpha_2, \alpha_3, \cdots$ is a periodic sequence of period n.

Now let us impose the condition of the hypothesis $q_{n+2}=q_2$. From the equations (3)

$$\begin{aligned} w_{n+2} &= v_{n+3} + q_{n+2}\, u_{n+2},\\ w_2 &= v_3 \quad + q_2 \quad u_2. \end{aligned}$$

Since $w_2 = 1$, and $u_{n+1} = w_{n+2}$, and $q_2 = q_{n+2}$, we find from (32)

$$u_{n+1} = \frac{w_{n+2}}{w_2} = \frac{\lambda v_3 + \lambda u_2 q_2}{v_3 + u_2 q_2} = \lambda,$$

or

$$N(u_{n+1}) = \alpha_{n+1} = 1. \tag{33}$$

Consequently the α's form a purely periodic sequence. Equation (17′) now becomes, for the special case under consideration

$$m_n = (\beta'_{n+1} - p\gamma'_{n+1})^3 + p(\beta'_{n+1} - p\gamma'_{n+1})^2 \gamma' + q(\beta'_{n+1} - p\gamma'_{n+1})\gamma'^2_{n+1} + r\gamma'^3_{n+1}. \tag{34}$$

By definition

$$v_{n+1} = u_{n+2} + p_{n+1} u_{n+1} = u_{n+1}(u_2 + p_{n+1}),$$

or

$$v_{n+1}/u_{n+1} = p_{n+1} + p - p_1 + \theta, \tag{35}$$

since

$$u_{n+2} = \lambda u_2,\ u_{n+1} = \lambda, \text{ and } u_2 = p - p_1 + \theta.$$

As

$$\alpha'_{n+1} + \beta'_{n+1}\theta + \gamma'_{n+1}\theta^2 = \frac{v_{n+1}}{u_{n+1}} N(u_{n+1}),$$

it follows that

$$\alpha'_{n+1} = p_{n+1} - p_1 + p;\ \beta'_{n+1} = 1;\ \gamma'_{n+1} = 0. \tag{36}$$

These last two values, when substituted in (34) give $m_n = 1$, or

$$N(C_n, B_n, A_n) = 1,$$

proving the theorem.

By a method similar to that used to find (36), we can find α''_{n+1}, β''_{n+1}, γ''_{n+1}, and as we will need them later, we evaluate them now.

$$w_{n+1} = v_{n+2} + q_{n+1}u_{n+1} = u_{n+3} + p_{n+2}u_{n+2} + q_{n+1}u_{n+1}.$$
$$w_{n+1}/u_{n+1} = u_3 + p_{n+2}u_2 + q_{n+1}.$$

Now

$$u_2 = -p_1 + p + \theta,$$
$$u_3 = (p_1 - p)p_2 - q_1 + q + (p - p_2)\theta + \theta^2,$$

and

$$\alpha''_{n+1} + \beta''_{n+1}\theta + \gamma''_{n+1}\theta^2 = \frac{w_{n+1}}{u_{n+1}} N(u_{n+1}).$$

Hence

$$\begin{aligned} \alpha''_{n+1} &= (p_1 - p)(p_2 - p_{n+2}) + q - q_1 + q_{n+1}, \\ \beta''_{n+1} &= p - p_2 + p_{n+2}, \\ \gamma''_{n+1} &= 1. \end{aligned} \tag{37}$$

14. *Special Cases.* It should be pointed out that the essential points of the hypothesis of theorem VI are that the q's are periodic after the first, and the p's are periodic after the second. No limitations are imposed upon q_1, p_1 and p_2, and in special cases these might be such that the q's are purely periodic, and the p's periodic after p_1 or even purely periodic. For example, it will be shown later, or may easily be verified directly, that if θ is a root of $x^3 + qx - 1 = 0$, then the expansion for $(1, \theta, q + \theta^2)$ is $(0, q; 0, q; \cdots)$, that is, purely periodic with $m = 1$. However, it may be interpreted as an expansion, which satisfies the hypothesis, so that $m_1 = 1$. Again, it is found that if θ is the real root of $x^3 + 5x - 3 = 0$, then

$$(1, \theta, 5 + \theta^2) = (0, 4; 2, 1; 2, 1; 5, 1; 2, 1; 2, 3; 2, 1; 2, 1; 5, 1; \cdots)$$

This expansion has 5 terms in the period, and may be considered as one which is periodic after the first partial quotient set. However, for the puropose of the theorem, we may consider that the q's after the first and the p's after the first and second are periodic, and hence $N(C_5, B_5, A_5) = 1$.

Let us further consider the numerical example previously given, $x^3 + x - 5 = 0$. It will be noticed that the part of the expansion obtained by deleting p_1, q_1; from the beginning and p_{n+1}, q_{n+1}; p_{n+2}, q_{n+2}; from the end is skew-palindromic. Consequently (theorem II), if the unit is expanded into a finite continued fraction using the same set of partial quotients, the u's will be the same sequence as the A's in reverse order. This is the key to experimental attempts to find expansions, for having first found a unit (not necessarily the fundamental one), we seek a finite expansion corresponding to it, which, after p_1, q_1; is skew-palindromic.

It should be noted that the period of partial quotient sets for the desired periodic expansion is

$$p_3, q_3; p_4, q_4; \cdots p_n, q_n; p_{n+1}, q_{n+1}; p_{n+2}, q_2;$$

while the $N(C_n, B_n, A_n) = 1$. That is, the expansion of the unit does not depend upon $p_{n+1}, p_{n+2}, q_{n+1}$. These numbers are involved in equations (36) and (37) and just what rôle they play in the expansion will be indicated in the next definition and theorem.

15. *Normal Ternary Continued Fractions.* The experimental determination of a large number of periodic expansions led to the following definition of a *Normal expansion.*

[illegible] θ, [illegible] θ', expands into a [illegible] periodic ternary continued fraction with n partial quotient sets in the period, such that the q's are periodic after the first and the p's after the second, and if that part of

the expansion indicated by $p_2, q_2; p_3, q_3; \cdots ; p_n, q_n;$ *is skew-palindromic, and further if* $p_{n+1} = p_{n+2} = p_1 + p_2 - p$ and $q_{n+1} = (p_1 - p)^2 + 2q_1 - q$, *then the expansion is normal.*

This definition was suggested by the following important theorem of which the last is a special case.

THEOREM VII. *If* 1, $p + \theta$, $q + p\theta + \theta^2$ *be expanded into a normal ternary continued fraction, then* $(C_{n-k+1}, B_{n-k+1}, A_{n-k+1})$ *is a solution of the normal cubic* $N(x, y, z) = \alpha_k$, $(k = 1, 2, \cdots, n)$.

The proof is long and we omit the details.* Under the assumptions that

$$p_{k+2} = p_{n-k+1}, \quad q_{k+1} = q_{n-k}, \qquad (k = 1, 2, \cdots, n), \tag{38}$$

$$p_{n+1} = p_{n+2} = p_1 + p_2 - p, \quad q_{n+1} = (p_1 - p)^2 + 2q_1 - q, \tag{39}$$

we first show by mathematical induction that

$$u_k = u_{n+1} (C_{n-k+1} + B_{n-k+1}\theta + A_{n-k+1}\theta^2). \tag{40}$$

To establish this for the special cases of $k = 1, 2, 3$, use is made of equations (18) and (19) and the special values of $\beta'_{n+1} = 1$, $\gamma'_{n+1} = 0$, $\gamma''_{n+1} = 1$, $\alpha_{n+1} = 1$, and the values of α''_{n+1}, α'_{n+1}, β''_{n+1} given by (36) and (37). Since $N(u_{n+1}) = 1$, the theorem readily follows.

16. *Fundamental Units.* It should be pointed out that n may be interpreted as the number of terms in one period or any number of periods, for if the expansion is normal, it is normal whether we consider only one period or m periods. That means, that instead of (40) we may write

$$u_k = u_{mn+1} (C_{mn-k+1} + B_{mn-k+1}\,\theta + A_{mn-k+1}\,\theta^2), \tag{41}$$

or in particular

$$u_1 = 1 = u_{mn+1} (C_{mn} + B_{mn}\theta + A_{mn}\theta^2). \tag{42}$$

Now by successive applications of the first equation of (32) and the value of $\lambda = u_{n+1}$, it readily follows that

$$u_{mn+1} = u^m_{n+1}. \tag{43}$$

Hence we have the two equations

$$\begin{aligned} u_1 &= u_{n+1} (C_n + B_n\theta + A_n\theta^2) = 1, \\ u_1 &= u^m_{n+1} (C_{mn} + B_{mn}\theta + A_{mn}\theta^2) = 1. \end{aligned} \tag{44}$$

Consequently,

$$(C_{mn} + B_{mn}\theta + A_{mn}\theta^2) = (C_n + B_n\theta + A_n\theta^2)^n. \tag{45}$$

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[*] For the method see the original paper, *loc. cit.*, pp. 292-294.

This proves that the unit obtained from the mth period of the expansion is the mth power of that obtained from the first period. If $(C_n + B_n\theta + A_n\theta^2)$ is a fundamental unit, say ϵ, and written $\epsilon = (C_n, B_n, A_n)$, then all positive powers of ϵ appear in the expansion as $\epsilon^m = (C_{mn}, B_{mn}, A_{mn})$, and the reciprocal of ϵ^m namely ϵ^{-m} is the mth power of the reciprocal of ϵ. If ϵ is not a fundamental unit, only some of the units may appear as convergents. We will now explain just what we mean by a fundamental unit.

Let us consider the case first, when there is only one real root and for convenience let it be positive. The units are essentially of two types, positive and reciprocal units. The positive units are such that their successive powers give points, which approach asymptotically the asymptotic line of the surface $N(x, y, z) = 1$, that is, C_{mn}, B_{mn}, A_{mn}, form positive and increasing sets of numbers as m increases, such that C_{mn}/A_{mn} and B_{mn}/A_{mn} approach $q+p\theta+\theta^2$ and $p+\theta$ as limits. The smallest set of integers (C_n, B_n, A_n) which satisfy this condition defines the fundamental unit $\epsilon = C_n + B_n\theta + A_n\theta^2$. From the very nature of our work C_n, B_n, A_n are integers and hence the fundamental unit in the field $K(\theta)$* and the fundamental unit as defined here, will be the same only if the minimal basis is $1, \theta, \theta^2$, and may or may not be the same if the basis is not $1, \theta, \theta^2$.† The numbers we deal with here form a subset of the numbers in the field and hence it is desirable to call our fundamental unit, as defined here, the *fundamental unit in the chain.* The fundamental unit may or may nor appear as a convergent set in the expansion and in the table at the end of this paper, this will be indicated by calling the unit which is determined from the first period ϵ' (C_n, B_n, A_n) if $\epsilon' \neq \epsilon$, and indicating the relation between ϵ' and ϵ.

If the cubic has three real roots, we will consider a fundamental unit corresponding to each root. Let $\epsilon_1 = C_n + B_n\theta_1 + A_n\theta_1^2$ be a unit corresponding to one root θ_1 of the given cubic, and let $\epsilon_1^m = (C_{mn} + B_{mn}\theta_1 + A_{mn}\theta_1^2)$. If C_n, B_n, A_n are such that $N(C_n, B_n, A_n) = 1$, and if their absolute values are the smallest integers which can be found, such that some power of ϵ, leads to a normal expansion, so that

$$\lim_{m\to\infty} \frac{B_{mn}}{A_{mn}} = p + \theta_1, \text{ and } \lim_{m\to\infty} \frac{C_{mn}}{A_{mn}} = q + p\theta_1 + \theta_1^2,$$

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* For a discussion of this subject, see for example L. W. Reid, *Tafel der Klassen*[illegible] [illegible] *Zahlkörper*, Dissertation, Göttingen (1899), pp. [illegible]

† The author has obtained periodic expansions corresponding to the case when the basis is not $1, \theta, \theta^2$, but they are not normal and will not be discussed here.

then ϵ_1, is defined as the *fundamental unit of the chain corresponding to* θ_1. Similarly ϵ_2 and ϵ_3 are defined as fundamental units corresponding to θ_2 and θ_3. These three units are not independent,[*] but are connected by a relation of the type

$$\epsilon_1^{m_1}\,\epsilon_2^{m_2}\,\epsilon_3^{m_3} = +1.$$

where m_1, m_2, m_3 are positive or negative integers. Even if the fundamental units, ϵ_1, ϵ_2, ϵ_3, are obtained from the corresponding convergent sets (C_n, B_n, A_n), all the positive units are not given by convergent sets, because every number of the type,

$$\eta = \epsilon_i^{m}\,\epsilon_j^{n},$$

is a unit and as it is not of a simple type it does not appear as a convergent set.

17. *Expansions with Given Periods.* It is possible to avoid some of the computation by determining the cubic, which has a given normal expansion. However, except in a few simple cases, the algebraic and numerical work involved is greater than the computation involved by other methods and not many of the resulting cubics are within the scope of the table given. We will consider two of the simple cases, which frequently occur, when $p = 0$, so that our cubic is of the type $x^3 + qx + r = 0$.

If the normal expansion of $(1, \theta, q + \theta^2)$ has only one partial quotient in the period, so that

$$(46)\qquad (1, \theta, q + \theta^2) = (p_1, q_1;\ p_2, p_1^2 + 2q_1 - q;\ |\ p_1 + p_2, p_1^2 + 2q_1 - q;),$$

then the purely periodic part is such that

$$(49)\qquad \bar{u}_3 : \bar{v}_3 : \bar{w}_3 = 1 : p_2 + \theta : q_1 + p_1\theta + \theta^2.$$

This comes from using the conditions (36) and (37), and (39) with $p = 0$.

If 1, $p_2 + \theta$, $q_1 + p_1\theta + \theta^2$, be expanded into a continued fraction with the partial quotient set $(p_1 + p_2, q_2;)$, the next line of the expansion, giving u, v, w is

$$-p_1 + \theta, \quad q - q_1 - p_1^2 + p_1\theta + \theta^2, \quad 1,$$

and these numbers are also in the ratio of $\bar{u}_3 : \bar{v}_3 : \bar{w}_3$. By equating the values of $\bar{v}_3/\bar{u}_3$ and $\bar{w}_3/\bar{u}_3$, we find

$$(48)\qquad -p_1^2 + q_1 + q + p_1\theta + \theta^2 = -p_1p_2 + (p_2 - p_1)\theta + \theta^2.$$

$$(49)\qquad 1 = -p_1q_1 - r.$$

From these equations, we find

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[*] See for example, L. W. Reid, *loc. cit.*, p. 34.

$$(50) \qquad p_2 = 2p_1; \quad q_1 = p_1^2 + q; \quad -r = p_1^3 + p_1q + 1.$$

In conclusion then, if θ is a root of $x^3 + qx + r = 0$, where

$$r = -(p_1^3 + p_1q + 1), \quad p_1 \text{ and } q \text{ being arbitrary, then}$$

$$(46') \qquad (1, \theta, q + \theta^2) = (p_1, p_1^2 + q; 2p_1, 3p_1^2 + q; \mid 3p_1, 3p_1^2 + q;)$$

and the fundamental unit in the chain is $(p_1^2 + q, p_1, 1)$. We will see later that this statement needs a slight modification. If, in particular, $p_1 = 0$, we have $p_1 = p_2 = p_3 = 0$; $q_1 = q_2 = q_3 = q$; $r = -1$, so that if θ is a root of $x^3 + qx - 1 = 0$, then

$$(46'') \qquad (1, \theta, q + \theta^2) = (0, \underset{0}{q};)$$

If $p_1 = 1$, $r = -q - 2$, and

$$(1, \theta, q + \theta^2) = (1, 1 + q; 2, 3 + q; \mid 3, 3 + q;).$$

If $p_1 > 1$, the equations obtained are beyond the scope of the table computed.

When p_1 is negative, besides (46′), we must take into consideration, the following theorem due to Lehmer.*

> *"If the characteristic cubic has one root ρ_1, whose modulus is greater than the modulus of either of the other two, then the fractions B_n/A_n and C_n/A_n, approach, as $n \to \infty$, limits which are cubic irrationalities $\cdots$. If the characteristic cubic has two imaginary roots, whose common modulus is greater than the absolute value of the real root, then these fractions do not approach any limit."*

In the case under consideration, the characteristic cubic $C(\rho) = 0$ is

$$(51) \qquad C(\rho) = \rho^3 - (3p_1^2 + q)\rho^2 - 3p_1\rho - 1 = 0.$$

When q is positive, it is readily seen that (51) has a real root greater than 1, since $C(1)$ is negative and $C(\infty)$ positive. Hence $C(\rho) = 0$ satisfies the first condition of the theorem, and C_n/A_n and B_n/A_n actually converge to $\theta^2 + q$ and θ as limits. When q is negative and p_1 positive and if $|q| < 3p_1^2 + 3p_1$, then $C(1)$ is negative and $C(\infty)$ positive, so that the first condition is satisfied. If $|q| > 3p_1^2 + 3p_1$, the discriminant of (51) is found to be positive, and hence (51) has three real roots and the first condition is again satisfied.

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* D. N. Lehmer, *Proceedings of the National Academy of Sciences*, Vol. 4 (1918), pp. 360-364.

The remaining case, q and p_1 both negative, sometimes leads to the first and sometimes to the second condition and under the second condition we do not have an expansion of cubic irrationalities. A numerical illustration will make this clear. Take $p_1 = -1$, so that equation under consideration is

$$x^3 + qx + q = 0, \qquad (q \text{ negative}). \tag{52}$$

The apparent expansion is

$$(1, \theta, q + \theta^2) = (-1, 1 + q; -2, 3 + q; \mid -3, 3 + q;), \tag{53}$$

and the fundamental unit is $(1 + q, -1, 1)$. The characteristic cubic is

$$C(\rho) = \rho^3 - (q + 3)\rho^2 + 3\rho - 1 = 0. \tag{54}$$

Its discriminant is

$$\Delta = q^2(-4q - 27).$$

It is readily seen that $C(0) = -1$; $C(1) = -q =$ positive number, and $C(\rho)$ has a root between 0 and 1. Also if $-q \leqq 6$, Δ is negative, while if $-q \geqq 7$, Δ is positive. In the former case $(-q \leqq 6)$, the second condition is satisfied and C_n/A_n and B_n/A_n do not approach limits and (53) does not represent an expansion in this case. It is worthy of note that $(1 + q, -1, 1)$ is the reciprocal of the fundamental unit, and hence the fundamental unit can be found. In the other case $(-q \geqq 7)$, the first condition of the theorem is satisfied, and it is found that (53) is the expansion corresponding to one of the negative roots of (52). In conclusion then, the statement of equation (46′) must be modified by adding the statement: providing A_n, B_n, C_n are actual convergents.

A second type of normal expansion which occurs frequently in the table is one with three sets of partial quotients in the period. We again consider the equation $x^3 + qx + r = 0$, such that the given expansion is

$$(1, \theta, q + \theta^2) = (p_1, q_1; p_2, q_2; \mid p_3, q_2;\ p_1 + p_2, q_4; p_1 + p_2, q_2;). \tag{55}$$

Let us consider the special case of $p_1 = 0$ so that $|\,\theta\,|$ is less than 1. Then the greatest integer in $q + \theta^2$ is q, and we select $q_1 = q$, and this also makes $q_4 = q$. The resulting expansion is periodic from the beginning. If we form the expansion, we have

$$\begin{array}{cccc}
1 & \theta & q + \theta^2 & 0, q; \\
\theta & \theta^2 & 1 & 0, q_2; \\
\theta^2 & 1 - q_2\theta & \theta & p_3, q_2; \\
1 - q_2\theta - p_3\theta^2 & \theta - q\theta^2 & \theta^2 & \cdots\cdot \\
\cdots & \cdots & \cdots &
\end{array}$$

From the relations $v_4/u_4 = \theta$ and $w_4/u_4 = (q + \theta^2)$, we readily find, by equating corresponding coefficients of like irrationalities, that

$$p_3 = 0; \quad q = -rq_2 = mq_2 \qquad (m = -r). \tag{56}$$

In conclusion, if θ is a root of $x^3 + mq_2x - m = 0$, then

$$(1, \theta, q + \theta^2) = (0, mq_2; 0, q_2; 0, q_2;), \tag{57}$$

provided A_n, B_n, C_n are actual convergents. It is possible to find other special cases for (55), but they do not occur frequently and it will not be done here.

18. *Negative Continued Fractions.* It was found convenient in certain cases to use what is called a negative continued fraction. This is defined by the relations

$$u_{n+1} = p_nu_n - v_n; \quad v_{n+1} = q_nu_n - w_n; \quad w_{n+1} = u_n. \tag{58}$$

The p's and q's are selected so the $|\,u\text{'s}\,|$ form a decreasing sequence of numbers. If the numbers are positive, this makes p_n the integer next above the greatest integer in v_n/u_n.

The corresponding recursion formulas are

$$\begin{aligned} A_n &= q_nA_{n-1} - p_nA_{n-2} + A_{n-3}, \\ B_n &= q_nB_{n-1} - p_nB_{n-2} + B_{n-3}, \\ C_n &= q_nC_{n-1} - p_nC_{n-2} + C_{n-3}, \end{aligned} \tag{59}$$

with the initial conditions

$$\begin{vmatrix} A_{-2} & A_{-1} & A_0 \\ B_{-2} & B_{-1} & B_0 \\ C_{-2} & C_{-1} & C_0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}. \tag{60}$$

The main theorems concerning normal expansions hold without modification and other results involve a change in sign, since the determinant of three successive convergent sets is now -1. For example, if θ is a root of $x^3 + qx - (p_1^3 + p_1q - 1) = 0$, then the negative expansion for $1, \theta, q + \theta^2$, is of the form (46′). No proofs of these facts will be given as they are readily derived by merely changing certain signs in the original proofs. It [illegible] readily shown that if a negative expansion for $(1, \theta, q + \theta^2)$ is known, then a positive expansion for $(1, -\theta, q + \theta^2)$ is obtained by changing the signs of the p's.

V. CONSTRUCTION AND EXPLANATION OF TABLE.

19. In making an extensive table of expansions, it may be pointed out that

$$(7) \qquad x^3 + px^2 + qx + r = 0,$$

need only be considered for values of $p = 0, 1, 2$.

Let θ be a root of (7), and let

$$(61) \qquad (1, p + \theta, q + p\theta + \theta^2) = (p_1, q_1; p_2, q_2; p_3, q_3; \cdots).$$

Decrease the roots of (7) by t, obtaining

$$(7') \qquad x^3 + p'x^2 + q'x + r' = 0,$$

where

$$p' = p + 3t, \quad q' = q + 2pt + 3t^2.$$

Let θ' be the corresponding root of (7′) and let

$$(61') \qquad (1, p' + \theta, q' + p'\theta' + \theta'^2) = (p'_1, q'_1; p'_2, q'_2; \cdots).$$

Then by actually constructing the expansion, it is found $p'_1 = p_1 + 2t$; and if we select $q'_1 = t^2 + p_1 t + q_1$; then $p'_2 = p_2 + t$, and the balance of the expansion (61′) is identical with (61). Hence we need consider the values of p in (7) which are less than 3, namely 0, 1, 2, although -1, 0, 1 would do just as well.

20. *Table for* $x^3 + qx + r = 0$. Because of the information already available in the paper by Reid, which we have previously mentioned, the author considered the case $p = 0$. The table at the end of this paper gives the units and the normal expansions corresponding to $x^3 + qx + r = 0$ for q between -9 and $+9$, and $-r$ between 1 and 9. The order and numbering of Reid's table is retained for reference. However, in this table the sign of the root is changed in order to make it positive, when there is only one real root. If there are three roots, they are called θ_1, θ_2, θ_3 in order of magnitude, θ_1 being the largest. The fundamental unit of the chain is designated by $\epsilon\,(x, y, z)$, and the relation between the unit which is determined by the first period $\epsilon'(C_n, B_n, A_n)$ and ϵ is given. If there are three real roots, the relation between ϵ_1, ϵ_2, ϵ_3, is also given. The period of the expansion is separated from the non-recurring sets by a vertical bar |, and a complete period is given or indicated. For example the entry corresponding to the numerical example given before $x^3 + x - 5 = 0$, would be

34. $x^3 + x - 5 = 0$ $\qquad$ (13, 6, 4) $\quad \epsilon' = \epsilon^2$

$$(1, 3; 0, 1; \mid 1, 1; 1, 1; 3, 4; 3, 1; 1, 1; 1, 1; 1, 6; 1, 1;).$$

In case the period is long, only part of the expansion is written, and the balance can be supplied from the palindromic relations and the conditions for a normal expansion (38, 39). The middle elements of the skew-palindromic part are enclosed in parentheses. If n is even, two p's and one q will be in parentheses, and if n is odd one p and two q's. For example the entry just written could also be written

$$(1, 3; 0, 1; \mid 1, 1; 1, 1; (3), (4); (3), \cdots).$$

If the expansion is purely periodic no vertical bar is used. A negative expansion is indicated by *NE* before the expansion, which means that formulas (58), (59), (60) must be used.

21. *Computation.* There are several methods of determining the expansion corresponding to a given equation. If the equation is not of a form, whose expansion is of a known type, and if a unit is known, we attempt to expand the unit into a finite fraction, which after p_1, q_1; is skew-palindromic. We do not usually use the fundamental unit, but some power of it, so that x, y, z are large enough to make x/z and y/z fair approximations to $q + \theta^2$ and θ. We also do not use the fundamental unit, because at times it does not occur in the expansion. The expansion can then be checked for the conditions of a normal expansion, by considering higher powers of the unit, or forming the rationalized complete quotient sets or other devices. The rationalization process is long and is avoided when possible.

As a numerical example, consider $x^3 - 3x - 6 = 0$. It is found that $\epsilon = (79, 73, 31)$, $\epsilon^2 = (33397, 30878, 13110)$. The expansion is below

				A	*B*	*C*	
				1	0	0	
				0	1	0	
u	*v*	*w*	*p*, *q*;	0	0	1	
13110	30878	33397	2, 2;	1	2	2	
4658	7177	13110	1, 2;	2	5	5	
2519	3794	4658	1, 1;	3	7	8	
1275	2139	2519	1, 1;	6	14	15	
864	1244	1275	1, 1;	11	26	28	
380	411	864	1, 2;	31	73	79	ϵ
31	104	380	3, 11;	380	. . .		
11	39	31	3, 2;	864			
6	9	11	1, 1;	1275			
3	5	6	1, 1;	2519			
2	3	3	1, 1;	4658			

The only place we did not take q as the greatest integer in w/u was for $q_7 = 11$ and this is done to make the expansion normal.

If a unit is not known, we take as $u_1 : v_1 : w_1$, numbers which are approximations to $1 : \theta : q + \theta^2$, and expand them, trying for a normal expansion, and computing the values of m as we proceed. Thus for $x^3 + 5x - 2 = 0$, we have $\theta \doteq .3883$; $\theta^2 \doteq .1508$. The expansion is below

				1	0	0	
				0	1	0	
				0	0	1	m
10000	3883	51508	0, 5;	1	0	5	4
3883	1508	10000	0, 2;	2	1	10	4
1508	2234	3883	1, 2;	5	2	26	2
726	867	1508	1, 2;	13	5	67	1
141	56	726	0, 5;		. . .		
56	21	141	0, 2;				
21	29	56	1, 2;				
8	14	21	1, 2;				
. . .	. . .						

We readily recognize that the expansion is normal and purely periodic. This may be checked by considering the expansion for some power of ϵ.

If the expansion is not obtained by this last method, we usually obtain values, which satisfy $N(x, y, z) = m$, for small values of m. By combining them according to various methods, we are able to find a unit and check to see whether it is a power of a unit.* Having a unit we expand it by the first method. In many cases the numerical calculations are long and tedious, and in the few cases where a blank appears in the table, the author feels that there is a normal expansion but that the effort necessary to find it, is probably not worth while.

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* For a discussion of these methods with reference to $x^3 - D = 0$, see Meissel, E., *Beitrag zur Pell'schen Gleichung höherer Grade*, or Wolfe, C., "On the Indeterminate Cubic Equation," *University of California Publications in Mathematics*, Vol. 1 (1923), pp. 359-369.

TABLE OF UNITS AND NORMAL EXPANSIONS.

	$\epsilon(x, y, z)$		*Expansion.*
1. $x^3-1=0$	reducible.		
2. $x^3+x-1=0$	(1, 0, 1)		(0, 1;)
3. $x^3-x-1=0$	(0, 1, 1)		(1, 0; 2, 2; | 3, 2;)
4. $x^3-2=0$	(1, 1, 1)		(1, 1; 2, 3; | 3, 3;)
5. $x^3+x-2=0$	reducible.		
6. $x^3-x-2=0$	(1, 1, 1)	$\epsilon'=\epsilon_1^4$	(1, 1; 0, 1; | 1, 1; 1, 1; 3, 2; 3, 1; 1, 1; 1, 1; 1, 4; 1, 1;)
7. $x^3+2x-2=0$	(3, 1, 1)		(0, 2; 0, 1; 0, 1;)
8. $x^3-2x-2=0$	(1, 1, 0)	$\epsilon'=\epsilon^2$	(1, 1; 0, 1; | 1, 5; 1, 1;)
9. $x^3+2x-1=0$	(2, 0, 1)		(0, 2;)
10. $x^3-2x-1=0$	reducible.		
11. $x^3-3=0$	(4, 3, 2)		(1, 2; 0, 2; | 1, 5; 1, 2;)
12. $x^3+x-3=0$	(2, 1, 1)		(1, 2; 2, 4; | 3, 4;)
13. $x^3-x-3=0$	(2, 2, 1)		(1, 1; 1, 1; | 2, 4; 2, 1;)
14. $x^3+2x-3=0$	reducible.		
15. $x^3-2x-3=0$	(2, 2, 1)	$\epsilon'=\epsilon^2$	(1, 1; 0, 1; | 0, 1; 2, 3; 2, 1; 0, 1; 1, 5; 1, 1;)
		or $\epsilon'=\epsilon$	*NE* (2, 2; 4, 10; | 6, 10;)
16. $x^3+3x-3=0$	(4, 1, 1)		(0, 3; 0, 1; 0, 1;)
17. $x^3-3x-3=0$	(1, 1, 0)	$\epsilon'=\epsilon^2$	(2, 1; 4, 9; | 6, 9;)
18. $x^3+3x-1=0$	(3, 0, 1)		(0, 3;)
19. $x^3-3x-1=0$	ϵ_1(1, 1, 0)	$\epsilon_1'=\epsilon_1^4$	(1, 0; 0, 1; | 0, 1; 2, 2; 2, 1; 0, 1; 1, 4; 1, 1;)
		or $\epsilon_1'=\epsilon_1^2$	*NE* (2, 1; 4, 9; | 6, 9;)
	ϵ_2(—3, 0, 1)		(0, —3;)
$\epsilon_1^2\epsilon_2^2\epsilon_3=1.$	ϵ_3(—1, —3, 2)		(—1, —1; 0, —1; | 1, —1; —1, 2; —1, —1;)
20. $x^3+3x-2=0$	(17, 3, 5)	$\epsilon'=\epsilon^2$	(0, 2; 2, 1; | 2, 2; 3, 1; 2, 1; 3, 2; 2, 1; 2, 1; 2, 1;)
21. $x^3-3x-2=0$	reducible.		
22. $x^3-4=0$	(5, 3, 2)		(1, 1; 2, 1; | 1, 1; 3, 3; 3, 1;)
23. $x^3+x-4=0$	(61, 29, 21)		(1, 2; 2, 2; | 1, 1; 2, 1; 1, 2; 3, 4; 3, 2;)
24. $x^3-x-4=0$	(11, 9, 5)	$\epsilon'=\epsilon^2$	(1, 2; 0, 1; | 0, 3; 0, 1; 0, 1; 3, 2; 3, 1; 0, 1; 0, 3; 0, 1; 1, 6; 1, 1;)
25. $x^3+2x-4=0$	(3, 1, 1)		(1, 3; 2, 5; | 3, 5;)
26. $x^3-2x-4=0$	reducible.		
27. $x^3+3x-4=0$	reducible.		
28. $x^3-3x-4=0$	(9, 11, 5)		(2, 2; —1, 5; | 1, 11; 1, 5;)
29. $x^3+4x-4=0$	(5, 1, 1)		(0, 4; 0, 1; 0, 1;)
30. $x^3-4x-4=0$	(1, 1, 0)	$\epsilon'=\epsilon^3$	(2, 1; 1, 1; | 2, 1; 3, 10; 3, 1;)

31. $x^3+4x-1=0$	$(4, 0, 1)$	$(0, 4;)$
32. $x^3-4x-1=0$	$\epsilon_1(0, 2, 1)$	$(2, 0; 4, 8; \mid 6, 8;)$
	$\epsilon_2(-4, 0, 1)$	$(0, -4;)$
$\epsilon_1\epsilon_2\epsilon_3=1.$	$\epsilon_3(0, -2, 1)$	$(-2, 0; -4, 8; \mid -6, 8;)$
33. $x^3+4x-2=0$	$(17, 2, 4)$	$(0, 4; 0, 2; 0, 2;)$
34. $x^3-4x-2=0$	$\epsilon_1(17, 42, 19)$	$(2, 0; 4, 4; \mid 3, 4; 6, 8; 6, 4;)$
	$\epsilon_2(-15, -2, 4)$	$(0, -4; 0, -2; 0, -2;)$
$\epsilon_1\epsilon_2^2\epsilon_3^3=1$	$\epsilon_3(1, 2, -1)$	$(-1, -1; 0, -1; \mid -1, 3; -1, -1;)$
35. $x^3+4x-3=0$	$(40, 6, 9)$	$(0, 4; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1;)$
36. $x^3-4x-3=0$	reducible.	
37. $x^3-5=0$	$(41, 24, 14)$	$(1, 2; 1, 1; \mid 1, 2; 3, 2; 1, 1; 2, 5; 2, 1;)$
38. $x^3+x-5=0$	$(13, 6, 4)$ $\epsilon'=\epsilon^2$	$(1, 3; 0, 1; \mid 1, 1; 1, 1; 3, 4; 3, 1; 1, 1; 1, 1; 1, 6; 1, 1;)$
	or $\epsilon'=\epsilon$	$NE(2, 4; 2, 2; \mid 0, 2; 4, 11; 4, 2;)$
39. $x^3-x-5=0$	$(3, 2, 1)$ $\epsilon'=\epsilon^2$	$(1, 2; 0, 1; \mid 0, 1; 2, 4; 2, 1; 0, 1; 1, 6; 1, 1;)$
	or $\epsilon'=\epsilon$	$NE(2, 3; 4, 11; \mid 6, 11;)$
40. $x^3+2x-5=0$	$(34, 12, 9)$ $\epsilon'=\epsilon^2$	$(1, 3; 2, 3; \mid 0, 3; 0, 7; 0, 3; 0, 3; 3, 5; 3, 3;)$
41. $x^3-2x-5=0$	$(2, 2, 1)$	$(2, 2; 4, 10; \mid 6, 10;)$
42. $x^3+3x-5=0$	$(4, 1, 1)$	$(1, 4; 2, 6; \mid 3, 6;)$
43. $x^3-3x-5=0$	$(2, 2, 1)$ $\epsilon'=\epsilon^2$	$(2, 1; 4, 3; \mid 2, 3; 6, 9; 6, 3;)$
44. $x^3+4x-5=0$	reducible.	
45. $x^3-4x-5=0$	$(4, 5, 2)$	$(2, 2; 0, 2; \mid 2, 12; 2, 2;)$
46. $x^3+5x-5=0$	$(6, 1, 1)$	$(0, 5; 0, 1; 0, 1;)$
47. $x^3-5x-5=0$	$(1, 1, 0)$ $\epsilon'=\epsilon^3$	$(2, 1; 1, 2; \mid -1, 2; 3, 11; 3, 2;)$
	or $\epsilon'=\epsilon^5$	$(2, 1; 1, 1; \mid 1, 2; 0, 1; 2, 1; 0, 2; 1, 1; 3, 11; 3, 1;)$
48. $x^3+5x-1=0$	$(5, 0, 1)$	$(0, 5;)$
49. $x^3-5x-1=0$	$\epsilon_1(4, 21, 9)$	$(2, 0; 1, 3; \mid 0, 3; 3, 9; 3, 3;)$
	$\epsilon_2(-5, 0, 1)$	$(0, -5;)$
$\epsilon_1\epsilon_2^2\epsilon_3^2=1.$	$\epsilon_3(-1, -2, 1)$	$NE(-2, -1; -4, 7; \mid -6, 7;)$
50. $x^3+5x-2=0$	$(67, 5, 13)$	$(0, 5; 0, 2; 1, 2; 1, 2;)$
51. $x^3-5x-2=0$	reducible.	
52. $x^3+5x-3=0$	$(85, 9, 16)$	$(0, 4; \mid 2, 1; 2, 1; 5, 1; 2, 1; 2, 3;)$
53. $x^3-5x-3=0$	$\epsilon_1(5, 10, 4)$ $\epsilon_1'=\epsilon_1^2$	$(2, 1; 0, 2; \mid 0, 2; 4, 9; 4, 2; 0, 2; 2, 11; 2, 2;)$
	$\epsilon_2(-4, -1, 1)$	$NE(-1, -4; -2, -2; \mid -3, -2;)$
$\epsilon_1\epsilon_2^2\epsilon_3=1.$	$\epsilon_3(-1, -2, 1)$	$(-2, -1; -4, 7; \mid -6, 7;)$
54. $x^3+5x-4=0$	$(801, 105, 145)$	$(0, 5; 0, 1; 0, 1; 0, 1; 1, 1; 1, 1; 5, 1; 1, 1; 1, 1; 0, 1; 0, 1;)$
55. $x^3-5x-4=0$	reducible.	

56. $x^3-6=0$	(109, 60, 33)	(1, 3; 0, 1; | 0, 1; 2, 1; 0, 1; 0, 1; 0, 1; 2, 1; 0, 1; 1, 7; 1, 1;)
57. $x^3+x-6=0$	(11, 5, 3)	(1, 2; 2, 1; | 0, 1; 0, 1; 0, 1; 3, 4; 3, 1;)
	or	NE(2, 4; 1, 3; | 3, 11; 3, 3;)
58. $x^3-x-6=0$	reducible.	
59. $x^3+2x-6=0$	(1129, 399, 274)	(1, 4; 0, 2; | 0, 2; 2, 1; 0, 1; 1, 1; 0, 1; 2, 2; 0, 2; 1, 11; 1, 2;)
60. $x^3-2x-6=0$	(77, 61, 28)	(2, 2; 4, 5; | 3, 5; 6, 10; 6, 5;)
61. $x^3+3x-6=0$	(11521, 3185, 2473)	(1, 4; 2, 3; | 1, 1; 3, 1; 0, 1; 0, 1; (1), (2); (1), · · ·)
62. $x^3-3x-6=0$	(79, 73, 31)	(2, 2; 1, 2; | 1, 1; 1, 1; 1, 1; 1, 2; 3, 11; 3, 2;)
63. $x^3+4x-6=0$	(5, 1, 1)	(1, 5; 2, 7; | 3, 7;)
64. $x^3-4x-6=0$	(95, 101, 40)	(2, 2; 0, 1; | 1, 1; 1, 1; 5, 1; 1, 1; 1, 1; 2, 12; 2, 1;)
65. $x^3+5x-6=0$	reducible.	
66. $x^3-5x-6=0$	(29, 35, 13)	(2, 2; 0, 1; | 1, 2; 2, 2; 1, 1; 2, 13; 2, 1;)
67. $x^3+6x-6=0$	(7, 1, 1)	(0, 6; 0, 1; 0, 1;)
	or	NE(1, 7; 2, 9; | 3, 9;)
68. $x^3-6x-6=0$	(1, 1, 0) $\epsilon'=\epsilon^5$	(2, 1; 1, 1; | 0, 1; 3, 1; 0, 1; 0, 1; 0, 1; 3, 1; 0, 1; 3, 12; 3, 1;)
69. $x^3+6x-1=0$	(6, 0, 1)	(0, 6;)
70. $x^3-6x-1=0$	ϵ_1(2974, 19020, 7521)	(2, 0; 0, 1; | 1, 1; 1, 4; 1, 1; 0, 1; 1, 1; (3), (2); (3), · · ·)
	ϵ_2(—6, 0, 1)	(0, —6;)
$\epsilon_1\epsilon_2^9\epsilon_3^2=1.$	ϵ_3(—14, —78, 33)	(—2, 0; 1, —2; | —4, 4; —4, —2; —1, 10; —1, —2;)
71. $x^3+6x-2=0$	(55, 3, 9)	(0, 6; 0, 3; 0, 3;)
72. $x^3-6x-2=0$	ϵ_1(13, 44, 17)	(2, 0; 1, 1; | 2, 1; 6, 1; 2, 1; 3, 10; 3, 1;)
	ϵ_2(—53, —3, 9)	(0, —6; 0, —3; 0, —3;)
$\epsilon_1\epsilon_2^2\epsilon_3=1.$	ϵ_3(—7, —18, 8)	(—2, —1; 0, —2; | 4, —2; —2, 8; —2, —2;)
73. $x^3+6x-3=0$	(25, 2, 4)	(0, 6; 0, 2; 0, 2;)
74. $x^3-6x-3=0$	ϵ_1(10, 24, 9)	(2, 0; 1, 1; | 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 3, 10; 3, 1;)
	ϵ_2(—23, —2, 4)	(0, —6; 0, —2; 0, —2;)
$\epsilon_1\epsilon_2\epsilon_3^2=1.$	ϵ_3(—2, —2, 1)	NE(—2, —2; —4, 6; | —6, 6;)
75. $x^3+6x-4=0$	(51, 5, 8)	(0, 5; | 2, 1; 2, 3; 2, 1; 2, 4;)
76. $x^3-6x-4=0$	reducible.	
77. $x^3+6x-5=0$	(8117, 938, 1231)	(0, 6; 0, 1; 0, 1; 0, 3; [illegible] 1, 1; 1, [illegible]; 1, [illegible] [illegible])
78. $x^3-6x-5=0$	reducible.	
79. [illegible] $=0$	(1, 2, 1) ϵ' [illegible] ϵ''	(1, 3; 0, 1; | 0, 1; 2, 5; 2, 1; 0, 1; 1, 7; 1, 1;)

80. $x^3+x-7=0$	$(16, 7, 4)$	$\epsilon'=\epsilon^2$	$(1, 3; 1, 1; \mid 0, 2; 0, 1; 0, 1; 3, 4; 3, 1; 0, 1; 0, 2; 0, 1; 2, 6; 2, 1;)$
		or $\epsilon'=\epsilon$	$NE(2, 4; 0, 4; \mid 2, 11; 2, 4;)$
81. $x^3-x-7=0$	$(3, 2, 1)$		$(2, 3; 4, 11; \mid 6, 11;)$
82. $x^3+2x-7=0$	$(290, 102, 95)$		$(1, 4; 0, 1; \mid 0, 1; 0, 1; 0, 3; 3, 3; 0, 1; 0, 1; 0, 1; 1, 7; 1, 1;)$
83. $x^3-2x-7=0$	$(372, 271, 120)$	$\epsilon'=\epsilon^2$	$(2, 3; 0, 3; \mid 2, 2; 2, 3; 3, 3; (6), (10); (6), \cdots)$
84. $x^3+3x-7=0$	$(25, 7, 5)$		$(1, 4; 2, 2; \mid 1, 2; 3, 6; 3, 2;)$
85. $x^3-3x-7=0$	$(6, 5, 2)$		$(2, 2; 2, 2; \mid 4, 11; 4, 2;)$
86. $x^3+4x-7=0$	$(1026, 231, 184)$	$\epsilon'=\epsilon^2$	$(1, 4; 6, 3; \mid 5, 5; 2, 2; 2, 3; -5, 11; -5, 3; 2, 2; 2, 5; 5, 3; 7, 5; 7, 3;)$
87. $x^3-4x-7=0$	$(46, 44, 17)$	$\epsilon'=\epsilon_2$	$(2, 2; 1, 1; \mid 3, 3; 3, 1; 1, 1; (6), (8); (6), \cdots)$
		or $\epsilon'=\epsilon$	$(2, 3; -1, 1; \mid 1, 1; 2, 1; 2, 1; 1, 1; 1, 4; 1, 1;)$
88. $x^3+5x-7=0$	$(6, 1, 1)$		$(1, 6; 2, 8; \mid 3, 8;)$
89. $x^3-5x-7=0$	$(3, 3, 1)$		$(2, 1; 2, 1; \mid 4, 11; 4, 1;)$
90. $x^3+6x-7=0$	reducible.		
91. $x^3-6x-7=0$	$(5, 6, 2)$		$(2, 2; 0, 2; \mid -2, 2; 2, 14; 2, 2;)$
92. $x^3+7x-7=0$	$(8, 1, 1)$		$(0, 7; 0, 1; 0, 1;)$
93. $x^3-7x-7=0$	$\epsilon_1(2, 3, 1)$		$(3, 2; 6, 20; \mid 9, 20;)$
	$\epsilon_2(-6, -1, 1)$		$(-1, -6; -2, -4; \mid -3, -4;)$
$\epsilon_1\epsilon_2\epsilon_3=1.$	$\epsilon_3(-3, -2, 1)$		$(-2, -3; -4, -5; \mid -6, 5;)$
94. $x^3+7x-1=0$	$(7, 0, 1)$		$(0, 7;)$
95. $x^3-7x-1=0$	$\epsilon_1(18, 133, 49)$		$(2, 0; 0, 1; \mid 0, 1; 0, 1; 1, 4; 1, 1; 1, 1; 0, 1; 2, 11; 2, 1;)$
	$\epsilon_2(-7, 0, 1)$		$(0, -7;)$
$\epsilon_1\epsilon_2{}^5\epsilon_3{}^2=1.$	$\epsilon_3(-2, -13, 5)$		$(-2, 0; 1, -3; \mid -4, -3; -1, 11; -1, -3;)$
96. $x^3+7x-2=0$	$(137865, 5501, 19473)$		$(0, 5; 7, 3; \mid 1, 2; 1, 1; 1, 1; 0, 1; 0, 1; 0, (1); (9), (1); \cdots)$
97. $x^3-7x-2=0$	$\epsilon_1(13, 50, 18)$		$(2, 0; 0, 1; \mid 0, 1; 0, 3; 0, 3; 0, 1; 0, 1; 2, 11; 2, 1;)$
	$\epsilon_2(97, 4, -14)$		$(0, -8; \mid -3, -3; 1, -1; 1, -3; -3, -9;)$
$\epsilon_1\epsilon_2\epsilon_3=1.$	$\epsilon_3(-3, -10, 4)$	$\epsilon_3'=\epsilon_3{}^2$	$(-2, -1; 0, -2; \mid 0, -2; -4, 7; -4, -2; 0, -2; -2, 9; -2, -2;)$
98. $x^3+7x-3=0$	$(86, 5, 12)$		$(0, 7; 0, 2; 0, 1; 1, 1; 0, 2;)$
99. $x^3-7x-3=0$	$\epsilon_1(1, 3, 1)$		$(2, 1; 0, 1; \mid 2, 13; 2, 1;)$
	$\epsilon_2(14, 1, -2)$		$(0, -6; \mid 2, -2; 2, -5;)$
$\epsilon_1\epsilon_2{}^2\epsilon_3=1$	$\epsilon_3(25, 48, -20)$		$(-2, -1; 1, -3; \mid -1, -3; -1, -3; -1, 9; -1, -3;)$

100. $x^3+7x-4=0$	(42115, 3161, 5769)	(0, 6; | 2, 1; 2, 2; 3, 4; 3, 2; 7, 2; 3, 4; 3, 2; 2, 1; 2, 5;)
101. $x^3-7x-4=0$	ϵ_1(105, 220, 76)	(2, 1; 0, 1; | 0, 2; 1, 3; 2, 3; 1, 2; 0, 1; 2, 13; 2, 1;)
	ϵ_2(33, 3, —5) $\epsilon'_2=\epsilon_2^2$	(0, —7; | —1, —2; 1, 3; 3, —8; 3, 3; 1, —2; —1, 7;)
$\epsilon_1\epsilon_2^2\epsilon_3^3=1$	ϵ_3(5, 7, —3)	(—2, —2; —1, —3; | —3, 7; —3, —3;)
102. $x^3+7x-5=0$	(67, 6, 9)	(0, 7; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1;)
103. $x^3-7x-5=0$	ϵ_1(2, 3, 1)	(2, 0; 1, 1; | 0, 1; 3, 11; 3, 1;)
	or	*NE*(3, 2; 6, 20; | 9, 20;)
	ϵ_2(—6, —1, 1)	*NE*(—1, —6; —2, —4; | —3, —4;)
$\epsilon_1\epsilon_2\epsilon_3=1$	ϵ_3(—3, —2, 1)	*NE*(—2, —3; —4, 5; | —6, 5;)
104. $x^3+7x-6=0$	(2719205, 280985, 356851)	(0, 7; 0, 1; 0, 1; 0, 2; 1, 1; 1, 3; 4, 1; 0, 1; 0, 1; 1, 2; (3), (5); (3), $\cdots$)
105. $x^3-7x-6=0$	reducible.	
106. $x^3-8=0$	reducible.	
107. $x^3+x-8=0$	(157, 66, 36)	(1, 3; 1, 1; | 0, 1; 1, 3; 0, 1; 0, 3; 1, 1; 0, 1; 2, 6; 2, 1;)
108. $x^3-x-8=0$	(133, 78, 36) $\epsilon'=\epsilon^2$	(2, 3; 4, 6; | 0, 6; 0, 13; 0, 6; 6, 11; 6, 6;)
109. $x^3+2x-8=0$	(43, 15, 9)	(1, 4; 1, 1; | 2, 4; 2, 1; 2, 7; 2, 1;)
110. $x^3-2x-8=0$	(31, 21, 9)	(2, 3; 1, 3; | 0, 3; 3, 12; 3, 3;)
111. $x^3+3x-8=0$	(21, 6, 4) $\epsilon'=\epsilon^2$	(1, 5; 0, 1; | 1, 1; 1, 1; (3), (6); (3), $\cdots$)
112. $x^3-3x-8=0$	(13, 10, 4) $\epsilon'=\epsilon^2$	(2, 3; 0, 2; | 0, 2; 4, 11; 4, 2; 0, 2; 2, 13; 2, 2;)
113. $x^3+4x-8=0$	(129, 30, 22)	(1, 5; 2, 3; | —1, 3; —1, 3; 3, 7; 3, 3;)
	or	(1, 2; 10, 1; | 2, 1; 0, 1; 0, 1; 0, 1; 2, 1; 11, 1; 11, 1;)
114. $x^3-4x-8=0$	(9, 8, 3)	(2, 2; 1, 1; | 0, 1; 0, 1; 0, 1; 3, 12; 3, 1;)
115. $x^3+5x-8=0$	(30949, 5842, 4754) $\epsilon'=\epsilon^2$	(1, 6; 2, 4; | 1, 3; 2, 1; 2, 3; 1, 1; 0, 1; 0, 1; 3, 2; 1, 4; (3), (8), (3), $\cdots$)
116. $x^3-5x-8=0$	(3, 3, 1)	(2, 2; 1, 1; | 3, 13; 3, 1;)
117. $x^3+6x-8=0$	(7, 1, 1)	(1, 7; 2, 9; | 3, 9;)
118. $x^3-6x-8=0$	(3, 3, 1)	(2, 1; 1, 1; | 0, 1; 3, 12; 3, 1;)
119. $x^3+7x-8=0$	reducible.	
120. $x^3-7x-8=0$	(5, [illegible], 2) $\epsilon'=\epsilon^2$	(2, 2; 0, 3; | 20, 5; 0, 20; 3, 5;)
121. $x^3+8x-8=0$	(9, 1, 1)	(0, 8; 0, 1; 0, 1;)
122. $x^3-8x-8=0$	reducible.	

123. $x^3+8x-1=0$	$(8, 0, 1)$	$(0, 8;)$
124. $x^3-8x-1=0$	$\epsilon_1(81, 676, 234)$	$(2, 0; 0, 1; \mid 0, 2; 1, 2; 1, 1; 0, 2; 0, 1; 1, 2; 1, 2; 0, 1; 2, 12; 2, 1;)$
	$\epsilon_2(-8, 0, 1)$	$(0, -8;)$
$\epsilon_1\epsilon_2^4\epsilon_3=1.$	$\epsilon_3(-26, -199, 72)$	$(-3, 0; -2, 4; \mid 0, 2; 0, 2; 0, 4; -5, 17; -5, 4)$
125. $x^3+8x-2=0$	$(129, 4, 16)$	$(0, 8; 0, 4; 0, 4;)$
126. $x^3-8x-2=0$	$\epsilon_1(1, 3, 1)$ $\epsilon_1'=\epsilon_1^2$	$(2, 0; 0, 1; \mid 0, 1; 4, 8; 4, 1; 0, 1; 2, 12; 2, 1;)$
	$\epsilon_2(-127, -4, 16)$	$(0, -8; 0, -4; 0, -4;)$
$\epsilon_1^3\epsilon_2^2\epsilon_3=1.$	$\epsilon_3(-951, -3451, 1281)$	$(-3, -1; 0, 2; \mid 1, 1; 0, 1; 2, 1; 2, 1; 0, 1; 0, 1; 0, 1; 2, 1; 2, 1; 0, 1; 1, 2; -3, 15; -3, 2;)$
127. $x^3+8x-3=0$	$(2764324, 125283, 339766)$	$NE(0, 8; 0, -2; -1, -2; 0, 1; 0, -1; -3, -10; 2, (5); (-5), (5); \ldots)$
128. $x^3-8x-3=0$	reducible.	
129. $x^3+8x-4=0$	$(33, 2, 4)$	$(0, 8; 0, 2; 0, 2;)$
130. $x^3-8x-4=0$	$\epsilon_1(1, 3, 1)$	$(3, 1; 6, 19; \mid 9, 19;)$
	$\epsilon_2(31, -2, 4)$	$(0, -8; 0, -2; 0, -2;)$
$\epsilon_1\epsilon_2\epsilon_3=1.$	$\epsilon_3(-3, -5, 2)$	$(-2, -2; 0, -1; \mid 1, -1; -2, 8; -2, -1;)$
131. $x^3+8x-5=0$	$(4764, 341, 570)$ $\epsilon'=\epsilon^2$	$(0, 7; 2, 1; \mid 2, 3; 1, 1; 2, 2; 0, 2; 1, 1; 2, 5; 2, (2); (3), (2); \cdots)$
132. $x^3-8x-5=0$	$\epsilon_1(776874, 1493568, 481729)$	$(3, 2; -4, 9; \mid 6, 6; 1, 1; 1, 2; 2, 9; 2, 1; 1, 1; 1, 9; 9, 9; (9), (19); (9), \cdots)$
	$\epsilon_2(-68, -6, 9)$	$(0, -8; \mid -1, -2; 2, 4; 2, -2; -1, -8;)$
	$\epsilon_3(4, 5, -2)$	$(-2, -2; 0, -2; \mid -2, 8; -2, -2;)$
133. $x^3+8x-6=0$	$(289, 24, 34)$ $\epsilon'=\epsilon^2$	$(0, 7; \mid 2, 1; 3, 7; 2, 2; 0, 1; 0, 1; (4), (2); (4), \cdots)$
134. $x^3-8x-6=0$	$\epsilon_1(427, 705, 224)$	$(4, 2; 0, -2; \mid 7, 9; 0, 2; 0, 9; 7, -2; 4, 28; 4, -2;)$
	$\epsilon_2(-7, -1, 1)$	$NE(-1, -7; -2, -5; \mid -3, -5;)$
$\epsilon_1\epsilon_3^3\epsilon_3=1$	$\epsilon_3(-19, -16, 7)$	$(-2, -4; -4, -2; \mid 3, -2; -6, 4; -6, -2;)$
135. $x^3+8x-7=0$	$(1177, 110, 136)$	$(0, 8; 0, 1; 0, 1; 0, 3; 0, 1; 0, 2; -1, 2; 0, 1; 0, 3; 0, 1; 0, 1;)$
136. $x^3-8x-7=0$	reducible.	
137. $x^3-9=0$	$(4, 2, 1)$	$(2, 4; 4, 12; \mid 6, 12;)$

138. $x^3+x-9=0$	(5, 2, 1)	$\epsilon'=\epsilon^2$	(1, 4; 0, 1; | 0, 1; 2, 6; 2, 1; 0, 1; 1, 8; 1, 1;)
		$\epsilon'=\epsilon$	*NE*(2, 5; 4, 13; | 6, 13;)
139. $x^3-x-9=0$	(16, 9, 4)		(2, 4; 0, 4; | 2, 13; 2, 4;)
140. $x^3+2x-9=0$	(817, 282, 160)		(1, 4; 1, 1; | 0, 2; 0, 1; 1, 2; 2, 1; 1, 1; 0, 1; 0, 2; 0, 1; 0, 1; (3), (5); (3), · · ·)
141. $x^3-2x-9=0$	(9550, 6104, 2545)		(2, 3; 1, 2; | 0, 1; 0, 1; 3, 4; 1, 2; 1, 2; 1, 4; 3, 1; 0, 1; 0, 1; 3, 15; 3, 2;)
142. $x^3+3x-9=0$	(28, 8, 5)		(1, 6; —1, 1; | 1, 2; 1, 1; 0, 10; 0, 1;)
143. $x^3-3x-9=0$	(7, 5, 2)		(2, 3; 0, 1; | 0, 1; 0, 1; 2, 13; 2, 1;)
144. $x^3+4x-9=0$	(172, 41, 28)		(1, 6; 0, 2; | 0, 2; 2, 2; 0, 2; 1, 9; 1, 2;)
145. $x^3-4x-9=0$	(113, 92, 34)		(2, 3; 0, 1; | 0, 1; 1, 1; 0, 3; 0, 1; 1, 1; 0, 1; 2, 14; 2, 1;)
146. $x^3+5x-9=0$	(61, 12, 9)	$\epsilon'=\epsilon^2$	(1, 6; 2, 3; | 0, 3; 0, 10; 0, 3; 0, 3; 3, 8; 3, 3;)
147. $x^2-5x-9=0$	(22, 20, 7)		(2, 2; 1, 1; | 0, 2; 1, 2; 0, 1; 3, 13; 3, 1;)
148. $x^3+6x-9=0$	(865, 140, 116)	$\epsilon'=\epsilon^2$	(1, 7; 2, 3; | 8, 4; 0, 1; 0, 1; 0, 4; (12), (5); (12), · · ·)
149. $x^3-6x-9=0$	reducible.		
150. $x^3+7x-9=0$	(8, 1, 1)		(1, 8; 2, 10; | 3, 10;)
151. $x^3-7x-9=0$	(20, 22, 7)		(3, 3; —1, 7; | 2, 22; 2, 7;)
152. $x^3+8x-9=0$	reducible.		
153. $x^3-8x-9=0$	(2, 1, 0)	$\epsilon'=\epsilon^5$	(3, 2; 2, 2; | 2, 1; 1, 2; 2, 2; 1, 1; 2, 2; 5, 21; 5, 2;)
154. $x^3+9x-9=0$	(10, 1, 1)		(0, 9; 0, 1; 0, 1;)
155. $x^3-9x-9=0$	ϵ_1(13, 17, 5)	$\epsilon'_1=\epsilon_1^2$	(3, 2; 1, 2; | 0, 1; 1, 1; (6), (20); (6), · · ·)
		or $\epsilon'_1=\epsilon_1$	(3, 3; —2, 2; | 0, 1; 0, 2; 1, 24; 1, 2;)
	ϵ_2(—8, —1, 1)		(—1, —8; —2, —6; | —3, —6;)
$\epsilon_1\epsilon_2^2\epsilon_3=1$	ϵ_3(16, 9, —4)		(—2, —4; 0, —4; | —2, 5; —2, —4;)
156. $x^3+9x-1=0$	(9, 0, 1)		(0, 9;)
157. $x^3-9x-1=0$	ϵ_1(0, 3, 1)		(3, 0; 6, 18; | 9, 18;)
	ϵ_2(—9, 0, 1)		(0, —9;)
	ϵ_3(0, —3, 1)		(—3, 0; —6, 18; | —9, 18;)
158. $x^3+9x-2=0$	(1475017, 36028, 163006)		(0, 9; 0, 4; 2, 4; 1, 2; 1, 2; 0, 1; 0, 1; 0, 1; (0), (1); (0), · · ·)
159. $x^3-9x-2=0$	ϵ_1(12479573, 60177508, 19377828)		
	ϵ_2(161, [illegible], 18)	$\epsilon'=$ [illegible]	(0, [illegible] 8; 1, [illegible] 1; | [illegible] 2, 4; 0, 8, —1, —4; 2, [illegible] 4; (0), ([illegible] 9); (0), · · ·)
	ϵ_3(—59, —245, 85)		(—3, [illegible] 1; 2, 6; —4, 1; 4, 6; —1, 16; 1, 6;)

160. $x^3+9x-3=0$	$(82, 3, 9)$	$(0, 9; 0, 3; 0, 3;)$
161. $x^3-9x-3=0$	$\epsilon_1(37, 123, 39)$	$(3, 0; 6, 6; \mid 3, 6; 9, 18; 9, 6;)$
	$\epsilon_2(-80, -3, 9)$	$(0, -9; 0, -3; 0, -3)$
$\epsilon_1\epsilon_2^2\epsilon_3^3=1.$	$\epsilon_3(1, 3, -1)$	$(-2, 1; 0, -1; \mid -2, 11; -2, -1;)$
162. $x^3+9x-4=0$	$(57387437, 2718291, 6244913)$	$(0, 8; \mid 2, 2; 0, 1; 0, 2; 0, 1; 0, 1; 4, 4; 1, 2; 1, 1; 0, 2; 3, (3); (0), (3); \cdots)$
163. $x^3-9x-4=0$	$\epsilon_1(25, 64, 20)$	$(3, 1; 1, 4; \mid 4, 4; 4, 20; 4, 4;)$
	$\epsilon_2(-4291, -222, 488)$	$(-1, -9; 0, 1; \mid 2, 2; 3, 4; 5, 4; 3, 2; 2, 1; -1, 8; -1, 1;)$
	$\epsilon_3(115, 217, -79)$	$NE(-3, -1; -2, -4; \mid 0, -5; 0, -4; -5, 16; -5, -4;)$
164. $x^3+9x-5=0$	$(6317, 366, 680)$	$(0, 9; 0, 1; 1, 1; 1, 1; 1, 2; 0, 1; (1), (1); (1), \cdots)$
165. $x^3-9x-5=0$	$\epsilon_1(31, 65, 20)$	$(3, 1; 2, 5; \mid -5, 5; 5, 20; 5, 5;)$
	$\epsilon_2(-43, -3, 5)$	$(-1, -8; -2, 2; \mid 1, 2; -3, -6; -3, 2;)$
$\epsilon_1\epsilon_2\epsilon_3=1.$	$\epsilon_3(-17, -24, 9)$ $\epsilon'_3=\epsilon_3^2$	$(-3, -2; 0, 3; \mid 0, 3; -6, 16; -6, 3; 0, 3; -3, 14; -3, 3;)$
166. $x^3+9x-6=0$	$(3489013, 236574, 370902)$	$(0, 9; 0, 1; 0, 1; 0, 1; 0, 1; 2, 4; 1, 1; 0, 3; 3, 2; 2, 1; 0, (1); (0), (1); \cdots)$
167. $x^3-9x-6=0$	$\epsilon_1(1, 3, 1)$	
	$\epsilon_2(289, 24, -34)$ $\epsilon'_2=\epsilon_2^2$	$(-1, -8; -1, 3; \mid -1, -2; 1, 2; 1, 3; 0, -8; 0, 3; 1, 2; 1, -2; 1, 3; -2, -6; -2, 3;)$
$\epsilon_1^2\epsilon_2\epsilon_3=1.$	$\epsilon_3(469, 522, -202)$ $\epsilon'_3=\epsilon_3^2$	$NE(-2, -2; 1, 2; \mid 1, 2; 0, 2; -1, 2; 1, 3; 1, -2; 1, -2; (1), (9); (1), \cdots)$
168. $x^3+9x-7=0$	$(143, 11, 15)$	$(0, 8; \mid 2, 1; 3, 8; 3, 1; 2, 7;)$
169. $x^3-9x-7=0$	$\epsilon_1(19, 30, 9)$ $\epsilon'=\epsilon^2$	$(3, 2; 0, 3; \mid 0, 3; 6, 20; 6, 3; 0, 3; 3, 22; 3, 3;)$
	$\epsilon_2(-8, -1, 1)$	$NE(-1, -8; -2, -6; \mid -3, -6;)$
$\epsilon_1\epsilon_2^3\epsilon_3=1$	$\epsilon_3(-17, -15, 6)$	$(-3, -3; 0, 3; \mid -3, 3; -3, 12; -3, 3;)$
170. $x^3+9x-8=0$	$(308121, 26292, 31822)$	$(0, 9; 0, 1; 0, 1; 0, 3; 1, 1; 1, 8; 0, 1; (2), (5); (2), \cdots)$
171. $x^3-9x-8=0$	reducible.	
172. $x^3-10=0$	$(181, 84, 39)$	$(2, 4; 4, 6; \mid 3, 6; 6, 12; 6, 6;)$

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Triads of Plane Curves.*

BY A. R. JERBERT.

1. *Introduction.* Papers have recently appeared by Professor E. P. Lane and A. F. Carpenter on ruled surfaces with generators in correspondence.† The following paper considers a somewhat similar problem, namely that of three plane curves with points in correspondence.

If the correspondence is such that the corresponding points are not collinear, we find that a projective theory of the configuration can be based upon a system of three ordinary linear first order differential equations.

In investigating the correspondence between the given curves it proves advantageous to introduce two associated triads of curves. They are the curves enveloped by the sides of the triangle formed by joining corresponding points of the given curves and the curves generated by the points in which the tangents to the given curves meet the opposite sides of this same triangle. We have called these curves the contact curves and tangential curves respectively.

A special case consists in showing that if the coefficients in the defining system of differential equations are constants the curves defined by the system, together with the associated contact and tangential curves, constitute a set of nine equiangular, logarithmic spirals.

2. *The Differential Equations.* Let the three homogeneous coördinates of an arbitrary point P_x on a curve C_x be given as analytic functions of an independent variable t. And in like manner let the three coördinates of each of two other points P_y and P_z be given as functions of the same variable t. Then, as t varies, the points P_x, P_y, P_z, describe three plane curves whose points are in correspondence. The object of this paper is to discover some of the implications of this correspondence.

We shall suppose that the points P_x, P_y, P_z, are not collinear. Then the determinant

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* Presented to the Society, San Francisco Section, June 18, 1927.

† E. P. Lane, "Ruled Surfaces with Generators in One to One Correspondence," *Transactions of the American Mathematical Society*, Vol. 25 (1923); A. F. Carpenter, "Triads of Ruled Surfaces," *Transactions of the American Mathematical Society*, Vol. 29 (April 1927).

$$\text{(D)} \qquad \Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

is different from zero, and it is possible to determine the coefficients in the system of differential equations

$$\text{(1)} \qquad \begin{aligned} x' &= a_1x + b_1y + c_1z, \\ y' &= a_2x + b_2y + c_2z, \\ z' &= a_3x + b_3y + c_3z, \end{aligned}$$

so that (x_i, y_i, z_i) $(i = 1, 2, 3)$ will be three sets of solutions. For example, we may substitute each of these three sets in turn in the first of system (1) and then solve the resulting three equations for the coefficients of the first of equations (1). Similarly we may determine the coefficients of each of the other three equations.

3. *A Canonical System.* If we have constructed a defining system (1) based on three points $(x) = (x_1, x_2, x_3)$, $(y) = (y_1, y_2, y_3)$, $(z) = (z_1, z_2, z_3)$, the question arises—what will be the effect on the coefficients in (1) if we multiply the x's by an arbitrary function $\lambda(t)$ the y's by $\mu(t)$, and the z's by $\nu(t)$? The factors λ, μ, ν, will not affect the points (x), (y), (z), since we are dealing with homogeneous coördinates, but they will affect the coefficients in the resulting system (1).

Accordingly, in (1) we shall put

$$\text{(2)} \qquad x = \lambda\bar{x}, \quad y = \mu\bar{y}, \quad z = \nu\bar{z}.$$

System (1) is transformed by (2) into

$$\text{(3)} \qquad \begin{aligned} \bar{x}' &= (a_1 - \lambda'/\lambda)\bar{x} + (b_1\mu/\lambda)\bar{y} + (c_1\nu/\lambda)\bar{z}, \\ \bar{y}' &= (a_2\lambda/\mu)\bar{x} + (b_2 - \mu'/\mu)\bar{y} + (c_2\nu/\mu)\bar{z}, \\ \bar{z}' &= (a_3\lambda/\nu)\bar{x} + (b_2\mu/\nu)\bar{y} + (c_3 - \nu'/\nu)\bar{z}. \end{aligned}$$

If we place

$$\lambda = k_1 \exp \int a_1dt, \quad \mu = k_2 \exp \int b_2dt, \quad \nu = k_3 \exp \int c_3dt,$$

then system (3) takes the canonical form

$$\text{(4)} \qquad \bar{x}' = b_1\bar{y} + \bar{c}_1\bar{z}, \quad \bar{y}' = \bar{a}_2\bar{x} + \bar{c}_2\bar{z}, \quad \bar{z}' = \bar{a}_3\bar{x} + b_3\bar{y}$$

where

$$\text{(5)} \qquad \begin{aligned} \bar{b}_1 &= (k_2/k_1)\, b_1 \exp \int (b_2 - a_1)\,dt, & \bar{c}_1 &= (k_3/k_1) c_1 \exp \int (c_3 - a_1)\,dt, \\ \bar{a}_2 &= (k_1/k_2)\, a_2 \exp \int (a_1 - b_2)\,dt, & \bar{c}_2 &= (k_3/k_2) c_2 \exp \int (c_3 - b_2)\,dt, \\ \bar{a}_3 &= (k_1/k_3) a_3 \exp \int (a_1 - c_3)\,dt, & \bar{b}_3 &= (k_2/k_1) b_3 \exp \int (b_2 - c_3)\,dt, \end{aligned}$$

[illegible] canonical form (4) is unique except for the [illegible]

[illegible] λ [illegible] λ, μ, ν, [illegible] λ [illegible] geometrically intuitive if we consider that a λ transformation [illegible] (x′), (y′), (z′) [illegible] λ [illegible] $-\lambda x'$. [illegible] reduces to zero if λ = const.

[illegible] meaning of the canonical form (4), expressed geometrically, is that [illegible] points corresponding to the vertices of the triangle (x), (y), (z) lie respectively in the opposite sides (yz), (zx), (xy).

It is to be noted that in the uncanonical form (1), the derivative points (x'), (y'), (z') were devoid of geometrical significance aside from the fact that they, together with the points (x), (y), and (z), respectively determined the tangent lines to the curves C_x, C_y, and C_z. In the canonical system (4) the derivative points are unique points on the tangents, namely, they are the points where the tangents to C_x, C_y, and C_z meet the opposite sides of the triangle (x), (y), (z).

In the uncanonical system (1) these points are given by the expressions

$$b_1y + c_1z,\quad a_2x + c_2z,\quad a_3x + b_3y. \tag{6}$$

The covariant character of the points given by (6) may readily be verified by means of equations (2) and (3). We shall call them the tangential points, and the curves generated by them, the tangential curves.

4. *Complete System of Invariants.* The complete system of invariants for the defining system (1) is readily computed by the well known Lie method of continuous transformations. The transformations, in this case, are the λ, μ, ν, transformations already mentioned and utilized in obtaining the canonical form (4).

The transformation on the parameter t is of no interest in this connection because under it the coefficients of the defining system (1) are merely multiplied by a factor. As a result, the functions of the coefficients which are invariant under the λ, μ, ν, transformations are relative invariants under [illegible]

The difference and the sum of the last two will prove to be of some importance. They are

$$\Delta_1 = H - G = a_2b_3c_1 - a_3b_1c_2,$$
$$\Delta_2 = H + G = a_2b_3c_1 + a_3b_1c_2. \tag{8}$$

If we consider invariants involving derivatives we find that there are 10 which are independent. Eight of these are accounted for by the original four together with the derivatives of these four. There remain two invariants to be found. For the latter we can take any two of the following three,

$$D_1 = a_1 - b_2 + (c_2/c_1)'(c_1/c_2),$$
$$D_2 = b_2 - c_3 + (a_3/a_2)'(a_2/a_3), \tag{9}$$
$$D_3 = c_3 - a_1 + (b_1/b_3)'(b_3/b_1).$$

Between the invariants (9) and those given in (7), there exists the following relation,

$$G/H = \exp \int (D_1 + D_2 + D_3)\,dt.$$

At this stage our system of invariants is complete in the sense that all other invariants of whatever order in the derivatives, may be obtained from the ones already gotten, by algebraic processes and by differentiation.

With respect to the covariants the discussion is simplified by the fact that there is no occasion to search for covariants involving derivatives of the variables, the latter being expressible in terms of the variables themselves by virtue of the defining system of differential equations.

A complete system of covariants is obtained by adjoining to the invariants already noted, any two of the following six,

$$a_2x/y,\ b_1y/x,\ a_3x/z,\ c_1z/x,\ b_3y/z,\ c_2z/y.$$

It is to be noted that the coefficients in the defining system (1) are relative invariants whose vanishing, in each case, has an obvious geometrical significance.

5. *Dual Considerations.* In proceeding with the Geometry of the triple of curves, we shall first find the adjoint system corresponding to the system (4). To this end, denote the line y, z as (u). The line coördinates of (u) i. e. (u_1, u_2, u_3), can be taken as any three numbers proportional to the cofactors of the x's in the determinant Δ (equation D). We shall take the cofactors themselves, thus making a choice of proportionality factor. In similar fashion the lines z, x and x, y are denoted by (v) and (w) respectively with the (v_1, v_2, v_3) and the (w_1, w_2, w_3) given by the cofactors of the y's and z's respectively, in Δ.

We find by direct computation that the adjoint system is

$$(10)\qquad \begin{aligned} u' &= \qquad\quad -a_2v - a_3w,\\ v' &= -b_1u \qquad\quad -a_3w,\\ w' &= -c_1u - c_2v \qquad\quad . \end{aligned}$$

The adjoint system corresponding to the uncanonical defining system (1) is the same as (10) with the diagonal terms involving a_1, b_2, c_3, restored.

In interpreting the adjoint system (10) it is important to keep in mind the fact that just as the join of x and x' is the tangent at (x), to the curve C_x traced by the point (x), so the intersection of (u) and (u') is the contact point on (u) i. e. the point where (u) touches its envelope curve. In the same manner $(v') = (v'_1, v'_2, v'_3)$, is a line through the contact point on (v) while (w') is a line through the contact point on (w). From the form of system (27) we see that the lines (u'), (v'), (w') pass respectively through the opposite vertices (x), (y), and (z). We shall refer to (u'), (v'), (w'), as the "contact lines."

Incidentally, by finding the intersections of (u), (u'); (v), (v'); (w), (w'); we learn that the contact points are given by the expressions

$$(11)\qquad \begin{aligned} \tau_x &= \qquad\quad a_3y - a_2z,\\ \tau_y &= -b_3x \qquad\quad + b_1z,\\ \tau_z &= c_2x - c_1y \qquad\quad . \end{aligned}$$

6. *Theorems involving Tangentials and Contact Points.* We shall denote by $_h\tau_x$ the harmonic conjugate of the contact point τ_x with respect to the points (y) and (z). In the same manner $_h\tau_y$ and $_h\tau_z$ denote the harmonic conjugates of the other two contact points with respect to z, x and x, y respectively.

We obtain the expression for $_h\tau_y$ from the expression for τ_x in (11) by changing the sign of the coefficient of z. A similar change of sign converts τ_y and τ_z into $_h\tau_y$ and $_h\tau_z$ respectively. We find accordingly,

$$(12)\qquad \begin{aligned} {}_h\tau_x &= \qquad\quad a_3y + a_2z,\\ {}_h\tau_y &= b_3x \qquad\quad + b_1z,\\ {}_h\tau_z &= c_2x + c_1y \qquad\quad . \end{aligned}$$

In a similar manner we indicate the harmonic conjugates of the tangentials x', y', and z' with respect to y, z; z, x; and x, y respectively, by $_hx'$, $_hy'$ and $_hz'$ and find

$$(13)\qquad \begin{aligned} {}_hx' &= \qquad\quad b\,y \quad c\,z,\\ {}_hy' &= -a_2x \qquad\quad + c_2z,\\ {}_hz' &= a_3x - b_3y \qquad\quad . \end{aligned}$$

If we designate by $\Delta_{x'}$ the determinant of the tangentials x', y', z', and by Δ_τ the determinant of the harmonic conjugates of the contact points, we find from (6) and (12)

$$\begin{aligned} \Delta_{x'} &= \Delta\ \Delta_2, \\ \Delta_\tau &= \Delta\ \Delta_2, \end{aligned} \tag{14}$$

where Δ is the determinant of the points x, y, z (equation D), and Δ_2 is the determinant defined by (8).

Since the vanishing of Δ is precluded, we have from (14)

THEOREM 1. *If the tangentials are collinear so also are the harmonic conjugates of the contact points and conversely. The necessary and sufficient condition for this is expressed by*

$$\Delta_2 = 0.\text{*}$$

Similar statements can be made in connection with the vanishing of the invariant Δ_1.

It will prove of interest to determine under what conditions the tangentials given by (16) are harmonic respectively to the contact points (11) and pairs of vertices of the triangle (x), (y), (z). The conditions are

$$\begin{aligned} C &= B, \\ A &= C, \\ B &= A, \end{aligned} \tag{15}$$

where A, B, and C are defined in (7).

Equation $(15)_1$ is the condition that the tangential $b_1y + c_1z$ shall be harmonic to the contact point $\tau_x = a_3y - a_2z$. Similar statements hold for $(15)_2$ and $(15)_3$. Since any two of the equalities (15) imply the third we have

THEOREM 2. *If any two of the tangentials are harmonic with respect to the corresponding contact points, then the third tangential is also harmonic with respect to its corresponding contact point.*

Theorems 1 and 2 have supplied geometric interpretations for the relative invariants A, B, C, Δ_1, and Δ_2. Interpretations for D_1, D_2, D_3, appear in the following situation. Denote by X, Y, and Z the contact points on the

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[*] From the known properties of triangles we can infer in theorem 1 that with the vanishing of Δ_2 not only will the tangentials be collinear but the harmonic conjugates of the tangent lines will be concurrent. In the same manner the collinearity of the contact point conjugates carries with it the concurrency of the contact lines.

lines joining τ_x, τ_y, τ_z, to the opposite vertices (x), (y) and (z). The following statements hold true.

$$
(16) \qquad \begin{array}{ll}
\Delta_2 = D_1(A),\ yZ \rightarrow \tau_y, & \Delta_2 = -D_2(B),\ yX \rightarrow \tau_y. \\
\Delta_2 = D_2(B),\ zX \rightarrow \tau_z, & \Delta_2 = -D_3(C),\ zY \rightarrow \tau_z. \\
\Delta_2 = D_3(C),\ xY \rightarrow \tau_x, & \Delta_2 = -D_1(A),\ xZ \rightarrow \tau_x.
\end{array}
$$

The notation is most readily understood by expanding the first statement of (16).

If and only if, $\Delta_2 = D_1(A)$, *will the line joining* (y) *to* (Z) *meet the opposite side of the triangle* (x), (y), (z) *in the point* τ_y.

7. *Defining Systems for the Tangential and Contact Curves.* By differentiating the expressions for the contact points given in (11) and eliminating x', y', z', by (4) and x, y, z by (11), we obtain expressions for τ_x', τ_y', τ_z', in terms of τ_x, τ_y, and τ_z. That is, we obtain for the contact points a defining system similar to (1).

By means of (7), (8) and (9) we can express the coefficients of τ_x, τ_y, τ_z, in terms of the fundamental invariants. The defining system is accordingly,

$$
(17) \qquad \begin{aligned}
\tau_x' &= A_1\tau_x + B_1\tau_y + C_1\tau_z, \\
\tau_y' &= A_2\tau_x + B_2\tau_y + C_2\tau_z, \\
\tau_z' &= A_3\tau_x + B_3\tau_y + C_3\tau_z.
\end{aligned}
$$

where

$$
(18) \quad \begin{array}{ll}
-b_1\Delta_1 B_1 = a_2 G(D_2 + G/C - H/B), & -c_1\Delta_1 C_1 = a_3 H(D_2 + G/C - H/B), \\
-a_2\Delta_1 A_2 = b_1 H(D_3 + G/A - H/C), & -c_2\Delta_1 C_2 = b_3 G(D_3 + G/A - H/C), \\
-a_3\Delta_1 A_3 = c_1 G(D_1 + G/B - H/A), & -b_3\Delta_1 B_3 = c_2 H(D_1 + G/B - H/A).
\end{array}
$$

From (35), we see that the following statements are true.

$$
(19) \qquad \begin{array}{lll}
D_2 + G/C - H/B = 0, & B_1 = C_1 = 0, & \tau_x = k_1, \\
D_3 + G/A - H/C = 0, & A_2 = C_2 = 0, & \tau_y = k_2. \\
D_1 + G/B - H/A = 0, & A_3 = B_3 = 0, & \tau_z = k_3.
\end{array}
$$

The complete statement corresponding to $(19)_1$, is—*the vanishing of the invariant* $D_2 + G/C - H/B$ *is the necessary and sufficient condition that the contact curve* τ_z *should degenerate into a point.*

For the sake of completeness the coefficients in the defining systems for the curves traced by the tangentials, tangential harmonics, and contact harmonics have been computed. They are as follows:

Tangentials.

$$\begin{aligned}
c_2\Delta_2 B_1 &= c_1 G(E_2 + G/B - H/C + AC/G + AB/G),\\
b_3\Delta_2 C_1 &= - b_1 H(E_2 + G/B - H/C - AB/H - AC/H),\\
c_1\Delta_2 A_2 &= - c_2 H(E_3 + G/C - H/A - AB/H - BC/H),\\
a_3\Delta_2 C_2 &= a_2 G(E_3 + G/C - H/A + AB/G + BC/G),\\
b_1\Delta_2 A_3 &= b_3 G(E_1 + G/A - H/B + AC/C + BC/C),\\
a_2\Delta_2 B_3 &= - a_3 H(E_1 + G/A - H/B - AC/H - BC/H).
\end{aligned}$$

Tangential Harmonics.

$$\begin{aligned}
c_2\Delta_1 B_1 &= c_1 G(E_2 + H/C - G/B - AC/G + AB/G),\\
b_3\Delta_1 C_1 &= b_1 H(E_2 + H/C - G/B - AC/H + AB/H),\\
c_1\Delta_1 A_2 &= c_2 H(E_3 + H/A - G/C - AB/H + BC/H),\\
a_3\Delta_1 C_2 &= a_2 G(E_3 + H/A - G/C - AB/G + BC/G),\\
b_1\Delta_1 A_3 &= b_3 G(E_1 + H/B - G/A - BC/G + AC/G),\\
a_2\Delta_1 B_3 &= a_3 H(E_1 + H/B - G/C - BC/H + AC/H).
\end{aligned}$$

Contact Harmonics.

$$\begin{aligned}
b_1\Delta_1 B_1 &= - a_2 G(D_2 + H/B - G/C - 2H/A),\\
c_1\Delta_1 C_1 &= a_3 H(D_2 + H/B - G/C + 2G/A),\\
a_2\Delta_2 A_2 &= b_1 H(D_3 + H/C - G/A + 2G/B),\\
c_2\Delta_1 C_2 &= - b_3 G(D_3 + H/C - G/A - 2H/B),\\
a_3\Delta_1 A_3 &= - c_1 G(D_1 + H/A - G/B - 2H/C),\\
b_3\Delta_2 B_3 &= c_2 H(D_1 + H/A - G/B + 2G/C).
\end{aligned}$$

8. *System of Linear Differential Equations with Constant Coefficients.* If the coefficients in system (1) are constants we can assume the solutions—

$$\begin{aligned}
x &= \alpha e^{rt},\\
y &= \beta e^{rt},\\
z &= \gamma e^{rt}.
\end{aligned} \tag{20}$$

where α, β, γ are to be determined. On substituting these values in (1) and suppressing the factor e^{rt} we find

$$\begin{aligned}
(a_1 - r)\,\alpha + b_1\beta + c_1\gamma &= 0,\\
a_2\alpha + (b_2 - r)\,\beta + c_2\gamma &= 0,\\
a_3\alpha + b_3\beta + (c_3 - r)\gamma &= 0.
\end{aligned} \tag{21}$$

For these equations to be consistent the determinant of the coefficients must vanish. The resulting cubic in r is satisfied, in general, by three distinct values $r = r_i$ $(i = 1, 2, 3)$. Substituting in turn each value of r in

(39) we obtain the corresponding values of α, β, γ. In this way we find the three sets of solutions

$$
\begin{aligned}
(x) &\equiv x_1 = \alpha_1 e^{r_1 t}, & x_2 &= \alpha_2 e^{r_2 t}, & x_3 &= \alpha_3 e^{r_3 t}, \\
(y) &\equiv y_1 = \beta_1 e^{r_1 t}, & y_2 &= \beta_2 e^{r_2 t}, & y_3 &= \beta_3 e^{r_3 t}, \\
(z) &\equiv z_1 = \gamma_1 e^{r_1 t}, & z_2 &= \gamma_2 e^{r_2 t}, & z_3 &= \gamma_3 e^{r_3 t}.
\end{aligned} \tag{22}
$$

Curves (x), (y), and (z) are equiangular logarithmic spirals. They possess the further property noted by Klein[*] of being transformable into themselves by the transformation

$$
\begin{aligned}
\bar{x}_1 &= e^{r_1 \lambda} x_1, \\
\bar{x}_2 &= \qquad e^{r_2 \lambda} x_2, \\
\bar{x}_3 &= \qquad\qquad e^{r_3 \lambda} x_3.
\end{aligned} \tag{T}
$$

If λ is regarded as a continuous variable then T constitutes a closed system of commutative linear transformations which carry a point of either (x), (y) or (z) with parameter t into a point of the same curve with parameter $t + \lambda$.

A set of curves transformable into themselves by a system T, has been named by Klein W-curves of a system.[†] The curves (x), (y), and (z) possess the further well known property of being projectively transformable into each other.

If we recall the expression (6) and (11) for the tangentials and the contact points we see that they are in the present discussion, constant linear combinations of the points (x), (y), and (z). From (22) it follows that the tangentials and the contact points describe logarithmic equiangular spirals belonging to the same system as the curves (x), (y) and (z), i. e., they are equiangular with the latter curves.

We can conclude the foregoing results in the

THEOREM 3. *A system of three first order linear homogeneous differential equations with constant coefficients defines, in general, three equiangular, logarithmic spirals under such a correlation, by virtue of the common parameter, that the associated tangential and contact curves are also logarithmic spirals equiangular with the given curves. The curves themselves and the associated tangential and contact curves are all W-curves of a system in Klein's sense.*

---

[*] *Mathematische Annalen*, Vol. 4 (1871), p. 51: "Uber diejenigen ebenen Curven, welche durch ein geschlossenes System von einfach unendlich vielen vertauschbaren, linearen, Transformationen in sich übergehen."

[†] [illegible]

We can summarize the present paper by stating that a system of three linear, homogeneous, first order, differential equations in one independent and three dependent variables defines three plane curves to within a projectivity. Furthermore, there is established between the curves by virtue of their expression in terms of a common parameter a point correspondence.

The properties of this triad of curves are a function, not alone of the curves themselves, but of the curves and the correlation established between them.

In conclusion we shall note a few of the problems which suggest themselves in connection with the present paper.

1. We have found the condition that the contact points be collinear, namely $\Delta_2 = 0$. We might ask what is the line locus described by the line of contact points when the condition $\Delta_2 = 0$, is fulfilled?

2. There is the obvious possibility of extending all of these results first to four equations in four dependent variables interpretable as defining four curves in three dimensions, and finally to n-equations which will define n curves in $(n-1)$ space.

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The Symbolic Development of the Disturbing Function.

By K. P. Williams.

1. In the development of the disturbing function, as carried out by LeVerrier, one becomes bewildered by the number of series with which he must deal. The method is elementary in principle, but extremely laborious and confusing in practice. The problem is preëminently one for which some symbolic method is desirable. This seems to have been first realized by Newcomb, who gave in the *American Journal of Mathematics* the first description of his method.[*] Later he published in detail, with the necessary numerical tables, a symbolic development in terms of the eccentric anomaly.[†] In practice the development in terms of the eccentric anomaly did not prove to possess the advantages that the author had anticipated, and he subsequently gave a briefer symbolic development in terms of the mean anomaly, which he used in an investigation of the secular variations of the four inner planets.[‡]

2. In the present paper a modification of Newcomb's method will be developed, which appears to the writer to possess certain distinct advantages. A group of fundamental operators are clearly isolated, and the development is effected by means of them. Incidentally certain symbols are introduced which possess a simple algebra, and the employment of them does away with the use of trigonometric functions, and their continual combination. The latter are reintroduced only at the end. The development proceeds according to the mean anomaly.

3. Let γ be the angle between the planes of the orbits of the two planets, and λ and λ' be the angular distances of the planets from their common node. Put $\sigma = \sin \frac{1}{2}\gamma$, $\alpha = a/a'$, where a and a' are the semi-major axes of the two orbits. Let $c_n^{(i)}$, $(n = 1, 3, 5, 7;\ i = 0, 1, 2, \cdots)$ be the constants called by Newcomb the *Gaussian coefficients.*[§] They are related to the familiar coefficients $b_s^{(i)}$ used by LeVerrier, by the equation

---

[*] Vol. 5, pp. 195-208.

[†] *Astronomical Papers of the American Ephemeris*, Vol. 3, pp. 1-200.

[‡] *Astronomical Papers*, Vol. 5, pp. 1-48, pp. 301-378.

[§] *Loc. cit.*, Vol. 5, p. 339 ff.

$$c_n^{(i)} = \alpha^{(n-1)/2}\, b_n^{(i)}.$$

Let

$$A_i = c_1^{(i)} - (1/2)\sigma^2(c_3^{(i-1)} + c_3^{(i+1)}) + (3/8)\,\sigma^4(c_5^{(i+2)} + 4c_5^{(i)} + c_5^{(i-2)}$$
$$-(5/16)\sigma^6(c_7^{(i+3)} + 9c_7^{(i+1)} + 9c_7^{(i-1)} + c_7^{(i-3)}),$$
$$B_i = (1/2)c_3^{(i)} - (3/4)\sigma^2(c_5^{(i+1)} + c_5^{(i-1)})$$
$$+ (15/16)\sigma^4(c_7^{(i+2)} + 3c_7^{(i)} + c_7^{(i-2)}),$$
$$C_i = (3/8)c_5^{(i)} - (15/16)\sigma^2(c_7^{(i+1)} + c_7^{(i-1)}),$$
$$D_i = (5/16)c_7^{(i)}.$$

Following Newcomb, put

$$V_i = i\lambda' - i\lambda,$$
$$V_i^{(1)} = (i+1)\lambda' - (i-1)\lambda,$$
$$V_i^{(2)} = (i+2)\lambda' - (i-2)\lambda,$$
$$\cdots\cdots\cdots\cdots$$

Then, if Δ_0 denotes the distance between the two planets, with the eccentricities neglected, we have*

$$a'\Delta_0^{-1} = \sum_{i=-\infty}^{\infty} (1/2)A_i \cos V_i \qquad \text{(called terms of class 0)}$$
$$+ \sigma^2 B_i \cos V_i^{(1)} \qquad \text{(called terms of class 1)}$$
$$+ \sigma^4 C_i \cos V_i^{(2)} \qquad \text{(called terms of class 2)}$$
$$\cdots\cdots\cdots$$

It is seen that we can write

$$\Delta_0^{-1} = \sum N \cos(\nu\lambda' + \mu\lambda), \tag{1}$$

with the following identifications:

Class	N	μ	ν
0	$(1/2)A_i/a'$	$-i$	i,
1	$\sigma^2 B_i/a'$	$-i+1$	$i+1$,
2	$\sigma^4 C_i/a'$	$-i+2$	$i+2$,
3	$\sigma^6 D_i/a'$	$-i+3$	$i+3$,

. . . .

Next let

g and g' be the mean anomalies,
f and f' be the true anomalies,
y and y' be the equations of the centers,
ω and ω' be the distances from the common node to the perihelia.

---

[*] Newcomb, *loc. cit.*, Vol. 3, pp. 14, 15.

Then

$$\lambda = \omega + f = \omega + g + (f - g) = \omega + g + y,$$
$$\lambda' = \omega' + g' + y'.$$

The quantity $\cos(\nu\lambda' + \lambda)$ in (1) then becomes

$$\cos(\nu\omega' + \mu\omega + \nu(g' + y') + \mu(g + y)),$$

which may be obtained by replacing g and g' by $g + y$ and $g' + y'$, respectively, in

$$\cos(\nu\omega' + \mu\omega + \nu g' + \mu g).$$

It will be seen that when the expansions are being made we can omit ω and ω', since it is possible to restore them in the final results. We thus work with

$$N \cos(\nu g' + \mu g).$$

When the eccentricities are to be taken into account we have merely to replace a, a', g, and g' by r, r', $g + y$, and $g' + y'$ respectively. Since r and r', as well as y and y' can be expressed in terms of the mean anomalies and the eccentricities, it is clear that an expansion in terms of the eccentricities and the mean anomalies can be found for Δ^{-1} where Δ is the distance between the two planets. The problem is clearly so complex that a symbolic treatment is desirable.

We first suppose that the eccentricity of the outer orbit is zero. Nothing is therefore to be done to a' and g', and accordingly they need not be exhibited. We therefore finally reduce the problem to the consideration of the change produced in

$$N \cos \mu g,$$

when the eccentricity of the inner orbit is introduced.

4. It is characteristic of Newcomb's method to regard N as a function of $\log a$ and $\log a'$. Remembering that the $c_n^{(i)}$ are all functions of $\alpha = a/a'$, and omitting the quantity σ, we have

$$N = (1/a')\, \phi\, (a/a').$$

It is easy to show that

$$\partial N/\partial \log a + \partial N/\partial \log a' = -N.$$

Therefore if we introduce the operators $\mathscr{D}$ and $\mathscr{D}'$ defined by

$$\mathscr{D} = \partial/\partial \log a, \mathscr{D}' = \partial/\partial \log a',$$

we have the simple operative relation

$$\mathfrak{D} + \mathfrak{D}' = -1, \quad \text{or} \quad \mathfrak{D}' = -1 - \mathfrak{D}. \tag{2}$$

If further we let

$$D = \partial/\partial \log \alpha,$$

we can show that

$$\mathfrak{D}\, N = DN, \tag{3}$$

and therefore by (2), it follows that

$$\mathfrak{D}' N = -(1 + D)N. \tag{4}$$

It may be remarked that Newcomb's development of the Gaussian constants gave methods of obtaining

$$D^m A_i, \quad D^m B_i, \quad D^m C_i, \quad D^m D_i,$$

so that the results of operating with $\mathfrak{D}$ and $\mathfrak{D}'$ upon N can be regarded as known.*

5. We turn now to a consideration of the analytic problem involved. We first develop a form for the nth derivative of a function of a function which is specially adapted to the present problem.

Suppose we define $F(t)$ from functions $G(x, t)$, $x = \phi(t)$, by putting

$$F(t) = \partial G/\partial x,$$

having placed $x = \phi(t)$ everywhere in the derivative $\partial G/\partial x$. It is seen that we can write

$$F'(t) = \frac{\partial}{\partial x}\,\frac{dG}{dt}, \quad F''(t) = \frac{\partial}{\partial x}\,\frac{d^2G}{dt^2}, \quad F'''(t) = \frac{\partial}{\partial x}\,\frac{d^3G}{dt^3}, \cdots$$

provided that dG/dt, d^2G/dt^2, $\cdots$, are all left properly expressed in terms of x and t.

Next consider

$$z = f(x), \quad x = \phi(t).$$

We have

$$\frac{dz}{dt} = \frac{\partial}{\partial x}\left(z\,\frac{dx}{dt}\right).$$

If we let $z\,\dfrac{dx}{dt}$ be the function $G(x, t)$ above, we have

$$\frac{d^{n+1}z}{dt^{n+1}} = \frac{\partial}{\partial x}\left[\frac{d^n}{dt^n}\left(z\,\frac{dx}{dt}\right)\right].$$

---

* *Loc. cit.*, Vol. 3, pp. 42-74.

Applying the formula of Leibnitz we obtain

$$(5)\qquad \frac{d^{n+1}z}{dt^{n+1}}=\frac{\partial}{\partial x}\left[\frac{d^nz}{dt^n}\,\frac{dx}{dt}+n\,\frac{d^{n-1}z}{dt^{n-1}}\,\frac{d^2x}{dt^2}+\cdots+z\,\frac{d^{n-1}x}{dt^{n+1}}\right],$$

the coefficients being the binomial coefficients.

In using (5) it is necessary to leave $dz/dt, \cdots, d^nz/dt^n$ expressed in terms of derivatives of z with regard to x, and of x with regard to t, and not replace x by its value in terms of t.

6. Consider next the following expansion problem. Given a function $F(a, g)$; replace a by $a+x$, and g by $g+y$, where

$$x=\alpha_1h+(\alpha_2/2\,!)h^2+(\alpha_3/3\,!)h^3+\cdots,$$
$$y=\beta_1h+(\beta_2/2\,!)h^2+(\beta_3/3\,!)h^3+\cdots.$$

Denote the result of the substitution by $F(h)$, and let it be required to expand $F(h)$ as a series in h.

Put

$$\mathfrak{D}=\partial/\partial a,\qquad \mathfrak{D}'=\partial/\partial g.$$

It is apparent that we can write symbolically

$$\begin{aligned}F(h)&=F(a+x,\ g+y)\\&=[1+h\,\mathcal{D}_1+(h^2/2\,!)\,\mathcal{D}_2+(h^3/3\,!)\,\mathcal{D}_3+\cdots]\,F(a,g),\end{aligned}$$

where $\mathcal{D}_1, \mathcal{D}_2, \cdots$ are operators, which are polynomials in $\mathfrak{D}$ and $\mathfrak{D}'$, with the customary convention that $\mathfrak{D}^nF=\partial^nF/\partial a^n$, etc.

We are to find the operators $\mathcal{D}_1, \mathcal{D}_2, \cdots$.

We can write symbolically *

$$F(a+x,\ g+y)=(\exp\,[x\mathfrak{D}+y\mathfrak{D}'])F.$$

This leads us to put

$$\exp\,[x\mathfrak{D}+y\mathfrak{D}']=E^z=G(h)$$

where

$$\begin{aligned}z=&[\alpha_1h+(\alpha_2/2\,!)h^2+\cdots]\,\mathfrak{D}\\&+[\beta_1h+(\beta_2/2\,!)h^2+\cdots]\,\mathfrak{D}'.\end{aligned}$$

We have accordingly

$$G(h)=G(0)+h(dG/dh)_{h=0}+(h^2/2\,!)(d^2G/dh^2)_{h=0}+\cdots.$$

---

* In accordance with astronomical usage we let E denote the base of the system

The quantity d^nG/dh^n is the nth derivative of a function of z, where z is a function of h. We can therefore employ (5) to calculate the coefficients in the expansion of $G(h)$. It is seen that in the expression for z we are to manipulate with $\mathfrak{D}$ and $\mathfrak{D}'$ as with ordinary numbers. We can therefore write

$$\frac{dG}{dh} = E^z \frac{dz}{dh}\,,$$

$$\frac{d^2G}{dh^2} = \frac{\partial}{\partial z}\left[\frac{dG}{dh}\,\frac{dz}{dh} + G\,\frac{d^2z}{dh^2}\right],$$

$$\frac{d^3G}{dh^3} = \frac{\partial}{\partial z}\left[\frac{d^2G}{dh^2}\,\frac{dz}{dh} + 2\,\frac{dG}{dh}\,\frac{d^2z}{dh^2} + G\,\frac{d^3z}{dh^3}\right],$$

A little attention to the expressions just given shows that the only way in which z enters the quantities within the signs [ ] is through the presence of E^z as a factor of $dG/dh,\ d^2G/dh^2, \cdots$. Since $z = 0$, when $h = 0$, and since

$$(d^nG/dh^n)_{h=0} = \mathcal{D}_n,$$

we obtain

$$\mathcal{D}_1 = (dz/dh)_{h=0},$$

$$\mathcal{D}_2 = \mathcal{D}_1(dz/dh)_{h=0} + (d^2z/dh^2)_{h=0},$$

$$\mathcal{D}_3 = \mathcal{D}_2(dz/dh)_{h=0} + 2\,\mathcal{D}_1(d^2z/dh^2)_{h=0} + (d^3z/dh^3)_{h=0},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

But

$$dz/dh = (\alpha_1 + \alpha_2 h + \cdots)\mathfrak{D} + (\beta_1 + \beta_2 h + \cdots)\mathfrak{D}',$$

$$d^2z/dh^2 = (\alpha_2 + \alpha_3 h + \cdots)\mathfrak{D} + (\beta_2 + \beta_3 h + \cdots)\mathfrak{D}',$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

so that we have finally

$$\mathcal{D}_1 = \alpha_1\mathfrak{D} + \beta_1\mathfrak{D}',$$

$$\mathcal{D}_2 = (\alpha_1\mathfrak{D} + \beta_1\mathfrak{D}')\mathcal{D}_1 + (\alpha_2\mathfrak{D} + \beta_2\mathfrak{D}'),$$

$$\mathcal{D}_3 = (\alpha_1\mathfrak{D} + \beta_1\mathfrak{D}')\mathcal{D}_2 + 2(\alpha_2\mathfrak{D} + \beta_2\mathfrak{D}')\quad {}_1 + (\alpha_3\mathfrak{D} + \beta_3\mathfrak{D}'),$$

$$\mathcal{D}_4 = (\alpha\mathfrak{D}_1 + \beta_1\mathfrak{D}')\mathcal{D}_3 + 3(\alpha_2\mathfrak{D} + \beta_2\mathfrak{D})\,\mathcal{D}_2 + 3(\alpha_3\mathfrak{D} + \beta_3\mathfrak{D}')\,\mathcal{D}_1 + (\alpha_4\mathfrak{D} + \beta_4\mathfrak{D}'),$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

It would be possible to express $\mathcal{D}_2,\ \mathcal{D}_3, \cdots$ explicitly in terms of $\mathfrak{D}$ and $\mathfrak{D}'$, but it would not be advantageous to do so.

7. We next define fundamental operators as follows:

$$
\begin{aligned}
&\theta_1 = \alpha_1\mathfrak{D} + \beta_1\mathfrak{D}', \\
(6) \qquad &\theta_2 = \alpha_2\mathfrak{D} + \beta_2\mathfrak{D}', \\
&\theta_3 = \alpha_3\mathfrak{D} + \beta_3\mathfrak{D}', \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

We can then write:

$$
\begin{aligned}
&\mathscr{D}_1 = \theta_1, \\
&\mathscr{D}_2 = \theta_1\mathscr{D}_1 + \theta_2, \\
(7) \qquad &\mathscr{D}_3 = \theta_1\mathscr{D}_2 + 2\theta_2\mathscr{D}_1 + \theta_3, \\
&\mathscr{D}_4 = \theta_1\mathscr{D}_3 + 2\theta_2\mathscr{D}_2 + 3\theta_3\mathscr{D}_1 + \theta_4, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

If it should happen that the function $F(a, g)$ on which we have to operate with $\mathscr{D}_1$, $\mathscr{D}_2$, $\cdots$ is such that the effect of operating on it with the fundamental operators $\theta_1\ \theta_2 \cdots$ is to merely replace a single term by a group of terms each with the general character of the original term, it is seen that the knowledge of the effect produced on F by θ_1, θ_2, $\cdots$ leads at once to the effect produced by $\mathscr{D}_1$, $\mathscr{D}_2$, $\cdots$. Such a condition is actually fulfilled in the case of the disturbing function, and it is due to this fact that the operators owe their power.

8. We shall proceed now to the application of the general method to the case of the disturbing function.

As stated above we consider the outer orbit to be circular and take for the function $F(a, g)$ of §§ 5-7 the function

$$N \cos \mu g,$$

of § 3, which represents a term of the expansion of Δ_0^{-1}, with the omission of symbols that are irrelevant for the moment.

We are to replace a by r, and g by the true anomaly f. Since we are considering N as a function of $\log a$, we must think of replacing $\log a$ by $\log r$. We can write

$$
\begin{aligned}
\log r &= \log a + x, \\
f &= g + y,
\end{aligned}
$$

and there are well known expansions for x and y in terms of the eccentricity e. We put

$$
\begin{aligned}
x &= e\alpha_1 + (e^2/2\,!)\alpha_2 + (e^3/3\,!)\alpha_3 + \cdots, \\
y &= e\beta_1 + (e^2/2\,!)\beta_2 + (e^3/3\,!)\beta_3 + \cdots.
\end{aligned}
$$

Instead of writing the α_i and the β_i in their usual trigonometric form, we shall introduce a convenient notation.

We put

$$(j)^+ = E^{jg\iota} + E^{-jg\iota},$$
$$(j)^- = E^{jg\iota} - E^{-jg\iota},$$

where
$$\iota = \sqrt{-1}.$$

The following relations, easy to derive, and simple to employ, form the symbolic algebra for the development.

$$(8)\qquad \begin{aligned} &(1)^+(j)^+ = (j+1)^+ + (j-1)^+, \quad &(1)^-(j)^- = (j+1)^+ - (j-1)^+,\\ &(2)^+(j)^+ = (j+2)^+ + (j-2)^+, \quad &(2)^-(j)^- = (j+2)^+ - (j-2)^+,\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}$$

In terms of the new symbol we have: *

$$(9)\qquad \begin{aligned} &\alpha_1 = -(1/2)(1)^+, \qquad \alpha_2 = -(3/4)(2)^+ + (1/2),\\ &\alpha_3 = -(17/8)(3)^+ + (9/8)(1)^+,\\ &\alpha_4 = -(71/8)(4)^+ + (11/2)(2)^+ + (3/4),\\ &\alpha_5 = -(1569/32)(5)^+ + (1155/32)(3)^+ + (15/16)(1)^+,\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}$$

$$(10)\qquad \begin{aligned} &\beta_1 = -\iota(1)^-, \qquad \beta_2 = -(5/4)\iota(2)^-,\\ &\beta_3 = -(13/4)\iota(3)^- + (3/4)\iota(1)^-,\\ &\beta_4 = -(103/8)\iota(4)^- + (11/2)\iota(2)^-,\\ &\beta_5 = -(1097/16)\iota(5)^- + (645/16)\iota(3)^- - (25/8)\iota(1)^-, \end{aligned}$$

9. Since we are considering N as a function of log a, the operator $\mathfrak{D}$ of the general theory stands for ∂/∂ log a, that is, $\mathfrak{D}$ becomes the $\mathcal{D}$ of § 4. Therefore by (3) we have

$$(11)\qquad \mathfrak{D}N = \mathcal{D}N = DN,$$

where $D = \partial/\partial \log \alpha$. As stated before there are methods for computing the values of DN, D^2N, $\cdots$ for any pair of planets.

The operator $\mathfrak{D}'$ of the general theory remains $\partial/\partial g$. In view of this and the definition of $(j)^+$ and $(j)^-$, it is easy to see that

---

* The symbolic expressions can be obtained by obvious changes made on the expansions given by Newcomb, *loc. cit.*, Vol. 5, p. 21. It might be said that the value of x, which is the least common of the two expansions, can be obtained directly quite easily in the form used here, without using the method of development that Newcomb used.

$$\mathfrak{D}(j)^{+} = \iota j(j)^{-}. \tag{12}$$

10. We shall now develop the fundamental operators θ_1, θ_2, $\cdots$ of the present theory.

We first write

$$N \cos \mu g = N'(\mu)^{+}, \qquad N' = N/2.$$

Using (6), (9), and (10) we see that

$$\begin{aligned} \theta_1 N \cos \mu g &= \theta_1 N'(\mu)^{+} \\ &= [-(1/2)(1)^{+}\mathfrak{D} - \iota(1)^{-}\mathfrak{D}']N'(\mu)^{+}. \end{aligned}$$

Remembering that $\mathfrak{D}$ operates only on N', and using (11) and (12) we find

$$\theta_1 N'(\mu)^{+} = [-(1/2)(1)^{+}(\mu)^{+}D + \mu(1)^{-}(\mu)^{-}]N'.$$

When this is simplified by means of (8) we obtain

$$\begin{aligned} \theta_1 N'(\mu)^{+} = [-(1/2)\{(\mu+1)^{+} + (\mu-1)^{+}\}D \\ + \mu\{(\mu+1)^{+} - (\mu-1)^{+}\}]N'. \end{aligned}$$

Since only symbolic parentheses with the $+$ sign remain, we can drop the distinguishing mark.* Collecting similar terms we have finally

$$\begin{aligned} \theta_1 N'(\mu) &= [-(1/2)D + \mu]N' \cdot (\mu+1) \\ &+ [-(1/2)D - \mu]N' \cdot (\mu-1). \end{aligned}$$

For θ_2 we find

$$\begin{aligned} \theta_2 N'(\mu)^{+} &= [\{-(3/4)(2)^{+} + (1/2)\}(\mu)^{+}D + \mu(5/4)(2)^{-}(\mu)^{-}]N' \\ &= [\{-(3/4)(\mu+2) - (3/4)(\mu-2) + (1/2)(\mu)\}D \\ &+ \mu\{(5/4)(\mu+2) - (5/4)(\mu-2)\}]N' \end{aligned}$$

which for actual employment is rearranged below. The values of θ_3, θ_4, $\cdots$ are found similarly.

---

* Parentheses ( ) are symbolic except when inclosing a numerical fraction.

$$
\begin{aligned}
\theta_1 N'(\mu) &= [-(1/2)D + \mu]N'\cdot(\mu + 1) \\
&\quad + [-(1/2)D - \mu]N'\cdot(\mu - 1), \\
\theta_2 N'(\mu) &= [-(3/4)D + (5/4)\mu]N'\cdot(\mu + 2) + (1/2)DN'\cdot(\mu) \\
&\quad + [-(3/4)D - (5/4)\mu]N'\cdot(\mu - 2), \\
\theta_3 N'(\mu) &= [-(17/8)D + (13/4)\mu]N'\cdot(\mu + 3) \\
&\quad + [(9/8)D - (3/4)\mu]N'\cdot(\mu + 1) \\
&\quad + [(9/8)D + (3/4)\mu]N'\cdot(\mu - 1) \\
&\quad + [-(17/8)D - (13/4)\mu]N'\cdot(\mu - 1), \\
\theta_4 N'(\mu) &= [-(71/8)D + (103/8)\mu]N'\cdot(\mu + 4) \\
&\quad + [(11/2)D - (11/2)\mu]N'\cdot(\mu + 2) + (3/4)DN'\cdot(\mu) \\
&\quad + [(11/2)D + (11/2)\mu]N'\cdot(\mu - 2) \\
&\quad + [-(71/8)D - (103/8)N']N\cdot(\mu - 4), \\
\theta_5 N'(\mu) &= [-(1569/32)D + (1097/16)\mu]N'\cdot(\mu + 5) \\
&\quad + [(1155/32)D - (645/16)\mu]N'\cdot(\mu + 3) \\
&\quad + [(15/16)D + (25/8)\mu]N'\cdot(\mu + 1) \\
&\quad + [(15/16)D - (25/8)\mu]N'\cdot(\mu - 1) \\
&\quad + [(1155/32)D + (645/16)\mu]N'\cdot(\mu - 3) \\
&\quad + [-(1569/32)D - (1097/16)\mu]N'\cdot(\mu - 5) \\
&\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot
\end{aligned}
$$

12. We are now ready to develop the operators $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$, $\cdots$. Using the relations (7) we have

$$\mathscr{D}_1 N'(\mu) = \theta_1 N'\cdot(\mu),$$

$$
\begin{aligned}
\mathscr{D}_2 N'(\mu) &= [-(1/2)D + \mu]^2 N'\cdot(\mu + 2) + [(1/4)D^2 - \mu^2]N'\cdot(\mu) \\
&\quad + [(1/4)D^2 - \mu^2]N'\cdot(\mu) + [-(1/2)D - \mu]^2 N'\cdot(\mu - 2) \\
&\quad + \theta_2,
\end{aligned}
$$

or using the value of θ_2 and collecting,

$$
\begin{aligned}
\mathscr{D}_2 N'(\mu) &= [(1/4)D^2 + \{-\mu - (3/4)\}D + \{\mu^2 + (5/4)\mu\}]N'\cdot(\mu + 2) \\
&\quad + [(1/2)D^2 + (1/2)D - 2\mu^2]N'\cdot(\mu) \\
&\quad + [(1/4)D^2 + \{\mu - (3/4)\}D + \{\mu^2 - (5/4)\mu\}]N'\cdot(\mu - 2).
\end{aligned}
$$

For $\mathscr{T}_3$ we find

$$
\begin{aligned}
\mathscr{D}_3 N'(\mu) &= [-(1/2)D + \mu][(1/4)D^2 + \{-\mu - (3/4)\}D \\
&\qquad\qquad + \{\mu^2 + (5/4)\mu\}]N'\cdot(\mu + 3) \\
&\quad + [-(1/2)D + \mu][(1/2)D^2 + (1/2)D - 2\mu^2]N'\cdot(\mu + 1)
\end{aligned}
$$

$$
\begin{aligned}
&+ [-(1/2)D + \mu][(1/4)D^2 + \{\mu - (3/4)D \\
&\qquad\qquad + \{\mu^2 - (5/4)\mu\}]N' \cdot (\mu - 1) \\
&+ [-(1/2)D - \mu][(1/4)D^2 + \{-\mu - (3/4)\}D \\
&\qquad\qquad + \{\mu^2 + (5/4)\mu\}]N' \cdot (\mu + 1) \\
&+ [-(1/2)D - \mu][(1/2)D^2 + (1/2)D - 2\mu^2]N' \cdot (\mu - 1) \\
&+ [-(1/2)D - \mu][(1/4)D^2 + \{\mu - (3/4)\}D \\
&\qquad\qquad + \{\mu^2 - (5/4)\mu\}]N' \cdot (\mu - 3) \\
&+ 2[-(3/4)D + (5/4)\mu][-(1/2)D + \mu]N' \cdot (\mu + 3) \\
&+ 2[-(3/4)D + (5/4)\mu][-(1/2)D - \mu]N' \cdot (\mu + 1) \\
&+ 2 \cdot (1/2)D[-(1/2)D + \mu]N' \cdot (\mu + 1) \\
&+ 2 \cdot (1/2)D[-(1/2)D - \mu]N' \cdot (\mu - 1) \\
&+ 2[-(3/4)D - (5/4)\mu][-(1/2)D + \mu]N' \cdot (\mu - 1) \\
&+ 2[-(3/4)D - (5/4)\mu][-(1/2)D - \mu]N' \cdot (\mu - 3) \\
&\qquad + \theta_3.
\end{aligned}
$$

Inserting the value of θ_3 and collecting we obtain,

$$
\begin{aligned}
\mathscr{D}_3 N'(\mu) = {} & [-(1/8)D^3 + \{(3/4)\mu + (9/8)\}D^2 \\
&\qquad -\{(3/2)\mu_2 + (33/8)\mu - (17/8)\}D \\
&\qquad\qquad + \mu^3 + (15/4)\mu^2 + (13/4)\mu]N' \cdot (\mu + 3) \\
&+ [-(3/8)D^3 + \{(3/4)\mu + (3/8)\}D_2 \\
&\qquad +\{(3/2)\mu^2 + (15/8)\mu + (9/8)\}D \\
&\qquad\qquad - 3\mu^3 - (15/4)\mu^2 - (3/4)\mu]N' \cdot (\mu + 1) \\
&+ [-(3/8)D_3 + \{-(3/4)\mu + (3/8)\}D^2 \\
&\qquad + \{(3/2)\mu^2 - (15/8)\mu + (9/8)\}D \\
&\qquad\qquad + 3\mu^3 - (15/4)\mu^2 - (3/4)\mu]N' \cdot (\mu - 1) \\
&+ [-(1/8)D^3 + \{-(3/4)\mu + (9/8)\}D^2 \\
&\qquad + \{-(3/2)\mu^2 + (33/8)\mu - (17/8)\}D \\
&\qquad\qquad - \mu^3 + (15/4)\mu^2 - (13/4)\mu]N' \cdot (\mu - 3).
\end{aligned}
$$

The actual calculations will not be given further than this. They become long on account of the inherent complexity of the problem. It is seen, however, that the work consists almost entirely of the multiplication of polynomials.

13. When we return to trigonometric functions, it is seen that

$$\cdots N'(\mu + 3). \quad N'(\mu + 2), \cdots$$

become, respectively,

$$\cdots N\cos(\mu g + 3g), \quad N\cos(\mu g + 2g), \cdots$$

and that operational coefficients on N' will be applied unaltered to N.

It is also recalled that $\mathfrak{D}_s/s!$ is the coefficient of e^s in the series that replaces the term $N\cos\mu g$, when the eccentricity of the inner orbit is taken into account.

We can summarize the results as follows:

When account is taken of the eccentricity e, the single term $N\cos\mu g$ of Δ_0^{-1} goes into an infinite series with terms of the form

$$\Pi_j{}^s N e^s \cos(\mu g + jg),$$

where $\Pi_j{}^s$ is an operational polynomial in D, the derivative with regard to $\log\alpha$. The letter s takes the values $0, 1, 2, \cdots$, while for a given s, the letter j takes the restricted set of values

$$s, \quad s-2, \quad s-4, \cdots, \quad -s+2, \quad -s.$$

The operators $\Pi_j{}^s$ are the *operators of Newcomb.*

It is seen that our operator $\mathfrak{D}_s$ gives a group of Newcomb operators. For instance, the operator $\mathfrak{D}_3$ gives $\Pi_3{}^3$, $\Pi_1{}^3$, $\Pi_{-1}{}^3$, $\Pi_{-3}{}^3$. It is to be observed that account must be taken of the factor $1/s!$ in passing from the operators to the operators Π.

A few of the operators of Newcomb are given below; they can all be obtained from the expressions previously given.*

$$\begin{aligned}
\Pi_0{}^0 &= 1,\\
2\Pi_1{}^1 &= 2\mu - D,\\
8\Pi_2{}^2 &= 4\mu^2 + 5\mu + (-4\mu - 3)D + D^2,\\
4\Pi_0{}^2 &= -4\mu^2 + D + D^2,\\
48\Pi_3{}^3 &= 8\mu^3 + 30\mu^2 + 26\mu + (-12\mu^2 - 33\mu - 17)D\\
&\qquad + (6\mu + 9)D^2 - D^3,\\
16\Pi_1{}^3 &= -8\mu^3 - 10\mu^2 - 2\mu + (4\mu^2 + 5\mu + 3)D\\
&\qquad + (2\mu + 1)D^2 - D^3.
\end{aligned}$$

The operators must, of course, be evaluated separately for terms of class $0, 1, 2, \cdots$. (See § 3). Thus for terms of class 0 we put $\mu = -i$. If this be done in the operators above, we shall find that the results agree with those given by Newcomb.[*]

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* Parentheses ( ) are no longer symbolic.

14. For the sake of completeness we shall show how to take into account the eccentricity e' of the outer orbit.

Since introducing the eccentricity e results in replacing every term of Δ_0^{-1} by a series, each term of which is of the same form as the term it replaces, and since the introduction of e' is, from the standpoint of the analytic processes involved, precisely the same as that of introducing e, it is seen we have merely to apply a second time the processes already developed.

Consider then the term $N \cos(\mu g + \nu g')$. When e and e' are both taken into account, this term will give rise to a series whose general term can be written

$$\Pi_j{}^s \, (\Pi_{j'}{}^{s'}) \;\; N \cdot e^s e'^{s'} \cos(\mu g + \nu g' + jg + j'g'),$$

where $(\Pi_{j'}{}^{s'})$ is obtained from $\Pi_j{}^s$ by replacing D by $\mathfrak{D}' = \partial/\partial \log a'$, and μ by ν.

The quantities s and s' take the values 0, 1, 2, $\cdot\cdot\cdot$, independently of each other, while for a fixed s and s',

$$\begin{aligned} j\, &= s,\; s-2,\; \cdots,\; -s+2,\; -s, \\ j' &= s',\; s'-2,\; \cdots,\; -s'+2,\; -s'. \end{aligned}$$

The operator $\mathfrak{D}'$ can be entirely eliminated, for we have by (4), $\mathfrak{D} = -1-D$.

For the operator $(\Pi_{j'}{}^{s'})$ when expressed in terms of D, Newcomb uses the symbol $\Pi^{0,\,s'}_{0,\,j'}$. We have the rule:

To obtain $\Pi^{0,\,s'}_{0,\,j'}$ replace D in $\Pi_{j'}{}^{s'}$ by $-1-D$, and μ by ν.

Some of the new operators are:

$$\begin{aligned} \Pi^{0,\,1}_{0,\,1} &= 1, \\ 2\Pi^{0,\,1}_{0,\,1} &= 2\nu + 1 + D, \\ 8\Pi^{0,\,2}_{0,\,2} &= 4\nu^2 + 9\nu + 4 + (4\nu + 5)D + D^2, \\ 4\Pi^{0,\,2}_{0,\,0} &= -4\nu^2 + D + D^2. \end{aligned}$$

Again, it is necessary to give to ν the value appropriate to the class of terms considered before comparing with the results of Newcomb.

It will merely be remarked that it is natural to combine the two operators by writing

$$\Pi^{s,\,s'}_{j,\,j'} = \Pi_j{}^s \; \Pi^{0,\,s'}_{0,\,j'} \,.$$

---

[*] *Loc. cit.*, Vol. 5, pp. 27 ff.

but it is unnecessary to go into the details. The process is apparent and the results are given by Newcomb.

15. Throughout the work the argument of the angle has been incompletely represented by the omission of ω and ω'. It is easy to see how they should be restored. For terms of class 0, 1, 2, $\cdots$, we replace

$$\cos\,(\mu g + \nu g + jg + j'g')$$

by

$$\begin{gathered}\cos\,(V_i + jg + j'g'),\\ \cos\,(V_i^{(1)} + jg + j'g'),\\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{gathered}$$

respectively, where

$$\begin{aligned}V_i &= i\lambda' - i\lambda = i(\omega' + g') - i(\omega + g),\\ V_i^{(1)} &= (i+1)(\omega' + g') - (i-1)(\omega + g),\\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}$$

The results can then be put in a form suitable for computation.

16. It is seen that the key to the development which has been given here lies in the fundamental operators, and the employment of equations (7), in which the coefficients are the binomial coefficients. There are only a small number of the fundamental operators and relations, and they are all comparatively simple. In Newcomb's work the existence of a group of fundamental operators does not appear to be recognized. The symbolic parentheses are of course a minor detail. The usual trigonometric functions could have been used. Yet the notation for the latter is at times unfortunately cumbersome. In a problem so inherently complex as that considered here, it seems desirable to adopt every possible convenience of writing. It appears to the writer that the symbols used contribute sensibly to ease of manipulation and to brevity.

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An Extension of Maschke's Symbolism.*

BY NOLA LEE ANDERSON.

1. *Introduction.* This paper considers a generalization of the symbolic method of Maschke for representing the invariants of a quadratic differential form $\sum g_{ij} dx^i dx^j$. According to this method, the coefficients g_{ij} are represented symbolically as products of $f_i f_j$.

Maschke regarded f_i and f_j as partial derivatives, $\partial f / \partial x^i$, $\partial f / \partial x^j$, of a symbolic function f, and placed

$$A = \sum g_{ij} dx^i dx^j = (df)^2 = (\sum f_i dx^i)^2. \tag{1}$$

The possibility of a similar symbolic representation in which it is not assumed that $\partial f_i / \partial x^j = \partial f_j / \partial x^i$ has been suggested by Wilson and Moore.† This is the generalization that is to be considered in this paper.

As might be expected this generalization does not affect the mode of representing first order differential parameters. It is found, however, that invariant forms of higher order can best be regarded as invariants of the set of quantities g_{ij} together with certain other sets.

The two dimensional case is discussed in some detail. A geometric interpretation is introduced,‡ and certain invariant vectors related to a surface are studied.

The $f_1(u, v)$ and $f_2(u, v)$ are here regarded as any pair of independent vectors, satisfying the relation (1), associated with the point u, v of the surface. We find that the plane (called the base plane) which they determine has many properties analogous to properties of the tangent plane in the ordinary case. The effect of varying the orientation of the vectors f_1 and f_2 in the base plane is considered only in a special case.

Normals to this plane which are expressible in terms of the first derivatives of f_1 and f_2 are called the first normals relative to the base plane. In general, there are four independent first normals to the base plane instead of three as in the usual case.*

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* Presented to the American Mathematical Society, November 27, 1926.

† "Differential Geometry of Two Dimensional Surface in Hyperspace," *Proceedings of the American Academy of Arts and Sciences*, Vol. 52 (1916), p. 294.

‡ A geometric interpretation of the Symbolic Method of Maschke was made and was discussed by L. Ingold in the paper "A Symbolic Treatment of the Geometry of Hyperspace." *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 574-599.

The analogues of the various curvature vectors associated with the curves $a(u, v) =$ constant are studied. These vectors in the main have properties very similar to the properties of the usual curvature vectors. There are, however, certain novel features. In particular we mention the appearance of a new invariant vector.

2. *First order invariants.* In order to represent expressions of higher than the first degree in g_{ij} by means of symbols, Maschke introduced symbols ϕ_i, ψ_i, etc. equivalent to the f_i. Thus, $g_{ij}g_{rs}$ might be written symbolically as $f_if_j \cdot \phi_r\phi_s$, and so on.

The fundamental theorem states that if $A^1, \cdots, A^n$ are any n invariants of the quadratic differential form $G \equiv \sum g_{ij}dx^idx^j$ in n variables, then β times their Jacobian is also an invariant where β is the reciprocal of the square root of the discriminant of G. This is written in the abbreviated form $(A^1, A^2, \cdots, A^n)$, where the factor β is understood.

In the usual sense, arbitrary functions are invariants so that some or all of the A's may stand for arbitrary functions of $x^1, \cdots, x^n$. This applies to the symbolic functions f, and, likewise, some or all of the A's may stand for any of the equivalent symbolic functions f, ϕ, ψ, etc. Again, the product of any number of the invariants is invariant. This fact enables us to construct invariants involving the g_{ij} by writing products in which the symbol f occurs in two distinct factors; therefore $(f A^1 \cdots A^{n-1})(f B^1 \cdots B^{n-1})$ is an invariant containing the quantities g_{ij}.

In generalizing these ideas, we shall continue to use the symbolism of Maschke and let $f_if_j = g_{ij}$, where f_i and f_j, however, are not regarded as partial derivatives. It becomes necessary, then, to give their law of transformation. It has been noticed that the argument by which it is shown that $(A^1, \cdots, A^r)$ is an invariant is based chiefly on the fact that the derivatives $\partial A^i/\partial x^j$ transform according to the law

$$\partial \bar{A}^i/\partial \bar{x}^s = \sum (\partial A^i/\partial x^j)(\partial x^j/\partial \bar{x}^s)$$

when the variables are changed from $x^1, \cdots, x^n$ to $\bar{x}^1, \cdots, \bar{x}^n$. In accordance with this, we assume that symbols f_i follow the same law of transformation

$$\bar{f}_r = \sum f_j(\partial x^j/\partial \bar{x}^r) \tag{2}$$

with similar equations for equivalent symbols.

---

[*] See Wilson and Moore, *loc. cit.*, p. 304.

It is evident that the invariant expressions used by Maschke are still formally invariant with this extended conception of the f_i, ϕ_i, etc. Throughout this discussion, it will be understood that parenthesis expressions containing the symbolic functions have the same meaning as in Maschke's theory except that f, ϕ, etc. represent columns $f_1, f_2, \cdots, f_n, \phi_1, \phi_2, \cdots, \phi_n$, etc. instead of derivatives. Some of the elements of the parenthesis expressions may be parenthesis expressions themselves, as $((fa)a)(fb)$. In such cases, $(fa)_1$, means $\partial(fa)/\partial x^1$, etc., and, in general, unless the contrary is specified, the elements of a column of any parenthesis will be understood to be the partial derivatives of the quantity at the head of the column excepting only the case of f or its equivalent.

It is seen immediately that any symbolic invariant expression represents exactly the same invariant whether the f_i are regarded as derivatives or not, *provided no derivatives of the* f_i, ϕ_i, *etc. appear.*

3. *Triple index symbols.* The invariant expressions involving first derivatives of the fundamental quantities or of arbitrary functions are called invariants of the first order. It is clear that the generalization proposed in this paper does not affect the representation of such invariants. But other invariant expressions may occur which contain derivatives of the f_i, that is, such quantities as f_{ij}, where f_{ij} denotes the partial derivative, $\partial f^i/\partial x^j$.* Then, invariants of this type are called invariants of the second order.

From the equations $f_j f_j = g_{jj}$, we find by differentiation with respect to x^i that

$$f_{ji} f_j = (1/2)\partial g_{jj}/\partial x^i. \tag{3}$$

The products $f_i f_{jk}$ will be denoted by $[jk, i]$; these are analogous to the usual Christoffel triple index symbols.

If the equation $f_i f_j = g_{ij}$, where $i \neq j$, is differentiated with respect to x^k, we obtain

$$f_i f_{jk} + f_j f_{ik} = \partial g_{ij}/\partial x^k. \tag{4}$$

In the usual treatment the expressions $[ij, k]$ and $[ji, k]$ are the same so that the equations (3) and (4) are sufficient to determine all of the triple index symbols in terms of the derivatives of the fundamental quantities.

Here, however, this is no longer the case. But if notations, P, Q, etc., are introduced for a sufficient number of these symbols, all may be expressed [illegible]

---

* Whenever double subscripts occur with a symbolic quantity, the second one always denotes differentiation.

$$P, Q, \cdots.$$

Furthermore, it is possible to express several combinations of $f_{ij}f_{km}$ and f_if_{jkm} in terms of the quantities g_{ij} together with the additional quantities just mentioned.

4. *Geometric interpretation.* A geometric interpretation of this symbolism for the two dimensional case will now be considered. Throughout the remainder of this paper, we shall let u and v denote the curvilinear coordinates of a two dimensional surface in space.* We shall also from this point on consider

$$g_{11} = E, \quad g_{12} = F, \quad g_{22} = G,$$

except in cases where it is advantageous to use the coefficients g_{ij} in the formulas.

As has been referred to previously, Maschke used the symbol f_i as $\partial f/\partial x^i$. To interpret this geometrically, $f(u, v)$ may be regarded as a vector from some arbitrary origin to points of a two dimensional surface in space. Hence, at each point, f_1 and f_2 will be considered as a pair of vectors which are the partial derivatives of f, with respect to u and v, and therefore are tangent to the parameter curves, $v =$ constant and $u =$ constant, respectively. Thus, f_1 and f_2 lie in the tangent plane to the surface. In this case, of course, $f_{12} = f_{21}$.

Under the proposed generalization, the $f_1(u, v)$ and $f_2(u, v)$ are interpreted as any two vectors associated with the point u, v of the surface subject to the conditions that $f_1^2 = E$, $f_1f_2 = F$, and $f_2^2 = G$. The products f_if_j will be regarded as scalar products. The plane which these vectors determine will be called the base plane † of the corresponding point of the surface, and the vectors themselves will occasionally be referred to as the fundamental vectors or base vectors. The base plane and the fundamental vectors $\mathbf{f}_1$, $\mathbf{f}_2$ ‡ will be called the basis $[\mathbf{f}_1, \mathbf{f}_2]$.

---

* This space is of unspecified number of dimensions.

† The introduction of an arbitrary base plane at each point of a surface in the place of the tangent plane is very similar to the introduction of an arbitrary line at each point of a curve instead of the tangent line. The effect of replacing the tangent vector to a curve by an arbitrary vector function of the arc length or other parameter was considered by W. Fr. Meyer in the paper, "Ausdehnung der Frenetschen Formeln und Verwandter auf den R_n," *Jahresbericht der Deutschen Mathematiker Vereinigung*, Vol. 19 (1910), pp. 160-169.

A similar method has been recently employed in the study of Riemannian geometry. See Eisenhart, *Riemannian Geometry*, p. 73 and p. 107 where further references will be found.

‡ From this point on bold-faced type will be used to indicate vectors.

It should be mentioned that the formulas of the theory will depend not only on the base plane but on the orientation of the fundamental vectors as well. The study of properties independent of this orientation is not included.

Since in the general case the restriction that $f_{12} = f_{21}$ is removed, the quantity f_{12} is considered as independent of the quantity f_{21}.

If in a special case the f_1, f_2 are the derivatives of a vector f, they are regarded as tangents to the parameter curves of the surface, and the base plane becomes the tangent plane. This case will be known as the ordinary or usual case. It is clear, however, that when the base plane coincides with the tangent plane, the fundamental vectors f_1, f_2 need not coincide with the tangents to the parameter curves. This will be illustrated in the special case which is considered in the next articles.

5. *A special case.* We consider here the case in which the base plane coincides with the tangent plane. There are first developed a number of general formulas which will be valuable later for purposes of reduction.

From the scalar products,

$$(5) \qquad f_1 f_1 = E, \quad f_1 f_2 = F, \quad f_2 f_2 = G,$$

we deduce by differentiation the following equations for the triple index symbols:

$$\begin{aligned}(6) \quad &[11,1] = (1/2)E_1, \quad [21,2] = (1/2)G_1, \quad [21,1] + [11,2] = F_1,\\ &[12,1] = (1/2)E_2, \quad [22,2] = (1/2)G_2, \quad [12,2] + [22,1] = F_2.\end{aligned}$$

It is convenient now to introduce the additional quantities, P, Q, etc. mentioned in section 3. Therefore, we shall let $[11,2] = P$ and $[22,1] = Q$.

The Christoffel triple index symbols of the second kind will be used throughout this paper and are defined in the usual manner by the equations

$$(7) \qquad \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = \sum g^{km} \, [ij, m].$$

That is,

$$\left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} = g^{11}[12,1] + g^{12}[12,2], \text{ etc., where}$$

$$(8) \qquad g^{11} = G\beta^2, \quad g^{12} = -F\beta^2, \quad g^{21} = -F\beta^2, \quad g^{22} = E\beta^2.$$

We may, also, express the triple index symbols of the first kind in terms of the triple index symbols of the second kind,

$$(9) \qquad [ij, k] = \sum g_{km} \left\{ \begin{matrix} m \\ ij \end{matrix} \right\},$$

and by this relation we have equations of the type

$$[11, 1] = g_{11} \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} + g_{12} \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} .$$

By taking the second derivatives of $f_1^2 = E$, $f_2^2 = G$, and $f_1 f_2 = F$, several combinations of $f_{ik} f_{lm}$ and $f_i f_{klm}$ can be expressed in terms of E, F, G, P, Q, and their derivatives, such as

$$f_{21} f_{11} + f_2 f_{111} = P_1, \; f_{11} f_{22} - f_{12} f_{21} = P_2 + Q_1 - F_{12}, \text{ etc.}$$

Now if the base plane coincides with the tangent plane the vectors f_1 and f_2 must be tangent vectors; they are therefore linearly expressible in terms of the vectors tangent to the parameter curves.

In the remaining portion of this section and in section 7 the notations φ_1 and φ_2 will be used to denote vectors tangent to the parameter curves $v =$ constant and $u =$ constant, respectively, subject to the conditions

$$\varphi_1^2 = E, \qquad \varphi_1 \varphi_2 = F, \quad \varphi_2^2 = G.$$

It is clear from these equations and equations (5) that the cosine of the angle between the vectors φ_1 and φ_2 is the same as the cosine of the angle between f_1 and f_2, and the angles themselves are either equal or supplementary. We consider the case in which they are equal.

Let $\theta(u, v)$ denote the angle between f_1 and φ_1 which is also the angle between f_2 and φ_2. We then have the formulas

$$f_1 \varphi_1 = E \cos\theta, \qquad f_2 \varphi_2 = G \cos\theta,$$

and therefore

$$(f_1 \varphi_1)/E = (f_2 \varphi_2)/G = \cos\theta.$$

These equations enable us to determine the coefficients p, q, m, n in the following equations

$$f_1 = p\varphi_1 + q\varphi_2, \tag{10}$$

$$f_2 = m\varphi_1 + n\varphi_2. \tag{11}$$

If (10) be multiplied by φ_1, and (11) by φ_2, the equations

$$p = \cos\theta - q(F/E)$$

$$\text{and} \quad n = \cos\theta - m(F/G)$$

are obtained.

If (10) be squared, we find after substituting the value of p, that $q = (E \sin\theta)/(EG - F^2)^{1/2}$;[*] consequently,

$$p = \cos\theta - (F \sin\theta)/(EG - F^2)^{1/2}.$$

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[*] The use of the opposite sign for q is equivalent to reversing the angle θ.

Again, when (10) is multiplied by (11) and the substitutions for p, q, and m are made, we have after simplifying, $m = (-G \sin\theta)/(EG - F^2)^{1/2}$, and consequently $n = \cos\theta + (F \sin\theta)/(EG - F^2)^{1/2}$.

Equations (10) and (11) may now be written

$$f_1 = (\cos\theta - \beta F \sin\theta)\varphi_1 + (\beta E \sin\theta)\varphi_2,$$
$$f_2 = -(\beta G \sin\theta)\varphi_1 + (\cos\theta + \beta F \sin\theta)\varphi_2$$
$$\text{where } \beta = 1/(EG - F^2)^{1/2}.$$

From these we can compute the derivatives of the base vectors, f_1 and f_2, with respect to u and v,

$$\begin{aligned}
f_{11} &= [\cos\theta - \beta F \sin\theta]\varphi_{11} + [\beta E \sin\theta]\varphi_{21} \\
&\quad + [\cos\theta - \beta F \sin\theta]_1\varphi_1 + [\beta E \sin\theta]_1\varphi_2; \\
f_{12} &= [\cos\theta - \beta F \sin\theta]\varphi_{12} + [\beta E \sin\theta]\varphi_{22} \\
&\quad + [\cos\theta - \beta F \sin\theta]_2\varphi_1 + [\beta E \sin\theta]_2\varphi_2; \\
f_{21} &= -[\beta G \sin\theta]\varphi_{11} + [\cos\theta + \beta F \sin\theta]\varphi_{21} \\
&\quad - [\beta G \sin\theta]_1\varphi_1 + [\cos\theta + \beta F \sin\theta]_1\varphi_2; \\
f_{22} &= -[\beta G \sin\theta]\varphi_{12} + [\cos\theta + \beta F \sin\theta]\varphi_{22} \\
&\quad - [\beta G \sin\theta]_2\varphi_1 + [\cos\theta + \beta F \sin\theta]_2\varphi_2.
\end{aligned}$$

If the scalar product of f_2 and f_{11} is computed, $P = [11, 2]$ is obtained in terms of E, F, G, and θ. In a similar manner, $Q = [22, 1]$ can be found, and thus by equations (6) all of the expressions $[ij, k]$ are known.

6. *Normal vectors.* By normal vectors we mean vectors that are orthogonal to the fundamental system of vectors, that is, in this case vectors orthogonal to the base plane. The vectors f_{ij} are in general oblique to the base plane, but there exist normal vectors expressible in terms of the fundamental vectors f_1, f_2, and the vectors f_{11}, f_{12}, f_{21}, f_{22}. Such vectors will be called first normals. Since f_{11}, f_{12}, f_{21}, f_{22} are as a rule independent, there are in general four independent first normals. This paper is for the most part concerned only with those cases in which these are independent.

One such set of four independent normals is given by

$$N_{ij} = f_{ij} - \sum \left\{ \begin{matrix} m \\ ij \end{matrix} \right\} f_m. \tag{12}$$

It is shown easily that N_{ij} and f_r are orthogonal, i. e. $N_{ij} f_r = 0$. If (12) is multiplied by f_k,

$$f_k N_{ij} = [ij, k] - \sum g_{km} \left\{ \begin{matrix} m \\ ij \end{matrix} \right\}.$$

Hence, it follows immediately from (9) that

$$\mathbf{f}_k\mathbf{N}_{ij} = 0.$$

There are, of course, other sets of independent first normals. In some cases the following set is more convenient,

$$\text{(13)} \qquad \boldsymbol{a} = \phi_1(\mathbf{f}\phi)_1, \quad \boldsymbol{b} = \phi_2(\mathbf{f}\phi)_2, \quad \boldsymbol{c} = \phi_1(\mathbf{f}\phi)_2, \quad \boldsymbol{d} = \phi_2(\mathbf{f}\phi)_1.^{*}$$

It is readily proved, for example, that $\phi_1(\mathbf{f}\phi)_1$ is a normal vector.

$$(\mathbf{f}\phi)_1 = \beta_1[\mathbf{f}_1\phi_2 - \mathbf{f}_2\phi_1] + \beta[\mathbf{f}_{11}\phi_2 - \mathbf{f}_{21}\phi_1 + \mathbf{f}_1\phi_{21} - \mathbf{f}_2\phi_{11}],$$

and if this be multiplied by ϕ_1,

$$\phi_1(\mathbf{f}\phi)_1 = \beta_1[F\mathbf{f}_1 - E\mathbf{f}_2] + \beta[F\mathbf{f}_{11} - E\mathbf{f}_{21} + [21,1]\mathbf{f}_1 - [11,1]\mathbf{f}_2].$$

Simplifying and substituting the values of $\mathbf{f}_{11}$ and $\mathbf{f}_{21}$ from (12),

$$\phi_1(\mathbf{f}\phi)_1 = \{\beta_1 F + \beta[21,1]\}\mathbf{f}_1 - \{\beta_1 E_2 + \beta[11,1]\}\mathbf{f}_2$$

$$+ \beta F\left(\mathbf{N}_{11} + \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} \mathbf{f}_1 + \begin{Bmatrix} 2 \\ 11 \end{Bmatrix} \mathbf{f}_2\right) - \beta E\left(\mathbf{N}_{21} + \begin{Bmatrix} 1 \\ 21 \end{Bmatrix} \mathbf{f}_1 + \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} \mathbf{f}_2\right)$$

$$= \left(\beta_1 F + \beta[21,1] + \beta F \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} - \beta E \begin{Bmatrix} 1 \\ 21 \end{Bmatrix}\right)\mathbf{f}_1$$

$$- \left(\beta_1 E + \beta[11,1] - \beta F \begin{Bmatrix} 2 \\ 11 \end{Bmatrix} + \beta E \begin{Bmatrix} 2 \\ 21 \end{Bmatrix}\right)\mathbf{f}_2 + \beta F\mathbf{N}_{11} - \beta E\mathbf{N}_{21}.$$

Since $\beta = 1/(EG - F^2)^{1/2}$,

$$\beta_1 = -(E_1G + EG_1 - 2FF_1)/2(EG - F^2)^{3/2}$$

$$= -\beta^3[(1/2)E_1G + (1/2)G_1E - FF_1],$$

and by (6),

$$\beta_1 = -\beta^3G[11,1] - \beta^3E[21,2] + \beta^3F[21,1] + \beta^3F[11,2].$$

Now by substituting this value β_1 in the last equation for $\phi_1(\mathbf{f}\phi)_1$, we find

$$\boldsymbol{a} = \phi_1(\mathbf{f}\phi)_1 = \beta(F\mathbf{N}_{11} - E\mathbf{N}_{21}).$$

In like manner,

$$\begin{aligned} \boldsymbol{b} &= \phi_2(\mathbf{f}\phi)_2 = \beta(G\mathbf{N}_{12} - F\mathbf{N}_{22}), \\ \text{(14)} \qquad \boldsymbol{c} &= \phi_1(\mathbf{f}\phi)_2 = \beta(F\mathbf{N}_{12} - E\mathbf{N}_{22}), \\ \boldsymbol{d} &= \phi_2(\mathbf{f}\phi)_1 = \beta(G\mathbf{N}_{11} - F\mathbf{N}_{21}). \end{aligned}$$

It is obvious that these are normal vectors.

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[*] $(\mathbf{f}\phi)_1$, and $(\mathbf{f}\phi)_2$ denote the partial derivatives $\partial(\mathbf{f}\phi)/\partial u$ and $\partial(\mathbf{f}\phi)/\partial v$. It seems preferable in formulas of this kind to print only the notation $\mathbf{f}$, which appears just once, in bold-faced type to indicate the vectorial character of the expression.

We can now show without difficulty that the invariant vectors

$$(15)\qquad (\phi a)((\boldsymbol{f}\phi)a),\quad (\phi a)(\psi a)((\boldsymbol{f}\phi)\psi),\quad (\psi\phi)(\psi a)((\boldsymbol{f}\phi)a),$$

where a denotes a function of u and v, are normal vectors since it is possible to express them linearly in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$.

For example,

$$\begin{aligned}(\phi a)((\boldsymbol{f}\phi)a) &= \beta[\phi_1 a_2 - \phi_2 a_1]\beta[(\boldsymbol{f}\phi)_1\, a_2 - (\boldsymbol{f}\phi)_2\, a_1]\\ &= \beta^2[\phi_1(\boldsymbol{f}\phi)_1\, a_2{}^2 - \phi_1(\boldsymbol{f}\phi)_2\, a_1a_2 - \phi_2(\boldsymbol{f}\phi)_1\, a_1a_2\\ &\qquad\qquad - \phi_2(\boldsymbol{f}\phi)_2\, a_1{}^2\\ &= a_2{}^2\boldsymbol{a} - a_1a_2(\boldsymbol{c} - \boldsymbol{d}) + a_1{}^2\boldsymbol{b}.\end{aligned}$$

By expanding in a similar manner it is found that the invariant vector $(\psi\phi)(\psi(\boldsymbol{f}\phi))$, is also a normal vector. (In the ordinary case this vector vanishes identically.)*

$$\begin{aligned}\text{Thus, }(\psi\phi)(\psi(\boldsymbol{f}\phi)) &= \beta^2\{[\psi_1\phi_2 - \psi_2\phi_1]\ [\psi_1(\boldsymbol{f}\phi)_2 - \psi_2(\boldsymbol{f}\phi)_1]\}\\ &= \beta^2(E\boldsymbol{b} - F\boldsymbol{d} - F\boldsymbol{c} + G\boldsymbol{a}).\end{aligned}$$

If the values of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ from equations (14) are substituted in this, we obtain

$$(16)\qquad (\psi\phi)(\psi(\boldsymbol{f}\phi)) = \beta[\boldsymbol{N}_{12} - \boldsymbol{N}_{21}] = \boldsymbol{j}.$$

There is another invariant normal vector,

$$((\boldsymbol{f}\phi)\phi) = \beta[(\boldsymbol{f}\phi)_1\phi_2 - (\boldsymbol{f}\phi)_2\phi_1],\ \text{or by (13)},$$

$$(17)\qquad ((\boldsymbol{f}\phi)\phi) = \beta(\boldsymbol{d} - \boldsymbol{c}) = \boldsymbol{h}^*.$$

It should be observed that the vectors $\boldsymbol{h}^*$ and $\boldsymbol{j}$ do not depend on an arbitrary function a.

7. *Application to special case.* It will be recalled that in the discussion of section 5 the base plane was made to coincide with the tangent plane to the surface and the fundamental vectors $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ to differ from the tangent vectors $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$ only by a rotation through the angle θ. In this special case, it is possible to express the invariant normal vectors $\boldsymbol{j}$ and $\boldsymbol{h}^*$ in terms of θ and $\boldsymbol{h}$, where $\boldsymbol{h} = \beta(\boldsymbol{d} - \boldsymbol{c})$ in the ordinary case.

First, let us expand $\boldsymbol{h}$, regarding the $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ as tangent vectors, in [illegible]

See L. Ingold, *loc. cit.*, p. 580.

$$\begin{aligned} \boldsymbol{h} = \beta(\boldsymbol{d} - \boldsymbol{c}) &= \beta[(\varphi\psi)_1\psi_2 - (\varphi\psi)_2\psi_2] \\ &= \beta\{\beta_1[\varphi_1\psi_2 - \varphi_2\psi_1]\psi_2 - \beta_2[\varphi_1\psi_2 - \varphi_2\psi_1]\psi_1 \\ &\qquad + \beta\psi_2[\varphi_{11}\psi_2 - \varphi_{21}\psi_1 + \varphi_1\psi_{21} - \varphi_2\psi_{11}] \\ &\qquad - \beta\psi_1[\varphi_{12}\psi_2 - \varphi_{22}\psi_1 + \varphi_1\psi_{22} - \varphi_2\psi_{12}]\}, \end{aligned}$$

where ψ_1 ψ_2 are equivalent to φ_1, φ_2.
Simplifying and collecting,

$$\begin{aligned} \boldsymbol{h} = {} & \beta\beta_1(G\varphi_1 - F\varphi_2) - \beta\beta_2(F\varphi_1 - E\varphi_2) \\ & + \beta^2(G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}) \\ & + \beta^2\varphi_1([21,2] + [22,1]) - \beta^2\varphi_2([11,2] + [12,1]), \end{aligned}$$

where we make use of the fact $\varphi_{12} = \varphi_{21}$.

Thus, $\boldsymbol{h} = \beta^2[G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}] +$ terms in φ_1 and φ_2.

Next, we shall consider the invariant normal vector $\boldsymbol{j} = (\psi\phi)(\psi(\boldsymbol{f}\phi)) = \beta[\boldsymbol{N}_{12} - \boldsymbol{N}_{21}]$. When the values for $\boldsymbol{N}_{12}$ and $\boldsymbol{N}_{21}$ from (12) are substituted in this equation,

$$\boldsymbol{j} = \beta[\boldsymbol{f}_{12} - \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} \boldsymbol{f}_1 - \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} \boldsymbol{f}_2 - \boldsymbol{f}_{21} + \left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} \boldsymbol{f}_1 + \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} \boldsymbol{f}_2].$$

Therefore,

$$\boldsymbol{j} = \beta[\boldsymbol{f}_{12} - \boldsymbol{f}_{21}] + \text{terms in } \boldsymbol{f}_1 \text{ and } \boldsymbol{f}_2.$$

From the values for $\boldsymbol{f}_{12}$ and $\boldsymbol{f}_{21}$ obtained in section 5,

$$\boldsymbol{f}_{12} - \boldsymbol{f}_{21} = \beta \sin\theta [G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}] + \text{terms in } \varphi_1 \text{ and } \varphi_2,$$

where φ_1 and φ_2 denote the tangent vectors to the parameter curves. It will, also, be recalled that $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ can be expressed in terms of φ_1 and φ_2. As a result of this, $\boldsymbol{j}$ can be written

$$\boldsymbol{j} = (\sin\theta)\boldsymbol{h} + \text{terms in } \boldsymbol{f}_1 \text{ and } \boldsymbol{f}_2.$$

Since $\boldsymbol{j}$ and $\boldsymbol{h}$ are normal vectors, the terms involving the tangent vectors, $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$, will vanish. Hence,

$$(18) \qquad \boldsymbol{j} = (\sin\theta)\boldsymbol{h}.$$

Lastly, we shall consider the normal vector $\boldsymbol{h}^* = ((\boldsymbol{f}\boldsymbol{f}')\boldsymbol{f}')$ with $\boldsymbol{f}'$ equivalent to $\boldsymbol{f}$. When it is expanded in a manner similar to the expansion of $\boldsymbol{j}$,

$$\boldsymbol{h}^* = \beta^2 \cos\theta \{G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}\} + \text{terms in } \boldsymbol{f}_1 \text{ and } \boldsymbol{f}_2,$$

which may be written

$$\boldsymbol{h}^* = (\cos\theta)\boldsymbol{h} + \text{terms in } \boldsymbol{f}_1 \text{ and } \boldsymbol{f}_2.$$

By the same reasoning as in the above case,

$$\boldsymbol{h}^* = (\cos\theta)\boldsymbol{h}. \tag{19}$$

8. *Polar directions.* We shall now consider the curves $a =$ constant on the two dimensional surface whose coördinates are u and v.

If the quantity V is a scalar function of u and v, the invariant $(Va)/[(fa)^2]^{1/2}$, is the same as it was before this generalization was made. That is, $(Va)/[(fa)^2]^{1/2} = dV/ds$, which is the derivative of the scalar quantity V with respect to arc length s along the curves $a =$ constant.*

But this is not true with $(\boldsymbol{f}a)/[(fa)^2]^{1/2}$, which will be denoted by $\boldsymbol{t}$, because $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are not $\partial\boldsymbol{f}/\partial u$ and $\partial\boldsymbol{f}/\partial v$. This, however, is an invariant vector lying in the base plane. Any scalar multiple of this vector $(\boldsymbol{f}a)$ will be called a polar vector of the curve $a =$ constant with respect to the basis. The direction of this polar vector may be called the polar direction. There is a polar direction corresponding to each point of the curve $a =$ constant.

It should be noted that by this definition of polar vector, the vectors $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are the polar vectors of the parameter curves $v =$ constant and $u =$ constant, respectively.

If this same operation is performed on $\boldsymbol{t}$, $(\boldsymbol{t}a)/[(fa)^2]^{1/2}$ results, which is really $d\boldsymbol{t}/ds$ since $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are $\partial\boldsymbol{t}/\partial u$ and $\partial\boldsymbol{t}/\partial v$, respectively. This vector is orthogonal to the polar vector, because $\boldsymbol{t}^2 = 1$ and $\boldsymbol{t}(d\boldsymbol{t}/ds) = 0$.

The vector $(\boldsymbol{t}a)/[(fa)^2]^{1/2}$ will be called a principal normal to $a =$ constant, relative to the basis, and its magnitude will be known as the first curvature relative to the basis. Usually these expressions will be called simply the principal normal, and the first curvature, the relation to the basis, being understood. In general, as has already been mentioned, a normal vector means a vector orthogonal to the base plane.

Instead of the polar vector $(\boldsymbol{f}a)$ to the curves $a =$ constant, we consider any vector function of u and v in the base plane, say $\alpha_1\boldsymbol{f}_1 + \alpha_2\boldsymbol{f}_2$, where α_1 and α_2 are functions of the coördinates. Suppose there exists a function b such that $\partial b/\partial u = -\lambda\alpha_2$ and $\partial b/\partial v = \lambda\alpha_1$, where λ is a proportionality factor. In this case, $(Vb)/[(fb)^2]^{1/2} = dV/ds$, where s is the arc length of the curve $b =$ constant. $(\boldsymbol{f}b) = \lambda(\alpha_1\boldsymbol{f}_1 + \alpha_2\boldsymbol{f}_2)$ which is the polar vector to $b =$ constant, and $(Vb) = \lambda(V_1\alpha_1 + V_2\alpha_2)$, hence,

$$(Vb)/[(fb)^2]^{1/2} = (V_1\alpha_1 + V_2\alpha_2)/[(f_1\alpha_1 + f_2\alpha_2)^2]^{1/2}.$$

If $\alpha_1\boldsymbol{f}_1 + \alpha_2\boldsymbol{f}_2$ is the polar vector of a curve $b =$ constant, it follows that the derivatives of any point function with respect to arc length along $b =$ constant is obtained by writing V_1, V_2 (that is $\partial V/\partial u$, $\partial V/\partial v$) instead of f_1, f_2 in the expression for the polar vector and dividing by the length of the vector.

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* L. Ingold, *loc. cit.*, p. 583.

As an example $(f\phi)(\phi a)$ is the polar vector of the orthogonal trajectories of $a =$ constant. If we denote arc length along one of these curves by σ,

$$dV/d\sigma = (V\phi)(\phi a)/[(fa)^2]^{1/2}.$$

We have used the relation $[(f\phi)(\phi a)]^2 = (fa)^2$ which is easily shown to be true.

Let $$k = (f\phi)(\phi a)(f\psi)(\psi a),$$

and let f in the third factor be interchanged with each element of the second factor; then

$$\begin{aligned} k &= (f\phi)(\phi f)(a\psi)(\psi a) + (f\phi)(fa)(\phi\psi)(\psi a) \\ &= 2(\psi a)^2 - k. \end{aligned}$$

Thus, $$k = [(f\phi)(\phi a)^2 = (fa)^2.$$

9. *The principal curvature vector.* The polar vector of the curves $a =$ constant may be written either in the form (fa) or $(f\phi)(\psi\phi)(\psi a)$, and from each of these forms it is possible to determine the curvature vector of the curves $a =$ constant.

If we find the curvature vector $d\mathbf{t}/ds$ from the equation

$$\mathbf{t} = (f\phi)(\psi\phi)(\psi a)/[(fa)^2]^{1/2},$$
$$d\mathbf{t}/ds = (\psi\phi)(\psi a)((f\phi)a)/(fa)^2 + \text{a vector in the base plane.}$$

The first term is normal to the base plane, and the others lie in the base plane.

The base component is more easily obtained from $\mathbf{t} = (fa)/[(fa)^2]^{1/2}$, and in this case

$$\frac{d\mathbf{t}}{ds} = \frac{((fa)a)}{(\phi a)^2} - \frac{(\phi a)((\phi a)a)}{[(\phi a)^2]^2}\,(fa).$$

Let $((fa)a)$ be resolved into its base and normal components,

$$((fa)a) = p(fa) + q(f\phi)(\phi a) + N.$$

Since (fa) and $(f\phi)(\phi a)$ are orthogonal to each other, we may multiply this equation first by (fa) and then by $(f\phi)(\phi a)$ and find that

$$p = \frac{((\psi a)a)(\psi a)}{(fa)^2}, \quad q = \frac{((\psi a)a)(\psi\phi)(\phi a)}{(fa)^2}.$$

Now,

$$\frac{d\boldsymbol{t}}{ds} = \frac{((\psi a)a)(\psi\phi)(\phi a)}{[(fa)^2]^2}(f\theta)(\theta a) + N.$$

From these two expressions for the curvature vector the base component and the normal component may be obtained, and

$$\frac{d\boldsymbol{t}}{ds} = \frac{(\psi\phi)(\phi a)((\psi a)a)}{[(fa)^2]^2}(f\theta)(\theta a) + \frac{(\psi\phi)(\psi a)((f\phi)a)}{(fa)^2}. \tag{20}$$

If we denote the vector of unit length in the direction of $(f\phi)(\phi a)$ by τ and the normal component of the curvature by α, (20) becomes $\Gamma_1 a\tau + \alpha$ where $\Gamma_1 a$ is the differential parameter

$$\frac{(\psi\phi)(\phi a)((\psi a)a)}{[\sqrt{(fa)^2}]^3}.$$

When $f_1 = \partial f/\partial u$, $f_2 = \partial f/\partial v$, the differential parameter $\Gamma_1 a$ is the geodesic curvature of the curves $a =$ constant. Generalizing this idea we call $\Gamma_1 a$ the geodesic curvature of the curves $a =$ constant *relative to the basis* $[f_1, f_2]$.

The curves $a =$ constant determined by $\Gamma_1 a = 0$ will be called the geodesic of the surface, relative to the basis $[f_1, f_2]$. These will be called merely *geodesics* and *geodesic curvature* hereafter, the relation to the basis being understood.

10. *Other curvature vectors.* If we use σ to denote arc length along the orthogonal trajectories of the curves $a =$ constant, we are able to find, by methods similar to those used above, an expression for $d\boldsymbol{t}/d\sigma$; also, expressions for $d\tau/ds$ and $d\tau/d\sigma$ may be found.

If we let

$$\Gamma_2 a = \frac{(f\phi)(fa)(\theta a)((\phi a)\theta)}{[\Delta_1 a]^{3/2}}, \quad \beta = \frac{(\phi a)(\theta a)((f\phi)\theta)}{\Delta_1 a},$$

$$\gamma = \frac{(\psi\phi)(\psi a)(\theta a)((f\phi)\theta)}{\Delta_1 a}, \quad \delta = \frac{(\phi a)((f\phi)a)}{\Delta_1 a}, \tag{21}$$

where $\Delta_1 a = (fa)^2$, the entire list of curvature vectors may be written

$$d\boldsymbol{t}/ds = \Gamma_1 a\tau + \alpha, \qquad d\boldsymbol{t}/d\sigma = -\Gamma_2 a\tau + \gamma,$$
$$d\tau/d\sigma = \Gamma_2 a\boldsymbol{t} + \beta, \qquad d\tau/ds = -\Gamma_1 a\boldsymbol{t} + \delta.$$

In these equations α and β denote the normal components of the curvature vectors of the curve $a =$ constant and its orthogonal trajectory, γ denotes the normal component of $d\boldsymbol{t}/d\sigma$, and δ the normal component of $d\tau/ds$.

It is obvious that the normal vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ are invariant under a change of parameters but depend on the function a.

We can readily obtain relations connecting these vectors.

If in $((f\phi)\phi)(\psi a)$, the last element in the first factor be exchanged with each of the elements of the second factor,

$$((f\phi)\phi)(\psi a) = ((f\phi)\psi)(\phi a) + ((f\phi)a)(\psi\phi).$$

It is recognized immediately that

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = ((f\phi)\phi)(\psi a)^2/\Delta_1 a = ((f\phi)\phi) = \boldsymbol{h}^*. \tag{23}$$

Thus, $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is an invariant proper and is independent of the function a. This is a well-known relation for the ordinary case.

Another relation for the ordinary case is

$$\boldsymbol{\delta} - \boldsymbol{\gamma} = 0.^*$$

In the general representation, this equation no longer holds as we shall now see.

If the numerator of the normal component of $d\boldsymbol{t}/d\sigma$ be simplified,

$$\begin{aligned}(\psi\phi)(\psi a)(\theta a)((f\phi)\theta) &= (\psi\phi)(\theta a)[((f\phi)\psi)(\theta a) + ((f\phi)a)(\psi\phi)] \\ &= (\psi\phi)((f\phi)\psi)(\theta a)^2 + (\psi\phi)(\psi\theta)(\theta a)((f\phi)a)\end{aligned}$$

where the θ of the last factor has been interchanged with each element of the second factor.

In accordance with the notation used in (16),

$$(\psi\phi)(\psi a)\theta a)((f\phi)\theta) = -\boldsymbol{j}(\theta a)^2 + (\phi a)((f\phi)a)$$

and $$(\psi\phi)(\psi a)(\theta a)((f\phi)\theta)/\Delta_1 a = -\boldsymbol{j} + [(\phi a)((f\phi)a)]/\Delta_1 a.$$

Therefore, it is noted in this general case that

$$\boldsymbol{\delta} - \boldsymbol{\gamma} = \boldsymbol{j}. \tag{24}$$

Thus the vector $\boldsymbol{\delta} - \boldsymbol{\gamma}$ is also independent of the function a.

11. *Polar vectors to other curves.* Any vector function which is a polar vector to a curve $a =$ constant on the surface may be written in the form $p\boldsymbol{t} + q\boldsymbol{\tau}$, where p and q are functions of the coördinates. Let $p^2 + q^2 = 1$, then the polar vector at each point of the curve is a unit vector.

In this discussion, we shall let $p\boldsymbol{t} + q\boldsymbol{\tau}$ be the polar vector of the curves

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[*] L. Ingold, *loc. cit.*, p. 588.

$b =$ constant, and shall denote by s and σ the arc length along b and along the orthogonal trajectories of $b =$ constant, respectively.

The curvature vector of the curves defined by this polar vector,

$$p\mathbf{t} + q\boldsymbol{\tau} = p\,\frac{(fa)}{[(fa)^2]^{1/2}} + q\,\frac{(f\phi)(\phi a)}{[(fa)^2]^{1/2}},$$

has the form

$$\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{ds} = p\,\frac{(p\mathbf{t} + q\boldsymbol{\tau},\, a)}{[(fa)^2]^{1/2}} + q\,\frac{(p\mathbf{t} + q\boldsymbol{\tau},\, \phi)(\phi a)}{[(fa)^2]^{1/2}}$$
$$= p\,\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{ds_a} + q\,\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{d\sigma_a}$$

where s_a and σ_a denote arc length along $a =$ constant and the orthogonal trajectories of $a =$ constant, respectively.

We may list the curvature vectors of the curves defined by the orthogonal unit vector, $q\mathbf{t} - p\boldsymbol{\tau}$, and the cross curvatures as follows:

$$\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{d\sigma} = q\,\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{ds_a} - p\,\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{d\sigma_a},$$
$$\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{ds} = p\,\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{ds_a} + q\,\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{d\sigma_a},$$
$$\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{d\sigma} = q\,\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{ds_a} - p\,\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{d\sigma_a}.$$

When these equations are expanded, the curvature vectors become

$$\begin{aligned}
\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{ds} &= G_1(q\mathbf{t} - p\boldsymbol{\tau}) + p^2\boldsymbol{\alpha} + pq(\boldsymbol{\gamma} + \boldsymbol{\delta}) + q^2\boldsymbol{\beta},\\
\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{d\sigma} &= G_2(p\mathbf{t} + q\boldsymbol{\tau}) + q^2\boldsymbol{\alpha} - (\boldsymbol{\gamma} + \boldsymbol{\delta}) + p^2\boldsymbol{\beta},\\
\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{ds} &= -G_1(p\mathbf{t} + q\boldsymbol{\tau}) + pq\boldsymbol{\alpha} + q^2\boldsymbol{\gamma} - p^2\boldsymbol{\delta} - pq\boldsymbol{\beta},\\
\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{d\sigma} &= -G_2(q\mathbf{t} - p\boldsymbol{\tau}) + pq\boldsymbol{\alpha} + q^2\boldsymbol{\delta} - p^2\boldsymbol{\gamma} - pq\boldsymbol{\beta},
\end{aligned} \tag{25}$$

where
$$G_1 = -\frac{(qa)}{[\Delta_1 a]^{1/2}} + \frac{(p\phi)(\phi a)}{[\Delta_1 a]^{1/2}} - p\Gamma_1 + q\Gamma_2,$$

and
$$G_2 = -\frac{(pa)}{[\Delta_1 a]^{1/2}} - \frac{(q\phi)(\phi a)}{[\Delta_1 a]^{1/2}} - q\Gamma_1 + p\Gamma_2.$$

If $p = \cos\theta$ and $q = \sin\theta$, the normal components of (25) in the new system are:

$$
(26)\qquad
\begin{aligned}
\bar{\boldsymbol{\alpha}} &= \boldsymbol{\alpha} \cos^2\theta + (\boldsymbol{\gamma} + \boldsymbol{\delta}) \sin\theta\cos\theta + \boldsymbol{\beta}\sin^2\theta,\\
\bar{\boldsymbol{\beta}} &= \boldsymbol{\alpha} \sin^2\theta - (\boldsymbol{\gamma} + \boldsymbol{\delta}) \sin\theta\cos\theta + \boldsymbol{\beta}\cos^2\theta,\\
\bar{\boldsymbol{\gamma}} &= (\boldsymbol{\alpha} - \boldsymbol{\beta}) \sin\theta\cos\theta + \boldsymbol{\gamma}\cos^2\theta - \boldsymbol{\delta}\sin^2\theta,\\
\bar{\boldsymbol{\delta}} &= (\boldsymbol{\alpha} - \boldsymbol{\beta}) \sin\theta\cos\theta + \boldsymbol{\delta}\cos^2\theta - \boldsymbol{\gamma}\sin^2\theta.
\end{aligned}
$$

From this we get

$$\bar{\boldsymbol{\alpha}} + \bar{\boldsymbol{\beta}} = \boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{h}^*,$$

$\boldsymbol{\alpha} + \boldsymbol{\beta}$ is therefore independent of the angle θ.

The vector $\bar{\boldsymbol{\delta}} - \bar{\boldsymbol{\gamma}}$ is also invariant, that is

$$\bar{\boldsymbol{\delta}} - \bar{\boldsymbol{\gamma}} = \boldsymbol{\delta} - \boldsymbol{\gamma} = \boldsymbol{j}.$$

In (25) we denoted by G_1 the geodesic curvature of the curves having $p\mathfrak{t} + q\tau$ as the polar vector and by G_2 the geodesic curvature of the curves having $q\mathfrak{t} - p\tau$ as the polar vector, i. e.,

$$
(27)\qquad
\begin{aligned}
G_1 &= -(dq/ds_a) + (dp/d\sigma_a) - p\Gamma_1 + q\Gamma_2,\\
G_2 &= -(dp/ds_a) - (dq/d\sigma_a) + q\Gamma_1 + p\Gamma_2.
\end{aligned}
$$

If $p = \cos\theta$ and $q = \sin\theta$,

$$
\begin{aligned}
(dp/ds_a) &= -\sin\theta(d\theta/ds_a) = -q(d\theta/ds_a),\\
(dq/d\sigma_a) &= \cos\theta\,(d\theta/d\sigma_a) = p(d\theta/d\sigma_a),\\
(dq/ds_a) &= \cos\theta(d\theta/ds_a) = p(d\theta/ds_a),\\
(dp/d\sigma_a) &= -\sin\theta(d\theta/d\sigma_a) = -q(d\theta/d\sigma_a).
\end{aligned}
$$

We know that

$$(dV/ds) = p(dV/ds_a) + q(dV/d\sigma_a)$$

and

$$(dV/d\sigma) = q(dV/ds_a) - p(dV/d\sigma_a).$$

Therefore, by substituting these results in the equations for G_1 and G_2,

$$
(28)\qquad
\begin{aligned}
G_1 &= -(d\theta/ds) - p\Gamma_1 + q\Gamma_2,\\
G_2 &= (d\theta/d\sigma) + q\Gamma_1 + p\Gamma_2.
\end{aligned}
$$

This is a generalization of a known result.*

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* See Eisenhart, *Differential Geometry*, p. 150, Ex. 17.

On the Approximate Solution of Fredholm's Homogeneous Integral Equation.*

BY RACHEL BLODGETT ADAMS.

Introduction. A number of papers † have appeared since Fredholm's memoir on integral equations which are concerned with the effect on the solution of a change in the kernel. In fact Fredholm ‡ himself obtained a formula for the first variation of the determinant $D(\lambda)$. With the exception of papers by Tricomi § and Viterbi, || however, the results leave one in doubt as to the bounds of the changes in the solutions corresponding to a given change in the kernel. The former considers only the non-homogeneous Fredholm integral equation, while the latter treats systems of non-homogeneous integral equations of the first and second kinds of the Volterra type. The present paper gives a method for determining bounds ¶ for the change in position of the zeros of the determinant $D(\lambda)$, and the changes in the solutions of the *homogeneous* Fredholm equations

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[*] Presented to the American Mathematical Society, May 1, 1926, under the title, "The solutions of Fredholm's homogeneous integral equation as functionals of the kernel." The results of this paper were obtained in 1921 and are to be found in a manuscript deposited at that time in the library of Radcliffe College. Inequalities were derived there which are practically equivalent to (4), (5), (7), and (8), published later by Tricomi (cf. note § of this page). His inequalities possess certain advantages over those obtained by the writer, first because of his use of an extension of Hadamard's determinent theorem where the writer used that theorem itself, and secondly because of his introduction of the function $\Omega(x)$ and its derivatives.

[†] H. Block, in "Några variationsformler inom integralekvationernas teori," *Acta Universitatis Lundensis, Lunds Universitets Årsskrift*, Nova Series, Medicin samt Matematiska och Naturvetenskapliga Ämnen, Vol. 7 (1911), pp. 3-34, finds formulas for the successive variations of the resolvent kernel, characteristic values, and fundamental solutions corresponding to a variation in the kernel and incidentally obtains essentially Theorems I and II of this paper. His method, however, gives no bounds for the changes in the solutions.

[‡] "Sur une classe d'équations fonctionnelles," *Acta Mathematica*, Vol. 27 (1903), pp. 379-381.

[§] "Sulla risoluzione numerica delle equazioni integrali di Fredholm," *Atti della Reale Accademia Nazionale dei Lincei, Rendiconti*, Classe di Scienze Fisiche, Matematiche, e Naturali, Series 5, Vol. 33 (1924), 1° Semestre, pp. 483-486; 2° Semestre, pp. 26-30.

[||] "Sulla risoluzione approssimata delle equazioni integrali di Volterra e della [illegible] *Rendiconti*, Series 2, Vol. 45 (1912), pp. 1024-1060.

[¶] The inequalities obtained are stronger than they need be theoretically: cf. G. C. Evans' review of W. V. Lovitt's "Linear Integral Equations," *The American Mathematical Monthly*, Vol. 34 (1927), p. 149.

$$\phi(x) + \lambda \int_a^b K(x,s)\,\phi(s)\,ds = 0 \tag{1}$$

$$\psi(y) + \lambda \int_a^b K(s,y)\,\psi(s)\,ds = 0 \tag{2}$$

corresponding to a small change in the kernel, where $K(x,y)$ is real and continuous in the region $a \leq x \leq b$, $a \leq y \leq b$.

It is well known that when $K(x,y)$ is expressible in the form $\sum_{i=1}^{n} \phi_i(x)\,\psi_i(y)$, the solutions of (1) [or (2)], which are in general infinite series involving multiple integrals, are particularly simple.* This fact suggests the possibility of obtaining approximate solutions for (1) [or (2)] by substituting for the given kernel an approximate one of the above type and solving the resulting equation. The results of this paper enable us in most of the interesting cases which arise in practice to determine bounds for the error thus incurred.†

1. *The change in a zero of $D(\lambda)$, $D(\lambda)$ being known.* Let $\lambda = \lambda_0$ be a zero of $D(\lambda)$ of multiplicity n, and let Γ be a circle about λ_0 of radius ρ to be determined later, such that Γ does not contain in its interior or on its boundary any other zero of $D(\lambda)$. Let $\bar{D}(\lambda)$ be the Fredholm determinant corresponding to a new kernel $\bar{K}(x,y)$. Then a necessary and sufficient condition that $D(\lambda)$ and $\bar{D}(\lambda)$ have the same number of zeros within Γ is given by the inequality

$$(1/2\pi i) \int_\Gamma [D'(\lambda)/D(\lambda) - \bar{D}'(\lambda)/\bar{D}(\lambda)]\, d\lambda < 1; \tag{3}$$

it is clear that this means that the integral on the left is zero, but for our purposes it is more convenient to write the condition in the form (3). We shall find an $\epsilon > 0$ so small that if

$$| K(x,y) - \bar{K}(x,y) | < \epsilon \qquad (a \leq x \leq b,\ a \leq y \leq b),$$

the inequality (3) will be satisfied.

Let $| K(x,y) |$ be $< K$ and $\epsilon \leq E$ in the region $a \leq x \leq b$, $a \leq y \leq b$. Then Tricomi ‡ deduces inequalities which we prefer to write in the following form:

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[*] T. Lalesco, *Introduction à la Théorie des Équations Intégrales*, Paris (1912), pp. 49-52.

[†] Cf. Note † p. 144 and Note † p. 148.

[‡] *Loc. cit.*, (16). In § 6, Tricomi gives a table of values of the function $\Omega(x)$ correct to four decimal places as x varies from 0 to 1 at intervals of .05.

(4) $$| D(\lambda) - \bar{D}(\lambda) | < \epsilon \, | \lambda | (b-a) \Omega' \, [| \lambda | (b-a)(K+E)]$$

(5) $$\begin{aligned} | D(x,y,\lambda) - \bar{D}(x,y,\lambda) | < \epsilon \, \{ & \Omega' \, [| \lambda | (b-a)(K+E)] \\ & + [| \lambda | (b-a)(K+E)] \, \Omega'' [| \lambda | (b-a)(K+E)] \} \end{aligned}$$

where $\Omega(x) = \sum_{n=0}^{\infty} (n^{n/2}/n!) \, x^n$. Similarly we could prove

(6) $$\begin{aligned} | D'(\lambda) - \bar{D}'(\lambda) | < \epsilon \, (b-a) \{ & | \lambda | (b-a)(K+E) \\ & \times \Omega'' [| \lambda | (b-a)(K+E)] + \Omega' [| \lambda | (b-a)(K+E)] \}. \end{aligned}$$

Tricomi also derives the relations[*]

(7) $$| D(\lambda) | < \Omega \, [| \lambda | (b-a) K]$$

(8) $$| D(x,y,\lambda) | < K \Omega' \, [| \lambda | (b-a) K].$$

To find a lower bound for $D(\lambda)$ on the circumference Γ of radius ρ, we suppose $D(\lambda)$ developed in a power series about the point λ_0; i. e.,

(9) $$\begin{aligned} D(\lambda) &= c_0 (\lambda - \lambda_0)^n + c_1 (\lambda - \lambda_0)^{n+1} + \cdots \\ &= (\lambda - \lambda_0)^n \, [c_0 + c_1 (\lambda - \lambda_0) + \cdots] \end{aligned}$$

where $c_0 \neq 0$. Thus if we denote $| c_0 |$ by c and a bound for $| c_i |$, $(i = 1, 2, \ldots)$, by C, and if we choose $\rho < c/(c+2C)$, we have, since $C \, | \lambda - \lambda_0 | / (1 - | \lambda - \lambda_0 |)$ is $< c/2$,

$$\begin{aligned} | D(\lambda)/(\lambda - \lambda_0)^n | &\geqq [c - C \{ | \lambda - \lambda_0 | + | \lambda - \lambda_0 |^2 + \cdots \}] \\ &= c - C \, | \lambda - \lambda_0 | / (1 - | \lambda - \lambda_0 |) \end{aligned}$$

or

(10) $$| D(\lambda)/(\lambda - \lambda_0)^n | > c/2.$$

Hence on a circle Γ about λ_0 of radius $\rho < c/(c+2C)$, $| D(\lambda) |$ is $> c\rho^n/2$ and furthermore, $D(\lambda)$ will have no second zero in this circle.

From (9) we have also on the circumference of Γ

(11) $$| D(\lambda) | \leqq \rho^n \, [c + C\rho/(1-\rho)].$$

Similarly if we denote $[c + C\rho/(1-\rho)]$ by $F(c, C, \rho)$ we obtain on Γ

(12) $$| D'(\lambda) | \leqq n\rho^{n-1} F(c, C, \rho) + \rho^n F'(c, C, \rho),$$

where F' indicates the derivative of F with respect to ρ.

Now $| D'(\lambda)/D(\lambda) - \bar{D}'(\lambda)/\bar{D}(\lambda) |$ may be written in the form

$$\left| \frac{D'(\lambda) [\bar{D}(\lambda) - D(\lambda)] + D(\lambda) [D'(\lambda) - \bar{D}'(\lambda)]}{D(\lambda) \bar{D}(\lambda)} \right| .$$

---

[*] *Loc. cit.* These are essentially inequalities (14).

Therefore if we set $L = (|\lambda_0| + \rho)(b - a)(K + E)$ and if ϵ is chosen so as to satisfy the condition

$$\epsilon < c\rho^n/\{2(|\lambda_0| + \rho)(b - a)\,\Omega'(L)\} \tag{13}$$

we obtain from (4), (6), (10), (11), and (12) the following inequality valid on the circumference of Γ:

$$|D'(\lambda)/D(\lambda) - \bar{D}'(\lambda)/\bar{D}(\lambda)|$$

$$< \frac{\{nF(c, C, \rho) + \rho F'(c, C, \rho)\}\,\epsilon(|\lambda_0| + \rho)(b - a)\,\Omega'(\mathrm{L}) + \rho F(c, C, \rho)\,\epsilon(b - a)\{\mathrm{L}\Omega''(\mathrm{L}) + \Omega'(\mathrm{L})\}}{(c\rho/2)\,\{(c\rho^n/2) - \epsilon\,(|\lambda_0| + \rho)(b - a)\,\Omega'(\mathrm{L})\}}\,.$$

It follows that condition (3) will certainly hold if ϵ satisfies the inequality,

$$\begin{aligned}\epsilon < c^2\rho^n/2(b - a)\,[(|\lambda_0| + \rho)\,\Omega'(\mathrm{L})\,\{c + 2nF(c, C, \rho)\\ + 2\rho F'(c, C, \rho)\} + 2\rho F(c, C, \rho)\{\mathrm{L}\Omega''(\mathrm{L}) + \Omega'(\mathrm{L})\}].\end{aligned} \tag{14}$$

Thus $D(\lambda)$ and $\bar{D}(\lambda)$ will have the same number of zeros inside the circle about λ_0 of radius $\rho < c/(c + 2C)$ or in other words $\bar{D}(\lambda)$ will have exactly n zeros $\bar{\lambda}_i$ satisfying the condition $|\lambda_0 - \bar{\lambda}_i| < \rho$.

2. *The change in a zero of $D(\lambda)$, $D(\lambda)$ being unknown.* In practice we do not know $D(\lambda)$, and so cannot determine c and C directly. This difficulty we meet by developing a method of trial and error which gives the desired information.

Let $\bar{\lambda}_1$ be a zero of $\bar{D}(\lambda)$ of multiplicity $\bar{n}$. Consider a circle $\bar{\Gamma}$ with center at $\bar{\lambda}_1$ and radius $\bar{\rho} < \bar{c}/(\bar{c} + 2\bar{C})$, where $\bar{c}$ and $\bar{C}$ have the meanings for $\bar{D}(\lambda)$ that c and C have for $D(\lambda)$. We thus obtain the condition under which $\bar{n}$ zeros of $D(\lambda)$ lie in the circle $|\lambda - \lambda_1| = \bar{\rho}$ by replacing λ_0, ρ, c, C, and n by $\bar{\lambda}_1$, $\bar{\rho}$, $\bar{c}$, $\bar{C}$, and $\bar{n}$ respectively in (14).

If this condition is not fulfilled, we take a better approximation to $K(x, y)$ and try again; in case of a second failure we take a still better approximation, and so on. If the multiplicity of $\bar{\lambda}_1$ is always the same as that of the corresponding zero of $D(\lambda)$, $\bar{c}$ will approach a limit not zero as $\bar{K}(x, y)$ approaches $K(x, y)$ and the above mentioned inequality can eventually be satisfied if we carry this process of repeated trial far enough. If, however, we are unable to satisfy the inequality, we conclude that the zero of $D(\lambda)$ corresponds to several zeros of $\bar{D}(\lambda)$ and so improve the approximation until it is clear that the zeros of $\bar{D}(\lambda)$ are gathering into groups. Suppose for convenience $\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\lambda}_3$ to be such a group.

Accordingly in the neighborhood of $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$, we have

$$(15)\qquad \bar{D}(\lambda) = (\lambda - \bar{\lambda}_1)^{\bar{n}_1} (\lambda - \bar{\lambda}_2)^{\bar{n}_2} (\lambda - \bar{\lambda}_3)^{\bar{n}_3} [\bar{c}_0 + \bar{c}_1 (\lambda - \bar{\lambda}_1) + \cdots].$$

In order to get a lower bound for $\lambda - \bar{\lambda}_2$ and $\lambda - \bar{\lambda}_3$ on a circle about $\bar{\lambda}_1$ of radius $\bar{\rho} < \bar{c}/(\bar{c} + 2\bar{C})$, we shall show that for a proper choice of ϵ, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ will lie in a circle of radius $\bar{\rho}/2$ about $\bar{\lambda}_1$.

From the considerations of § 1, we know that n zeros of $\bar{D}(\lambda)$ cluster about a zero of $D(\lambda)$ of multiplicity n. If we are right in our guess that $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$ are such a group, then $D(\lambda)$ will be of the form

$$(16)\qquad (\lambda - \lambda_0)^{\bar{n}_1+\bar{n}_2+\bar{n}_3} [c_0 + c_1 (\lambda - \lambda_0) + \cdots],$$

and as $\bar{D}(\lambda)$ approaches $D(\lambda)$, $\bar{c}_0$ approaches c_0, $\bar{c}_1$ approaches c_1, etc. Therefore $\bar{c}/2\,(\bar{c} + 2\bar{C})$ does not grow indefinitely small but approaches a limit not zero; namely, $c/2(c + 2C)$. Therefore for a sufficiently close approximation, we have

$$(17)\qquad |\bar{\lambda}_1 - \bar{\lambda}_i| < \bar{\rho}/2, \qquad (i = 2, 3).$$

Furthermore $\bar{c}_0 + \bar{c}_1 (\lambda - \bar{\lambda}_1) + \cdots$ will have no zeros in a circle about $\bar{\lambda}_1$ of radius $\bar{\rho} < \bar{c}/(\bar{c} + 2\bar{C})$ and on the circumference of such a circle we shall have

$$(18)\qquad \bar{c}_0 + \bar{c}_1 (\lambda - \bar{\lambda}_1) + \cdots > \bar{c}/2.$$

Thus on this circle we find

$$\begin{aligned} |\bar{D}(\lambda)| &> |(\lambda - \bar{\lambda}_1)^{\bar{n}_1} (\lambda - \bar{\lambda}_2)^{\bar{n}_2} (\lambda - \bar{\lambda}_3)^{\bar{n}_3}|\, \bar{c}/2 \\ (19)\qquad &> \bar{c}\bar{\rho}^n/2^{1+\bar{n}_2+\bar{n}_3}, \end{aligned}$$

and $\bar{D}(\lambda)$ will have no zeros other than $\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\lambda}_3$ within the circle.

As in (11) and (12), we have on $\bar{\Gamma}$

$$(20)\qquad |\bar{D}(\lambda)| \leq \bar{\rho}^n (3/2)^{\bar{n}_2+\bar{n}_3} \mathrm{F}(\bar{c}, \bar{C}, \bar{\rho})$$

and

$$(21)\qquad |\bar{D}'(\lambda)| \leq \bar{\rho}^{n-1} (3/2)^{\bar{n}_2+\bar{n}_3} [nF(\bar{c}, \bar{C}, \bar{\rho}) + \bar{\rho}F'(\bar{c}, \bar{C}, \bar{\rho})].$$

It can be easily verified now that we obtain the condition under which $D(\lambda)$ and $\bar{D}(\lambda)$ have the same number of zeros in the circle of radius $\bar{\rho}$ if in (11), in addition to replacing λ_0, ρ, c, and C by $\bar{\lambda}_1$, $\bar{\rho}$, $\bar{c}$, and $\bar{C}$ respectively, [illegible] ρ^n, $F(c, C, \rho)$ and $F'(c, C, \rho)$ the values $\rho^n/2$ [illegible] $F(c, C, \rho)$ and $3^{\bar{n}_2+\bar{n}_3} F'(\bar{c}, \bar{C}, \bar{\rho})$ respectively.

Now since $\bar{c}$ approaches a definite limit not zero as $\bar{K}(x, y)$ approaches

$K(x,y)$, the inequality mentioned in the preceding paragraph together with (17) can eventually be satisfied. Since $\bar{\lambda}_2$ and $\bar{\lambda}_3$ lie in a circle of radius $\bar{\rho}/2$ about $\bar{\lambda}_1$, we shall then have

$$(22) \qquad | \lambda_0 - \bar{\lambda}_i | < 3\bar{\rho}/2 \qquad (i = 1, 2, 3).$$

3. *The change in the fundamental solutions for the case that the characteristic values, both for $K(x,y)$ and $\bar{K}(x,y)$, are simple poles of the corresponding resolvent kernels.* It is well known that equations (1) and (2) have non-zero solutions when and only when λ is a zero of $D(\lambda)$. For such values of λ the resolvent kernel always has a pole. These zeros of $D(\lambda)$ are called *characteristic values* and the solutions *fundamental solutions.*[*] We shall now determine the change in these solutions, when $K(x,y)$ is replaced by $\bar{K}(x,y)$. We shall first restrict ourselves to the case that, both for $K(x,y)$ and $\bar{K}(x,y)$, the characteristic values are simple poles of the corresponding resolvent kernels. This case, however, includes three large classes of equations; namely, all those whose kernels are symmetric, skew-symmetric, or symmetrisable.[†] When the resolvent kernel has a simple pole the *index* or number of linearly independent solutions of (1) or (2) is equal to the multiplicity of the corresponding zero of $D(\lambda)$, and conversely.[‡]

Now let $\bar{\lambda}_i$ of multiplicities $\bar{n}_i$ $(i = 1, 2, \ldots, k; \sum_{i=1}^{k} \bar{n}_i = n)$ be the zeros of $\bar{D}(\lambda)$ corresponding to a zero λ_0 of $D(\lambda)$ of multiplicity n, and let $\bar{\Gamma}$ be a circle of radius $\bar{\rho}$ about $\bar{\lambda}_1$ chosen as in § 2, so that $D(\lambda)$ does not vanish on its circumference and the zeros of $D(\lambda)$ and $\bar{D}(\lambda)$ correspond within it. Then in $\bar{\Gamma}$ we have

$$(23) \qquad \frac{D(x,y,\lambda)}{D(\lambda)} = \frac{\phi_1(x,y)}{\lambda - \lambda_0} + \phi_0(x,y,\lambda)$$

$$(24) \qquad \frac{\bar{D}(x,y,\lambda)}{\bar{D}(\lambda)} = \frac{\bar{\phi}_{11}(x,y)}{\lambda - \bar{\lambda}_1} + \frac{\bar{\phi}_{12}(x,y)}{\lambda - \bar{\lambda}_2} + \cdots + \frac{\bar{\phi}_{1k}(x,y)}{\lambda - \bar{\lambda}_k} + \bar{\phi}_0(x,y,\lambda),$$

where $\phi_0(x,y,\lambda)$ and $\bar{\phi}_0(x,y,\lambda)$ are analytic in λ within and on $\bar{\Gamma}$. Then $\phi_1(x,y)$ and $\bar{\phi}_{1i}(x,y)$ can be written in the forms[§]

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[*] For definitions of terms see Lalesco, *loc. cit.*, pp. 31, 51.

[†] Cf. Lalesco, *loc. cit.*, pp. 66, 74, 81. It is interesting to note that the integral equations arising out of the generalized first and second boundary value problems both in the plane and in space can be treated by the method of this section. Cf. J. Plemelj, "Über lineare Randwertaufgaben der Potentialtheorie," I Teil, *Monatshefte für Mathematik und Physik*, Vol. 15 (1904), pp. 374-383.

[‡] Lalesco, *loc. cit.*, p. 57.

[§] Lalesco, *loc. cit.*, p. 58.

$$(25)\qquad \phi_1(x,y)=\sum_{j=1}^{n}\phi_j(x)\,\psi_j(y)$$

$$(26)\qquad \bar{\phi}_{1i}(x,y)=\sum_{j=1}^{\bar{n}_i}\bar{\phi}_{ij}(x)\,\bar{\psi}_{ij}(y)\qquad (i=1,2,\cdots,k),$$

where the ϕ's and $\bar{\phi}$'s are solutions of (1) for $K(x,y)$ and $\bar{K}(x,y)$ respectively and similarly the ψ's and $\bar{\psi}$'s are solutions of the adjoint equation (2). Also we have*

$$(27)\qquad \int_a^b \phi_i(x)\,\psi_j(x)\,dx=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}\qquad (i,j=1,2,\cdots,n)$$

$$(28)\qquad \int_a^b \bar{\phi}_{ij}(x)\,\bar{\psi}_{pl}(x)\,dx=\begin{cases}1, & i=p,\quad j=l\\ & i\neq p\quad\text{or}\\ 0, & \\ & i=p,\quad j\neq l\end{cases}\qquad \begin{pmatrix} i,p=1,2,\cdots,k\\ j=1,2,\cdots,\bar{n}_i\\ l=1,2,\cdots,\bar{n}_p\end{pmatrix}.$$

From their method of formation it is clear that the ϕ's and $\bar{\phi}$'s are finite and not identically zero.† Accordingly we may replace $\bar{\phi}_{ij}(x)$ by $\bar{\phi}_{ij}(x)/(\int_a^b|\bar{\phi}_{ij}(x)|^2dx)^{1/2}$ and $\bar{\psi}_{ij}(y)$ by $\bar{\psi}_{ij}(y)(\int_a^b|\bar{\phi}_{ij}(x)|^2dx)^{1/2}$. We shall say that the $\bar{\phi}$'s are then *normalized.* Evidently this definition reduces to the usual one for real functions. We observe that the relations (26) and (28) remain unaltered. Similarly we may normalize the ϕ's.

By the theory of residues we have

$$(29)\qquad (1/2\pi i)\int_{\bar{\Gamma}}[D(x,y,\lambda)/D(\lambda)]\,d\lambda=\sum_{j=1}^{n}\phi_j(x)\,\psi_j(y)$$

and

$$(30)\qquad (1/2\pi i)\int_{\bar{\Gamma}}[\bar{D}(x,y,\lambda)/\bar{D}(\lambda)]\,d\lambda=\sum_{i=1}^{k}\sum_{j=1}^{\bar{n}_i}\bar{\phi}_{ij}(x)\,\bar{\psi}_{ij}(y).$$

Since this last sum has just n terms, we shall write it for convenience in the form

$$(31)\qquad \sum_{j=1}^{n}\bar{\phi}_j(x)\,\bar{\psi}_j(y),$$

the terms being ordered as in (30).

We shall show that given an η_1, an ϵ can be found such that, if $|K(x,y)-\bar{K}(x,y)|$ is $<\epsilon$ in the region $a\leq x\leq b$, $a\leq y\leq b$, we shall have

$$(32)\qquad \left|\sum_{j=1}^{n}\phi_j(x)\,\psi_j(y)-\sum_{j=1}^{n}\bar{\phi}_j(x)\,\bar{\psi}_j(y)\right|<\eta_1.$$

We already have on the circumference of $\bar{\Gamma}$, as in (19),

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[illegible]

$$(33) \qquad |\bar{D}(\lambda)| > \bar{c}\bar{\rho}^{n}/2^{1+\bar{n}_2+\cdots+\bar{n}_k},$$

where

$$\bar{D}(\lambda) = (\lambda-\bar{\lambda}_1)^{\bar{n}_1}(\lambda-\bar{\lambda}_2)^{\bar{n}_2}\cdots \times(\lambda-\bar{\lambda}_k)^{\bar{n}_k}\,[\bar{c}_0+\bar{c}_1(\lambda-\bar{\lambda}_1)+\cdots].$$

If, therefore, we set $\bar{\mathrm{L}} = (|\bar{\lambda}_1|+\bar{\rho})(b-a)(K+E)$ and if ϵ is chosen to satisfy the condition,

$$\epsilon < \bar{c}\bar{\rho}^{n}/\{2^{1+\bar{n}_2+\cdots+\bar{n}_k}(|\bar{\lambda}_1|+\bar{\rho})(b-a)\,\Omega'(\bar{\mathrm{L}})\},$$

we obtain from (4), (5), (7), (8), and (33) the inequality,

$$(34) \qquad \left|(1/2\pi i)\int_{\bar{\Gamma}}[D(x,y,\lambda)/D(\lambda)-\bar{D}(x,y,\lambda)/\bar{D}(\lambda)]\,d\lambda\right| =$$

$$\left|(1/2\pi i)\int_{\bar{\Gamma}}\frac{D(x,y,\lambda)[\bar{D}(\lambda)-D(\lambda)]+D(\lambda)[D(x,y,\lambda)-\bar{D}(x,y,\lambda)]}{D(\lambda)\,\bar{D}(\lambda)}d\lambda\right|$$

$$< \frac{2^{2(n-\bar{n}_1+1)}\,\epsilon\bar{\rho}\{\bar{L}\,[\Omega'(\bar{L})]^2+\Omega(\bar{L})\Omega'(\bar{L})+\bar{L}\Omega(\bar{L})\Omega''(\bar{L})\}}{\bar{c}^2\rho^{2n}-2^{n-\bar{n}_1+1}\,\bar{c}\bar{\rho}^{n}\,\epsilon\,(|\bar{\lambda}_1|+\bar{\rho})(b-a)\,\Omega'(\bar{L})}.$$

It follows that (32) will hold if ϵ is made to satisfy the inequality,

$$(35) \quad \epsilon < \bar{c}^2\bar{\rho}^{2n-1}\eta_1/\Big[2^{2(n-\bar{n}_1+1)}\{\bar{L}\,[\Omega'(\bar{L})]^2+\Omega(L)\Omega'(\bar{L}) +\bar{L}\Omega(\bar{L})\Omega''(\bar{L})\}+2^{n-\bar{n}_1+1}\,\bar{c}\rho^{n-1}\,\eta_1(|\bar{\lambda}_1|+\bar{\rho})(b-a)\Omega'(\bar{L})\Big].$$

But we know that $\bar{c}$ approaches a limit not zero as $\bar{K}(x,y)$ approaches $K(x,y)$. Therefore, as on pp. 143-4, an ϵ can be found to satisfy (35).

Finally we shall show that given an η_2, an η_1 and consequently an ϵ can be found such that each $\bar{\phi}_p(x)$ $(p=1,2,\cdots,n)$ differs from a linear combination of the $\phi_p(x)$ by less than η_2, or in other words, each $\phi_p(x)$ differs from a solution of (1) by less than η_2. By Schwartz' inequality[*] we have

$$\int_a^b|\bar{K}(x,s)|\,|\bar{\phi}_p(s)|\,ds \leq \left|\left(\int_a^b|\bar{\phi}_p(s)|^2ds\int_a^b|\bar{K}(x,s)|^2ds\right)^{1/2}\right|$$

$$\leq\left|\left(\int_a^b|\bar{K}(x,s)|^2ds\right)^{1/2}\right|$$

$$<|\bar{K}(b-a)^{1/2}|,$$

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[*] E. Schmidt, "Zur Theorie der linearen und nichtlinearen Integralgleichungen," *Mathematische Annalen*, Vol. 63 (1907), p. 440.

where $|\bar{K}(x,y)|$ is $< \bar{K}$ in the region $a \leqq x \leqq b$, $a \leqq y \leqq b$. If we denote by $\bar{\lambda}$ a bound for $|\bar{\lambda}_i|$, $(i = 1, 2, \ldots, k)$, we then have

$$|\bar{\phi}_p(x)| \leqq \bar{\lambda} \int_a^b |\bar{K}(x,s)|\,|\bar{\phi}_p(s)|\,ds$$
$$< \bar{\lambda}\bar{K}\,|(b-a)^{1/2}| \qquad (p = 1, 2, \cdots, n).$$

Multiplying (32) by $|\bar{\phi}_p(y)|$ and integrating with respect to y, we obtain

$$(36) \qquad \left|\sum_{j=1}^{n} \phi_j(x) \int_a^b \psi_j(y)\,\bar{\phi}_p(y)\,dy - \bar{\phi}_p(x)\right| < \eta_1 \bar{K}\bar{\lambda}\,|(b-a)^{3/2}|.$$

Consequently the left hand member of (36) is less than η_2 provided we have

$$\eta_1 \bar{K}\bar{\lambda}\,|(b-a)^{3/2}| < \eta_2$$

or

$$(37) \qquad \eta_1 < \eta_2/\{\bar{K}\,\bar{\lambda}\,|(b-a)^{3/2}|\}.$$

We see that if ϵ is made to satisfy (35) and (37) in addition to (17) and the other inequality mentioned in § 2, we have

$$(38) \qquad \left|\sum_{j=1}^{n} \phi_j(x) \int_a^b \psi_j(y)\,\bar{\phi}_p(y)\,dy - \bar{\phi}_p(x)\right| < \eta_2 \qquad (p = 1, 2, \cdots, n).$$

In the same manner we can show that given an η_3, an ϵ can be found such that each $\phi_p(x)$ differs by less than η_3 from a suitable linear combination of the $\bar{\phi}_p(x)$. A similar discussion is possible for the $\psi_p(y)$ and $\bar{\psi}_p(y)$. Thus we have proved

THEOREM I. *Let* λ_0 *of multiplicity* n *and* $\bar{\lambda}_i$ $(i = 1, 2, \cdots, k)$ *of multiplicities* $\bar{n}_i$ *be corresponding characteristic values for* $K(x, y)$ *and* $\bar{K}(x, y)$ *respectively. If the corresponding resolvent kernels have only simple poles at these points, then given an arbitrarily small positive quantity* δ, *a positive number* ϵ *can be found such that, if* $|K(x, y) - \bar{K}(x, y)|$ *is* $< \epsilon$ *in the region* $a \leqq x \leqq b$, $a \leqq y \leqq b$, *the difference between each of the* n *normalized fundamental solutions of* (1) [*or* (2)] *for either kernel and a suitable linear combination of the* n *normalized fundamental solutions for the other is in absolute value less than* δ.

4. *The change in the fundamental solutions for the general case.* We shall consider briefly the general case in which no hypothesis is made regarding the orders of the poles of the resolvent kernels. As before let λ_0 be a zero of $D(\lambda)$ of multiplicity n, and $\bar{\lambda}_i$ of multiplicities $\bar{n}_i$ $(i = 1, 2,$ $\cdots, k)$ be the corresponding zeros of $\bar{D}(\lambda)$.

[illegible]

$$([illegible]) \qquad \int_\Gamma [D(x,y,\lambda) \quad D(\lambda)]\,d\lambda \qquad \sum_{j=1}^{n} \phi_j(x)\,\psi_j(y).$$

where the ϕ's and ψ's are called *fundamental functions*, r of the ϕ's being solutions of (1) and r of the ψ's solutions of (2) for $\lambda=\lambda_0$. Similarly, we have

$$(1/2\pi i)\int_{\bar{\Gamma}}[\bar{D}(x,y,\lambda)/\bar{D}(\lambda)]\,d\lambda=\sum_{j=1}^{n}\bar{\phi}_j(x)\,\bar{\psi}_j(y),$$

where the $\bar{\phi}$'s and $\bar{\psi}$'s are fundamental functions for $\bar{K}(x,y)$, $\bar{r}$ of the $\bar{\phi}$'s and $\bar{\psi}$'s being solutions of (1) and (2) respectively for $\bar{K}(x,y)$. As in § 3, the ϕ's and ψ's form a biorthogonal system [*]; so also do the $\bar{\phi}$'s and $\bar{\psi}$'s. It is a simple matter to normalize the ϕ's and $\bar{\phi}$'s; in what follows we shall suppose this to have been done.

Exactly as before we can show that given an η, an ϵ can be found such that, for

$$|K(x,y)-\bar{K}(x,y)|<\epsilon \qquad (a\leq x\leq b,\ a\leq y\leq b),$$

we shall have

$$(39)\qquad \Big|\sum_{j=1}^{n}\phi_j(x)\int_a^b\psi_j(y)\,\bar{\phi}_p(y)\,dy-\bar{\phi}_p(x)\Big|<\eta,$$

where $\bar{\phi}_p(x)$ is any one of the $\bar{r}$ fundamental solutions of (1) for $\bar{K}(x,y)$. Evidently this statement still holds if we interchange ϕ and $\bar{\phi}$, ψ and $\bar{\psi}$ and replace $\bar{r}$ and $\bar{K}(x,y)$ by r and $K(x,y)$ respectively. A similar discussion is possible for the $\psi_p(y)$ and $\bar{\psi}_p(y)$. Thus a proof is obtained for [†]

THEOREM II. *Let λ_0 of multiplicity n and $\bar{\lambda}_i$ $(i=1,2,\cdots,k)$ of multiplicities $\bar{n}_i$ be corresponding characteristic values for $K(x,y)$ and $\bar{K}(x,y)$ respectively. If no hypothesis is made regarding the orders of the poles of the resolvent kernels, then given an arbitrarily small positive quantity δ, a positive number ϵ can be found such that, if $|K(x,y)-\bar{K}(x,y)|$ is $<\epsilon$ in the region $a\leq x\leq b$, $a\leq y\leq b$, the difference between each of the normalized fundamental solutions of* (1) [*or* (2)] *for either kernel and a suitable linear combination of the n normalized* fundamental functions *for the other is in absolute value less than δ.*

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[*] Lalesco, *loc. cit.*, pp. 48, 54, 56.

[†] Since not every linear combination of the ϕ's is a solution of (1), we do not know here as in § 3 that each $\bar{\phi}_p(x)$ differs from a solution of (1) by less than η. Furthermore the inequality obtained from (39) by interchanging ϕ and $\bar{\phi}$, ψ and $\bar{\psi}$ does not enable us actually to find approximate solutions of (1) since we cannot calculate $\int_a^b\bar{\psi}_j(y)\phi_p(y)\,dy$. We simply know such approximate solutions exist.

Functions of *Écart Fini.*

BY H. E. BRAY.

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1. *Introduction.* A function $f(x)$ is said to be of *écart fini*[*] in the interval $-\pi \leq x \leq \pi$, if it is summable (Lebesgue) and if the integrals

$$\int_a^b f(x) n \sin nx\, dx, \quad \int_a^b f(x) n \cos nx\, dx$$

are bounded, independently of the integer n and the numbers a, b; where $-\pi \leq a < b \leq \pi$. This property is of importance in the study of the behavior of an analytic function in the neighborhood of its circle of convergence.[†]

In this paper we shall study some of the interesting conditions which are necessary or sufficient in order that a given function be of écart fini.

Without loss of generality we shall assume that $f(x)$ is periodic, of period 2π; then the property stated above is equivalent to the statement that the quantity,

$$\int_a^b f(x+k) \cos nx\, dx$$

is uniformly of the order $(1/n$, i. e., $O(1/n)$, for all n, k, a, b, where $|b-a| \leq 2\pi$. In fact n need not be an integer; for if $n = N + \theta$, where N is an integer and $0 \leq \theta \leq 1$ we have after integrating by parts

$$\begin{aligned}\int_a^b f(x) n \sin nx\, dx &= \int_a^b f(x)\,(N+\theta)\,(\sin Nx \cos \theta x + \cos Nx \sin \theta x)\, dx \\ &= (N+\theta) \Big[\cos \theta x \int_x^{-\pi} f(x) \sin Nx\, dx \Big]_a^b \\ &\quad + (N+\theta) \int^b \Big\{ \int_{-\pi}^x f(t) \sin Nt\, dt \Big\}\, \theta \sin \theta x\, d\theta \\ &\quad + \cdots\end{aligned}$$

Consequently if our integral is bounded when $n = N$, it is also bounded when $n = N + \theta$.

2. *Elementary properties.* We first recall the following elementary property:

THEOREM 1. *Every function $f(x)$, of bounded variation, $-\pi \leq x \leq \pi$, is a function of écart fini.*

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[*] Hadamard, *Journal de Liouville*, Ser. 4, Vol. 8 (1892), p. 154 et seq.

[†] Mandelbrojt, *The Rice Institute Pamphlet*, Vol. 14 (Oct., 1927), No. 4, p. 203 et seq.

For on integrating by parts, we have

$$\int_a^b f(x+k)n\cos nx\,dx = \sin nx\, f(x+k)\Big]_a^b + \int_a^b \sin nx\, d_x f(x+k).$$

The Stieltjes integral in the second member is numerically $\leqq T$, the total variation of f in the interval $-\pi \leqq x \leqq \pi$. Since the other term is evidently bounded, the theorem is proved.

It was this theorem which suggested to Hadamard the converse question, whether a function of écart fini is necessarily of bounded variation. We shall show later that this question must be answered in the negative, even if we consider continuous functions exclusively. As a corollary of the foregoing theorem we have the result:

If $f(x, \alpha)$ is uniformly of bounded variation in x, and uniformly bounded for all values of the parameter α, then $f(x, \alpha)$ is uniformly of écart fini.

THEOREM 2. *If $f(x)$ is of écart fini it is equivalent to a bounded function.*

We shall show, in fact, that the derivative of $F(x)$, the integral of $f(x)$, is bounded wherever this derivative exists, i. e. almost everywhere; from which it will follow that, except on a point set of measure zero, $f(x)$ is bounded.

Consider then the integral

$$\int_0^{\pi/2n} f(x+t)n\cos nt\,dt$$

which, by hypothesis, is bounded for all values of n and x. On integrating by parts we obtain, after setting $F(x+t) = F(x) + \int_0^t f(x+t)\,dt$,

$$-nF(x) + \int_0^{\pi/2n} F(x+t)n^2\sin nt\,dt$$
$$= \int_0^{\pi/2n} [F(x+t) - F(x)]\, n^2 \sin nt\,dt.$$

If now we suppose that the derivative of $F(x)$ exists at the point x and is equal to $f(x)$ we may replace $F(x+t) - F(x)$ by $t(f(x) + \eta(t))$ where $\eta(t)$ approaches zero with t. We obtain on integrating

$$\int_0^{\pi/2n} (f(x) + \eta(t))\,t\,n^2 \sin nt\,dt$$
$$= f(x) + \int_0^{\pi/2n} \eta(t)\; t\,n^2 \sin nt\,dt.$$

The last integral is not greater numerically than the upper bound of $\eta(t)$ in the interval $[0, \pi/2n]$, and it approaches zero as $n \rightarrow \infty$, since

$$\int_0^{\pi/2n} t\,n^2 \sin nt\,dt = 1.$$

Hence $f(x)$ is bounded at all points where $f(x) = dF(x)/dx$.

It is interesting to see whether the last theorem will be valid if we set aside the assumption that $f(x)$ is summable in the Lebesgue sense and assume merely that $f(x)$ is measurable and integrable in the Denjoy sense, and that the *Denjoy integral*

$$\int_a^b f(x+t) n \cos nt \, dt$$

is bounded for all n, x, a, b, $a \leq b \leq a + 2\pi$. Since Denjoy integrals are not in general absolutely convergent integrals it is conceivable that this integral could be bounded and yet permit $f(x)$ not to be equivalent to a bounded function. We shall show, however, that this is not possible. In our proof, we require the following lemma:

LEMMA. *If $f(x)$ is integrable (D) and if $\alpha(x)$ is a function of bounded variation, then*

$$\int_c^d dx \int_a^b f(x+t)\alpha(t)\,dt = \int_a^b \alpha(t)\,dt \int_c^d f(x+t)\,dx.$$

Since a Denjoy integral is a continuous function of its limits of integration the integral $\int_c^d f(x+t)\,dx = F(d+t) - F(c+t)$ is a continuous function of t. Let us approximate to $\alpha(t)$ by means of a bounded step function $\alpha_n(t)$ of uniformly bounded variation, where $\alpha_n(t)$ is constant on each of a finite number k_n of intervals, for each n; and $\lim \alpha_n(t) = \alpha(t)$; $a \leq t \leq b$.

Since the inner integral is continuous and $\alpha_n(t)$ is bounded we have, first,

$$\lim_{n=\infty} \int_a^b dt\,\alpha_n(t) \int_c^d f(x+t)\,dx = \int_a^b dt\,\alpha(t) \int_c^d f(x+t)\,dx.$$

Let

$$\alpha_n(t) = a_i^{(n)} \text{ when } t_i^{(n)} \leq t < t_{i+1}^{(n)},\ i = 1, 2, \cdots k_{(n-1)}$$

$$= a_{k_n}^{(n)} \text{ when } t_{k_n}^{(n)} \leq t \leq t_{k_{n+1}}^{(n)},$$

$$\text{where } t_1^{(n)} = a,\ t_{k_n+1}^{(n)} = b, \quad n = 1, 2, 3, \cdots.$$

Then

$$\int_a^b dt\,\alpha_n(t) \int_c^d f(x+t)\,dx = \sum_{i=1}^{k_n} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} dt\, a_i^{(n)} \int_c^d f(x+t)\,dx$$

$$= \int_c^d dx \left(\sum_{i=1}^{k_n} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} a_i^{(n)} f(x+t)\,dx \right)$$

$$= \int_c^d dx \int_a^b \alpha_n(t) f(x+t)\,dt$$

$$= \int_c^d dx \left\{ [\alpha_n(b)F(x+b) - \alpha_n(a)F(x+a)] - \int_a^b F(x+t)\,d\alpha_n(t) \right\}$$

the last step being obtained by the valid process of integrating by parts.[*] Now, as n becomes infinite, each term within the square bracket approaches boundedly a continuous function of x, and the Stieltjes integral approaches boundedly the Stieltjes integral in which $\alpha_n(t)$ is replaced by $\alpha(t)$, on account of the hypothesis on α_n.[†] Consequently

$$\begin{aligned}
&\lim \int_a^b dt\, \alpha_n(t) \quad \int_c^d f(x+t)\,dx \\
&= \int_d^c dx \quad \left\{ \alpha(b)F(x+b) - \alpha(a)F(x+a) - \int_a^b F(x+t)\,d\alpha(t) \right\} \\
&= \int_c^d dx \quad \int_a^b \alpha(t) f(x+t)\,dt.
\end{aligned}$$

Hence

$$\int_a^b dt\, \alpha(t) \quad \int_c^d f(x+t)\,dx = \int_c^d dx \quad \int_a^b f(x+t)\alpha(t)\,dt.$$

In order to prove that $f(x)$ is nearly everywhere bounded assuming that it is of écart fini in the wider (Denjoy) sense we proceed as follows. Consider the approximating function

$$f_\mu(x) = (1/\mu) \int_0^\mu f(x+t)\,dt.$$

It is well known that $\lim_{\mu=0} f_\mu(x) = f(x)$ almost everywhere when $f(x)$ is integrable (D),[‡] and also that $f_\mu(x)$ is a continuous function of x for each $\mu > 0$. Also

$$\begin{aligned}
&\int_a^b f_\mu(x+t)\, n \cos nt\, dt \\
&\qquad = (1/\mu) \int_a^b n \cos nt\, dt \int_0^\mu f(x+t+t')\,dt' \\
&\qquad = (1/\mu) \int_0^\mu dt' \quad \int_a^b f(x+t+t')\, n \cos nt\, dt
\end{aligned}$$

by the foregoing lemma. But the inner integral is, by hypothesis, uniformly bounded for every value of t', x, a, b. Hence its average value in the interval $0 \leq t' \leq \mu$ is uniformly bounded, which signifies that the function $f_\mu(x)$ is uniformly of écart fini. It follows, therefore, by Theorem 2, that $f_\mu(x)$ is uniformly bounded for all μ. Hence $f(x)$ which is equal almost everywhere to $\lim_{\mu=0} f_\mu(x)$ is bounded except on a set of measure zero. We have now proved that every function of écart fini in the Denjoy sense is nearly everywhere bounded, therefore integrable (L), and of écart fini in the ordinary sense.

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[*] Hobson, *The Theory of Functions of a Real Variable*, Second Edition, Vol. 1, p. 648.

[†] H. E. Bray, *Annals of Mathematics*, Vol. 20 (1919), p. 180.

[‡] Hobson, *loc. cit.*, p. 629 et seq.

3. *Necessary and sufficient condition.* In the next theorem we shall show that the condition that $f(x)$ be a function of écart fini may be expressed in terms of $F(x)$, the indefinite integral of $f(x)$, without introducing explicitly the trigonometric functions used in the definition.

Let δ be any positive number $\leq 2\pi$ and let m be any positive integer such that $m\delta \leq 2\pi$, and let

$$\begin{aligned} E(x, m, \delta) &= \sum_{r=1}^{m} (-1)^r \{F(x + r\delta) - F(x + \overline{r - 1}\delta)\} \\ &= F(x) - 2F(x + \delta) + 2F(x + 2\delta) - \ldots + (-1)^m F(x + md). \end{aligned}$$

THEOREM 3. *A necessary and sufficient condition in order that $f(x)$ be of écart fini is that $E(x, m, \delta)$ be uniformly of the order δ for all pairs of values of m, δ, subject to the condition $m\delta \leq 2\pi$ and for every x, $-\pi \leq x < \pi$.*

To prove that this condition is sufficient we observe first that if $F(x + \delta) - F(x) = O(\delta)$ it follows that $f(x)$ is bounded nearly everywhere and therefore in considering the boundedness of the quantity $n\int_a^b f(x + t) \cos nt\, dt$ a part of the interval $[a, b]$ whose length is $O(1/n)$ may be disregarded. We therefore consider the integral

$$\int_0^a f(x + t) \cos nt\, dt,$$

for our integral is the difference of two integrals of this kind, and let $a = m\pi/n$. We have, on integrating by parts,

$$\begin{aligned} \int_0^{m\pi/n} f(x + t) \cos nt\, dt = (-1)^m F(x + m\pi/n) - F(x) \\ + \int_0^{m\pi/n} F(x + t) n \sin nt\, dt. \end{aligned}$$

But

$$\begin{aligned} \int_0^{m\pi/n} F(x + t) n \sin nt\, dt \\ = (1/2) \int_0^{\pi/n} [F(x + t) + (-1)^{m-1} F(x + t + (m - 1)\pi/n)] n \sin nt\, dt \\ + (1/2) \int_0^{\pi/n} E(x + t, m, \pi/n) n \sin nt\, dt. \end{aligned}$$

Since $E(x+t, m, \pi/n)$ is $O(1/n)$ the last integral is $O(1/n)$ and we can write

$$\begin{aligned}
\int_0^{m\pi/n} & f(x+t)\cos nt\,dt \\
= (1/2) \int_0^{\pi/n} & [F(x+t)+(-1)^{m-1}F(x+t+(m-1)\pi/n)]n \sin nt\,dt \\
& -F(x)+(-1)^m F(x+m\pi/n)+O(1/n) \\
& = (1/2)\int_0^{\pi/n}[F(x+t)-F(x)]n\sin nt\,dt \\
& + \frac{(-1)^{m-1}}{2}\int_0^{\pi/n}[F(x+t+(m-1)\pi/n) \\
& \qquad\qquad -F(x+m\pi/n)]n\sin nt\,dt \\
& + O(1/n).
\end{aligned}$$

In the last two integrals the integrands are both bounded, since in the first integral, for example, $F(x+t)-F(x)=O(t)=O(1/n)$, since $0 \leqq t \leqq \pi/n$. Hence the last expression is $O(1/n)$. This establishes the sufficiency of the condition.

To show that our condition is necessary, we consider the periodic function $p(x)$ of period 2π, defined in the interval $-\pi \leqq x \leqq \pi$ as follows:

$$\begin{aligned}
p(x) &= 1, & 0 < x < \pi \\
&= -1, & -\pi < x < 0 \\
&= 0 \quad \text{otherwise.}
\end{aligned}$$

The Fourier expansion for $p(x)$ is given by

$$p(x) = (4/\pi)\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin(2m-1)x$$

and if n is a positive number, not necessarily an integer,

$$p(n\pi x) = (4/\pi)\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin n(2m-1)\pi x$$

and since $p(x)$ is of bounded variation we can calculate by termwise integration the integral

$$\int_a^b f(k+x)p(n\pi x)\,dx = (4/\pi)\sum_{m=1}^{\infty}\frac{1}{2m-1}\int_a^b \sin n(2m-1)\pi x\, f(k+x)\,dx.$$

Therefore

$$\left|\int_a^a f(k+x)p(n\pi x)\,dx\right| \leqq (4/n\pi)\sum_{m=1}^{\infty}\frac{K}{(2m-1)^2} \leq \frac{8K}{n\pi}$$

since the respective integrals in the second member are uniformly of the order $1/(n(2m-1))$. Consequently

$$E(k, r, 1/n) = \int_0^{r/n} f(k+x)\,p(n\pi x)\,dx$$

is of the order $1/n$, i. e. $E(k, r, \delta)$ is $O(\delta)$. The theorem is now proved.

The method used in proving Theorem 3 is applicable, if $f(x)$ is assumed to be bounded, to the case where the index m is such that the length of the interval of integration $m\pi/n$ is equal to 2π. We thus obtain a necessary and sufficient condition that the integral

$$\int_0^{2\pi} f(k+x)\,n\cos nx\,dx$$

remain bounded, in other words, a necessary and sufficient condition that the Fourier coefficients of a bounded measurable function be of the order $1/n$. We may therefore state without further proof:

THEOREM 4. *If $f(x)$ is bounded and measurable, then a necessary and sufficient condition that its Fourier coefficients,*

$$a_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

be of the order $1/n$, is that the quantity

$$E(x, 2n, \pi/n) = F(x) - 2F(x+\pi/n) + 2F(x+2\pi/n) - \ldots \\ - 2F(x+(2n-1)\pi/n) + F(x+2\pi)$$

be of the order $1/n$, uniformly for every x.

4. *Necessary conditions.* It is interesting to observe that in the latter part of the proof of Theorem 3, the only properties required of the function denoted by $p(x)$ are these:

i) $p(x)$ is periodic, of period 2π.

ii) $p(x)$ is of bounded variation in the interval $-\pi \leq x \leq \pi$, its total variation being $\leq T$,

iii) $\int_{-\pi}^{\pi} p(x)\,dx = 0$.

It then follows that the Fourier coefficients of $p(x)$ satisfy the inequalities

$$|a_k| \leq T/\pi k, \qquad |b_k| \leq T/\pi k.$$

As before, we have

$$\int_0^{2\pi} f(k+x)\,p(nx)\,dx = \sum_{p=1}^{\infty}\int_0^{2\pi} f(k+x)(a_p\cos pnx - b_p\sin pnx)\,dx$$

and, since $f(x)$ is of écart fini, it follows that a constant K exists, such that

$$\left| \int_a^b f(x) \cos nx \, dx \right| \leq K/n, \quad \left| \int_a^b f(x) \sin nx \, dx \right| \leq K/n.$$

Consequently,

$$\left| \int_a^b f(x) p(nx) dx \right| \leq \frac{2TK}{\pi n} \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \frac{4TK}{\pi n}.$$

We thus obtain

THEOREM 5. *A necessary condition in order that $f(x)$ be of écart fini, is that a constant K exist, such that*

$$\left| \int_a^b f(x) p(nx) dx \right| \leq 4TK/n\pi$$

where $p(x)$ is any function satisfying conditions i), ii) and iii).

5. *Sufficient conditions.* In order to show that a function of écart fini is not in general of bounded variation, even if continuous, nor equivalent to a function of bounded variation, we proceed to establish sufficient conditions which are easily applicable to a wide class of functions. These conditions apply to functions $f(x)$ possessing derivatives of the first order which are integrable, except in the neighborhood of a finite number of points in the interval $-\pi \leq x \leq \pi$. For example, we shall show that every function of the form $x^\alpha \sin x^{-\alpha}$, $\alpha > 0$, is of écart fini though its derivative is not summable in the whole interval $-\pi \leq x \leq \pi$ and therefore the function itself is not of bounded variation in the whole interval. In any interval which contains the point $x = 0$, the function is of unbounded variation. For the sake of clearness we will state our theorems for the case in which there is just one point in the neighborhood of which the function $f(x)$ is of unbounded variation. This point we shall take as the point $x = 0$.

THEOREM 6. *If $g(x)$ is a function of finite variation in the interval $\epsilon \leq x \leq 2\pi - \epsilon$; and if the total variation of $g(x)$ in this interval is $O(1/\epsilon)$; and if $f(x)$ is bounded and satisfies the equation,*

$$f(x) = f(x_0) + \int_{x_0}^{x} g(t) dt, \qquad 0 < x_0 \leqq x < 2\pi,$$

then $f(x)$ is of écart fini.

Consider the integral

$$\int_a^b f(x) n \cos n(x + k) dx.$$

Since $f(x)$ is bounded, we may discard a portion of the interval $[a, b]$ of magnitude of the order $1/n$. Hence we may assume $1/n \leq a < b \leq 2\pi - 1/n$. Integrating by parts we have

$$\int_a^b f(x) n \cos n(x+k)\, dx = f(x) \sin n(x+k) \Big] - \int_a^b \sin n(x+k) g(x)\, dx.$$

The term between limits is bounded since $f(x)$ is bounded. On integrating the second term by parts we obtain

$$\int_a^b \sin n(x+k) g(x)\, dx = -\cos n(x+k) \cdot g(x)/n \Big]_a^b + (1/n) \int_a^b \cos n(x+k)\, dg(x).$$

The term between limits is again bounded, since $g(x)$ is at most of the order n, its total variation being of the order n. The last integral is at most equal to $1/n$ times the total variation of $g(x)$ and is therefore bounded. The theorem is thus proved.

As an example of this type of function, consider

$$f(x) = \sin \log (1/x), \qquad 0 < x \leq 2\pi.$$

Here, $g(x) = -x^{-1} \cos (\log x^{-1})$, and the total variation of $g(x)$ in the interval $[\epsilon, 2\pi]$ is given by:

$$\int_\epsilon^{2\pi} | dg/dx |\, dx = \int_\epsilon^{2\pi} | (1/x^2) [\cos(\log x^{-1}) + \sin(\log x^{-1})] |\, dx < 2^{1/2}/\epsilon.$$

Evidently $f(x)$ is of unbounded variation in the interval $[0, 2\pi]$. It is of écart fini but is discontinuous at $x = 0$.

In the next theorem we relax somewhat the restriction on the order of the total variation of the function $g(x)$ while restricting the behavior of the function $f(x)$ in the neighborhood of the point $x = 0$.

THEOREM 7. *If $g(x)$ is a function of finite variation in the interval $[\epsilon, 2\pi]$, $0 < \epsilon < 2\pi$, its total variation being of the order $(1/\epsilon^\beta)$, and $f(x)$ is given by*

$$f(x) = f(x_0) + \int_{x_0}^x g(t)\, dt \qquad 0 < x_0 < x \leq 2\pi$$

and if $f(x)$ satisfies at $x = 0$ the condition,

$$| f(x) - f(0) | \leq Ax^\alpha, \qquad \alpha > 0,$$

[illegible]

We shall show that the integral

$$\int^b f(x) n \cos n(k+x)\, dx$$

is bounded; whether n is such that, first $0 < a < b \leq n^{-1/(1+\alpha)}$ or, second, $n^{-1/(1+\alpha)} \leq a < b \leq 2\pi$. In case $a < n^{-1/(1+\alpha)} < b$ the interval may be split up into two parts which come under these two cases. In the first case

$$\int_a^b f(x) n \cos n(k+x)\,dx = \int_a^b [f(x) - f(0)]\,d_x \sin n(k+x) \\ + f(0)\{\sin n(k+b) - \sin n(k+a)\}.$$

The last term is bounded. The integral in the second member is not greater numerically than max $| f(x) - f(0) |$ times the total variation of $\sin n(k+x)$, when $x \leq n^{-1/(1+\alpha)}$. But max $| f(x) - f(0) | \leq Ab^{\alpha} \leq An^{-\alpha/(1+\alpha)}$, and the total variation of the function $\sin nx$ in an interval of length $n^{-1/(1+\alpha)}$ is of the order $n^{\alpha/(1+\alpha)}$, being at most equal to $2 + 2n^{\alpha/(1+\alpha)}/\pi$. Hence the integral in question is less than $2An^{-\alpha/(1+\alpha)} + 2A/\pi$, which is less than $3A$.

Secondly, suppose $n^{-1/(1+\alpha)} \leq a < b$. We have

$$\int_a^b f(x) n \cos n(k+x)\,dx = f(x) \sin n(k+x) \Big]_a^b - \int_a^b g(x) \sin n(k+x)\,dx.$$

The first term is evidently bounded. The second we integrate again by parts:

$$\int_a^b g(x) \sin n(k+x)\,dx = -\cos n(k+x) \cdot g(x)/n \Big]_a^b \\ + (1/n) \int_a^b \cos n(k+x)\,dg(x).$$

The total variation of $g(x)$ in the interval $[n^{-1/(1+\alpha)}, 2\pi]$ is of the order $(n^{1/(1+\alpha)})^{\beta}$, by hypothesis, and is therefore $O(n)$ since $\beta \leqq 1 + \alpha$. Consequently the last integral is bounded. A fortiori, $g(x)$ itself, taken at the limits a, b, is $O(n)$; the quantity between limits is therefore bounded. This completes the proof of the theorem.

As an example of this theorem consider the function:

$f(x) = x^{\alpha} \sin x^{-\alpha}, \qquad 0 \leq x \leq 2\pi, \quad \alpha > 0.$

$g(x) = \alpha[x^{\alpha-1} \sin x^{-\alpha} - x^{-1} \cos x^{-\alpha}]$

$dg/dx = \alpha[(\alpha - 1)x^{\alpha-2} \sin x^{-\alpha} + (1 - \alpha)x^{-2} \cos x^{-\alpha} - \alpha x^{-2-\alpha} \sin x^{-\alpha}]$

$| dg/dx | < Kx^{-2-\alpha}.$

Consequently the total variation of $g(x)$ in the interval $[\epsilon, 2\pi]$ is less than $K(1+\alpha)(1/\epsilon^{1+\alpha})$, as required by the conditions of the theorem. Hence for every value of $\alpha > 0$, the function $f(x)$ is of écart fini.

The reader will see readily that the last two theorems could be generalized so as to cover cases in which, instead of one point in the neighborhood of which $f(x)$ is of unbounded variation, there are any finite number of such points in the interval $0 \leq x \leq 2\pi$, or even a denumerable infinitude of such points.

6. *Construction of functions of écart fini.* Having given a function of écart fini it is possible to construct another function of écart fini by multiplying the given one by a function $\alpha(x)$ of bounded variation. We have, in fact, on integrating by parts

$$\int_a^b \alpha(x)f(x)n\cos n(x+k)\,dx = \alpha(b)\int_a^b f(t)n\cos n(t+k)\,dt$$
$$-\int_a^b d\alpha(x)\int_a^x f(t)n\cos n(t+k)\,dt.$$

The first term of this expression is evidently bounded, since the quantity $\left|\int_a^b f(t)n\cos n(t+k)\,dt\right| \leq K$, a constant. The second term is not numerically greater than this constant times the total variation of $\alpha(x)$ in the interval $[a, b]$. We have thus proved

THEOREM 8. *If $f(x)$ is of écart fini, and if $\alpha(x)$ is of bounded variation, then the product $\alpha(x)f(x)$ is of écart fini.*

In a similar manner we can prove

THEOREM 9. *If $f(x)$ is a continuous function of écart fini then the function,*

$$g(x) = \int_{-\pi}^{x} \alpha(t)\,df(t), \qquad -\pi \leq x \leq \pi,$$

is a function of écart fini, continuous in the closed interval $-\pi \leq x \leq \pi$.[*]

For

$$g(x) = \alpha(t)f(t)\Big]_{-\pi}^{x} - \int_{-\pi}^{x} f(t)\,d\alpha(t).$$

The term between limits is of écart fini, by the preceding theorem. The Stieltjes integral is a function of bounded variation, which is also of écart fini. The fact that $g(x)$ is continuous appears from the fact that if $\alpha(t)$ is assumed to be monotone—this will not affect the argument—we can write, by the law of the mean,

$$g(x'') - g(x') = \alpha(x'')f(x'') - \alpha(x')f(x') - f(\xi)[\alpha(x'') - \alpha(x')]$$

where $x' \leq \xi \leq x''$, since $f(x)$ is continuous. Therefore

$$g(x'') - g(x') = \alpha(x')(f(\xi) - f(x')) + \alpha(x'')(f(x'') - f(\xi)),$$

and both of these terms approach zero with $x'' - x'$.

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[*] Of course, if the definition of $g(x)$ is extended so as to render it periodic the [illegible] defined, will not in general be continuous at $x = \pi$ or at congruent points.

7. *Fourier series.* In this section we shall study the order of approximation to a continuous function $f(x)$, of écart fini, by its Fourier sum S_n. For this purpose we make use of the quantity $\omega(\delta)$, called the *modulus of continuity* of $f(x)$, which is defined as follows:

$$\omega(\delta) = \max | f(x') - f(x'') |, \qquad | x' - x'' | \leq \delta.$$

For the sake of brevity we shall say that $f(x)$, of écart fini, *admits the constant K*, if K is such that the inequalities

$$\left| \int_a^b f(x+k) n \sin nx \, dx \right| \leq K, \left| \int_a^b f(x+k) n \cos nx \, dx \right| \leq K$$

are satisfied for all values of $n \geqq 1$, whether integers or not, where $| b - a | \leq 2\pi$, and where k is arbitrary.

We denote by R_n the difference $R_n = S_n - f$.

THEOREM 10. *If $f(x)$ is of écart fini, admitting the constant K, and if $| f(x) | \leq M$, then there exist positive constants A, B, independent of $f(x)$, such that*

$$| S_n(x) | \leq M(A \log \phi(n) + B) + 2K/\phi(n)$$

where $\phi(n)$ is any non-decreasing function such that $1 \leq \phi(n) \leq n$.

Consider the expression

$$S_n(x) = (1/\pi) \int_0^x [f(x+t) + f(x-t)] \frac{\sin (n + \frac{1}{2}t)}{2 \sin \frac{1}{2}t} dt$$

$$= (1/\pi) \int_0^{\phi(n)/n} + \int_{\phi(n)/n}^{\pi} [f(x+t) + f(x-t)] \frac{\sin (n+\frac{1}{2})t}{2 \sin \frac{1}{2}t} dt.$$

It follows that

$$| S_n(x) | \leq (M/\pi) \int_0^{\phi(n)/n} \left| \frac{\sin (n + \frac{1}{2})t}{\sin \frac{1}{2}t} \right| dt$$

$$+ (1/\pi) \left| \int_{\phi(n)/n}^{\pi} [f(x+t) + f(x-t)] \frac{\sin (n+\frac{1}{2})t}{2 \sin \frac{1}{2}t} dt. \right|$$

The first integral is not greater than

$$\int_0^{1/n} \left| \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} \right| dt + \int_{1/n}^{\phi(n)/n} (4/t) \, dt,$$

which is less than $(2n+1)/n + 4 \log \phi(n)$. Hence the first term is not greater than

$$(M/\pi)(3 + 4 \log \phi(n)).$$

If we apply the second law of the mean to the second integral, noting that $\csc (t/2)$ is a decreasing function, we find that the second term is equal to:

$$(1/2\pi)\csc(\phi(n)/2n)\int_{\phi(n)/n}^{t_n}[f(x+t)+f(x-t)]\sin(n+\tfrac{1}{2})t\,dt,$$
$$\phi(n)/n\leq t_n\leq\pi,$$

and since f is of écart fini this is not greater, numerically, than

$$(1/2\pi)\cdot(4n/\phi(n))\cdot 2K/(n+\tfrac{1}{2})\leq 2K/\phi(n).$$

These two results show that

$$|S_n|\leq(M/\pi)(4\log\phi(n)+3)+2K/\phi(n),$$

and our theorem is proved. As a corollary of this theorem we have

THEOREM 11. *Under the hypothesis of Theorem* 10, *there exist positive constants A, B, independent of f, such that*

$$|R_n|\leq M(A\log\phi(n)+B)+2K/\phi(n).$$

In fact

$$|R_n|=|S_n-f|\leq|S_n|+|f|$$
$$\leq M(A\log\phi(n)+(3/\pi)+1)+2K/\phi(n).$$

THEOREM 12. *If $f(x)$ possesses derivatives of the first and second orders, $f'(x)$ and $f''(x)$, and if $f''(x)$ is of écart fini, admitting the constant K'', and if $|f''(x)|\leq M''$, then there exists constants $A, B, > 0$, independent of $f(x)$, such that*

$$|R_n|\leq(M''/n^2)(A\log\phi(n)+B)+4K''/n^2\phi(n),$$

where $\phi(\mu)$ is any function satisfying the inequalities

$$1\leq\phi(n)\leq n,\qquad 1\leq\phi(n+1)/\phi(n)\leq(n+1)/n.$$

Let

$$S_n=\sum_0^\infty A_k,\qquad A_k=a_k\cos kx+b_k\sin kx,$$

and let the Fourier sum for $f''(x)$ be denoted by

$$\sigma_n=\sum_1^n B_k=-\sum_1^n k^2(a_k\cos kx+b_k\sin kx)$$
$$=-\sum_1^n k^2A_k.$$

We have

$$R_n=\sum_{n+1}^\infty A_k=-\sum_{n+1}^\infty\frac{B_k}{k^2}=-\sum_{n+1}^\infty\frac{1}{k^2}(\sigma_k-\sigma_{k-1})$$
$$=\frac{\sigma_n}{(n+1)^2}-\sum^\infty\sigma_k\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right).$$

Now, by Theorem 10,

$$|\sigma_k| \leq M''(A\log\phi(k) + B) + 2K''/\phi(k).$$

Hence

$$\begin{aligned}|R_n| \leq {} & [M''/(n+1)^2]\,(A\log\phi(n)+B) + [2K''/(n+1)^2\phi(n)] \\ & + BM''\sum_{n+1}^{\infty}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right) \\ & + 2K''\sum_{n+1}^{\infty}(1/\phi(k))\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right) \\ & + M''A\sum_{n+1}^{\infty}\log\phi(k)\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)\end{aligned}$$

The third term of these five is equal to

$$[BM''/(n+1)^2].$$

The fourth term is not greater than

$$\frac{2K''}{\phi(n+1)}\sum_{n+1}^{\infty}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)=\frac{2K''}{(n+1)^2\phi(n+1)}.$$

The last term can be written

$$\begin{aligned}& M''A\left(\frac{\log\phi(n+1)}{(n+1)^2}+\sum_{n+1}^{\infty}\log\frac{\phi(k+1)}{\phi(k)}\cdot\frac{1}{(k+1)^2}\right) \\ & \leq M''A\left(\frac{\log\phi(n+1)}{(n+1)^2}+\sum_{n+1}^{\infty}\frac{1}{k(k+1)^2}\right) \\ & \leq M''A\left(\frac{\log\phi(n+1)}{(n+1)^2}+\frac{1}{2n^2}\right)\end{aligned}$$

since

$$\log[\phi(k+1)/\phi(k)] \leq \log[(k+1)/k] \leq 1/k.$$

Hence, uniting these results, we find

$$|R_n| \leq (M''/n^2)[2A\log\phi(n)+2B+(A/2)] + [4K''/n^2\phi(n)],$$

and this expression is of the form specified in the theorem.

In order to prove the next theorem we shall make use of the approximating function *

$$f_\mu(x) = (1/\mu^2)\int_0^\mu dt\int_0^\mu f(x+t+t')\,dt'.$$

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* Some of the properties and applications of this kind of approximating function are given in an earlier paper of the author [*Bulletin of the American Mathematical Society*, Vol. 29, No. 6 (June, 1923)]. The present form is a natural development of a well known idea of Riemann.

It is easy to show that, if $f(x)$ is continuous,

$$(d/dx)f_\mu(x) = (1/\mu^2) \int_0^\mu [f(x+\mu+t) - f(x+t)]\, dt$$

$$(d^2/dx^2)f_\mu(x) = (1/\mu^2)[f(x+2\mu) - 2f(x+\mu) + f(x)].$$

The following facts are especially useful. If we assume that $f(x)$ is of écart fini, admitting the constant K, and that $f(x)$ is continuous and has $\omega(\delta)$ as its modulus of continuity, then

a) $f_\mu(x)$ is of écart fini and admits the constant K.
b) $(d^2/dx^2)f_\mu(x)$ is of écart fini and admits the constant $4K/\mu^2$.
c) $|(d^2/dx^2)f_\mu(x)| \leq 2\omega(\mu)/\mu^2$.
d) $|f_\mu(x) - f(x)| \leq \omega(2\mu)$.

We are now prepared to prove

THEOREM 13. *If $f(x)$ is a function of écart fini, admitting the constant K, and is continuous, having $\omega(\delta)$ as its modulus of continuity, then there exist positive constants A, B, C, independent of $f(x)$, such that*

$$|R_n| \leq \omega(1/n)(A \log \phi(n) + B) + CK/\phi(n),$$

where $\phi(n)$ is any non-decreasing function of n satisfying the inequalities

$$n \geq \phi(n) \geq 1, \quad (n+1)/n \geq \phi(n+1)/\phi(n) \geq 1.$$

We express $f(x)$ as the sum of two functions as follows:

$$f(x) = (f(x) - f_\mu(x)) + f_\mu(x).$$

Then

$$R_n = R_n' + R_n''$$

where R_n', R_n'' are the Fourier remainders of order n corresponding to $f(x) - f_\mu(x)$ and $f_\mu(x)$ respectively. Since $f(x) - f_\mu(x)$ is of écart fini, admitting the constant $2K$ we have, by Theorem 11, using the constants A, B, of Theorem 12,

$$|R_n'| \leq \omega(2\mu)(A \log \phi(n) + B) + 4K/\phi(n).$$

Since $f_\mu''(x)$, the second derivative of $f_\mu(x)$, is of écart fini, admitting the constant $4K/\mu^2$ and $|f_\mu''(x)| \leq 2\omega(\mu)/\mu^2$, we have by Theorem 12,

$$|R_n''| \leq (2\omega(\mu)/n^2\mu^2)(A \log \phi(n) + B) + (16K/n^2\mu^2\phi(n)).$$

Hence, if we choose $\mu = 1/2n$, we have,

$$|R_n''| \leq 8\omega(1/2n)(A \log \phi(n) + B) + (64K/\phi(n)),$$

and, since $\omega(1/2n) \leq \omega(1/n)$, we have

$$|R_n| \leq |R_n'| + |R_n''| \leq 9\omega(1/n)(A \log \phi(n) + B) + (68K/\phi(n)).$$

In other words, we have found constants $A, B, C, > 0$, independent of $f(x)$, such that

$$|R_n| \leqq \omega(1/n)(A \log \phi(n) + B) + (CK/\phi(n)).$$

The theorem which has just been proved, together with the method of proof, was suggested by a theorem given by de la Vallée Poussin[*] on the order of approximation to a continuous function by its Fourier sum, and proved by a method due to Dunham Jackson.[†] The formula arrived at in that case is

$$|R_n| \leqq \omega(1/n)(A \log n + B).$$

It is to be noted that the formula of Theorem 13 does not give any better result than the one just cited in the case where $\omega(\delta)$ is of the form $k\delta^\alpha$ $(0 < \alpha \leqq 1)$, since in this case, to apply our formula, we should have to take $\phi(n)$ of the same order as n^α (so as to make the two terms infinitesimals of about the same order). The main purpose of our theorem is to establish a formula applicable to cases in which the quantity

$$\omega(1/n) \log n$$

does not approach zero as n becomes infinite.

Given now any continuous function of écart fini we can choose the function $\phi(n)$ in such a way as to satisfy the conditions of Theorem 13 and also to satisfy the equations

$$\lim_{n=\infty} \phi(n) = \infty,$$
$$\lim_{n=\infty} \omega(1/n) \log\phi(n) = 0,$$

whence it follows that R_n approaches zero uniformly. Therefore we may state in conclusion

THEOREM 14. *The Fourier development of a continuous (periodic) function of écart fini converges uniformly to that function.*

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[*] C. J. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, Paris, Gauthier Villars, 1919, p. 25.

[†] Dunham Jackson, *Dissertation*, Göttingen, 1911, p. 40.

Singular Points of Vector Fields under General Boundary Conditions.[*]

By Marston Morse.

I.

1. *Introduction.* In his work on differential equations Poincaré[†] found it necessary to study the distribution of singular points of vector fields. Following Poincaré this study has been continued by many.[‡] Recently Hopf[§] and Lefschetz[¶] have made notable additions. The work of Hopf is based on Brouwer's[‖] fundamental notion of the degree of a transformation.

The theory as developed centers around the sum of the indices of the singularities. Up to the present paper this sum has been determined on general manifolds only for the special case where the vectors on the boundary are all interiorly or exteriorly directed. The evaluation for this case has been obtained both by Hopf and Lefschetz.

The principal advance of the present paper is the evaluation of this index-sum under general boundary conditions. The result obtained was suggested by an analogous problem in the study of critical points of functions.[**] The index-sum turns out to be a topological invariant not of the complex C_n on which the field is defined, but of a sequence of $n+1$ complexes associated with the field, a *sequence characterizing the field* as it will be called. A minor advance consists in giving a new definition of the index of infinite sets of singularities. This definition does not rest simply on an existence theorem, but is in a form immediately applicable to problems in analysis, for example, to the case where one has a regular curve of singular points.

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[*] Presented at the summer meeting of the Society at Amherst on September 6, 1928.

[†] Poincaré, "Sur les Courbes Définies par une Équation Differentielle," *Journal de Mathématiques*, Ser. 3, Vol. 7 (1881), pp. 375-421.

[‡] For the older references see von Kerékjarto, *Vorlesungen über Topologie*, I, pp. 265-266.

[§] Hopf I, "Curvatura integra geschlossener Hyperflächen," *Mathematische Annalen*, Vol. 95 (1925), pp. 340-351; Hopf II, "Abbildungsklassen n-dimensionaler Mannigfaltigkeiten," *Mathematische Annalen*, Vol. 96 (1296), pp. 209-224; Hopf III, "Vectorfelder in n-dimensionalen Mannighaltigkeiten," *Mathematische Annalen*, Vol. 96 (1926), pp. 225-249.

[¶] Lefschetz, "Manifolds with a Boundary and Their Transformations," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 429-463.

[‖] Brouwer, "Über Abbildung von Mannigfaltigkeiten," *Mathematische Annalen*, [illegible]

[**] Morse, "The Analysis and Analysis Situs of Regular n-Spreads in [illegible]," *Proceedings of the National Academy of Sciences*, Vol. 13, pp. 813-817. Here an abstract is given of results to be published in full later.

2. *The complexes. Let $\sum_n$ be a closed manifold in the sense of Analysis Situs.*[*] *We suppose also that $\sum_n$ possesses a Riemannian metric,*[†] that is, that every sufficiently small neighborhood of each point of $\sum_n$ is homeomorphic to the neighborhood of a point in an Euclidian space of coördinates $(x)=(x_1,\cdots,x_n)$ and thereby has a metric defined by a positive definite quadratic form,

$$ds^2=\sum_{ij} g_{ij}\,dx_i\,dx_j \qquad (i,j=1,2,\cdots,n).$$

We shall not assume that the g_{ij}'s are analytic in (x), but simply of class C'', that is, continuously differentiable up to the second order. The restriction that $\sum_n$ have a Riemannian metric could be removed by the substitution of suitable abstractions, but the writer is unwilling to sacrifice brevity, transparency, and applicability for the slight advantage thereby to be gained. Manifolds with a Riemannian metric include all the ordinary r-manifolds in Euclidian $(n+r)$-space, as well as the so-called characteristic manifolds of dynamical systems.

Let $(u)=(u_1,\cdots,u_r)$ be a set of rectangular coördinates in an auxiliary Euclidian r-space. By an *r-element* on $\sum_n$, $o<r\leqq n$, will be meant a set of points on $\sum_n$ homeomorphic with an r-simplex in the space (u). An r-element will be called *regular* if on the element the Riemannian coördinates (x) of the preceding section are functions of the u's of at least class C'', and if at least one set of r of the variables (x) has a jacobian with respect to the u's which is not zero. A *closed r-manifold* on $\sum_n$ will be called *regular* if each of its points possesses a neighborhood which is a regular r-element. An *r-manifold with boundary B_{r-1}* will be called *regular* if it is a connected subcomplex of a regular closed r-manifold, and if its boundary B_{r-1} consists of a finite set of distinct, regular, closed $(r-1)$-manifolds. An *r-complex* will be called *regular* if it consists of a finite set of distinct, regular r-manifolds, either with or without boundary.

3. *The vector field.* Let there be given on $\sum_n$ any regular n-manifold

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[*] Veblen, "Analysis Situs," *The Cambridge Colloquium,* Part II.

[†] Eisenhart, *Riemannian Geometry.*

The first deformation D_1 is one that can be affirmed to be possible by virtue of Lemmas A and B of § 4. In it we deform the vectors on Z_{n-1} at which $x_n = 0$, and only these, and for these vectors we leave the component h_n unchanged. The remaining components form a vector field, H_{n-1} on $x_n = 0$, within and on the $(n-2)$-sphere S_{n-2}, defined by the equality in (3). Without altering H_{n-1} on S_{n-2} we continuously deform H_{n-1} interior to S_{n-2} into a field K_{n-1} with the following properties. Its only singularity is to be at P_k. On the sphere S'_{n-2} concentric with S_{n-2} and with radius $r/2$, the vectors K_{n-1} are to be of length equal to $r/2$ and such that K_{n-1} defines a transformation of S'_{n-2} onto a similarly oriented unit $(n-2)$-sphere $S_{n-2}{}^0$, into a Riemann hypersurface of $|v_k|$ leaves on $S_{n-2}{}^0$, by a transformation whose degree is v_k. On each sphere concentric with S_{n-2} and of radius ρ less than $r/2$, the field is to be similar to that on S_{n-2} but with its vectors all of length ρ.

The second deformation D_2 is one in which we continuously change h_n on Z_{n-1} so as to make h_n a positive constant in a neighborhood N of P_k, instead of simply positive. This can be effected without altering h_n outside of a neighborhood of P_k slightly larger than N.

The vector field H'_n on Z_{n-1}, resulting from the deformations D_1 and D_2 defines a transformation F', of Z_{n-1} on a unit $(n-1)$-sphere R_{n-1}, whose index is the required index of Q_k. Under F' the point P_k goes into some point P on R_{n-1}, and the neighborhood of P_k on Z_{n-1} goes into a neighborhood of P on R_{n-1} covered $|v_k|$ times. Further, under F' no other point of Z_{n-1} goes into a sufficiently small sub-neighborhood of P, since no other point of Z_{n-1} besides P_k bears a vector of H_n parallel to that at P_k. Finally from the fact that the vector at P_k on Z_{n-1} is interiorly normal to Z_{n-1}, and from the conventions regarding orientations, it follows that the desired index of $Q_k = -v_k$. Thus the lemma is proved.

Suppose in the present section that the singularities of H_n are isolated.

The extended field H_n will have isolated singularities and be interiorly directed on the boundary of the extended domain C'_n. According to Theorem 1 the index-sum of the singularities of H_n on C_n equals (-1) times the Euler-Poincaré characteristic of C_n. Since the characteristic of C_n is clearly that of C'_n we have the following theorems.

THEOREM A_0. *If H_n is exteriorly normal to B_{n-1} at most at isolated points, the index-sum I_n of H_n is given as follows:*

$$I_n = (-1)^n E_n + I_{n-1} \qquad n > 0 \tag{4}$$

The field H_{n-1} will have isolated singularities at the above points P_k. We denote the index of H_{n-1} at P_k by v_k. For any two sets of our geodesic coördinates the component h_n of H_n at any point of B_{n-1} is the same function of the point on B_{n-1}. For h_n is readily seen to be the product of the length of H_n and the cosine of the angle at P between H_n and the geodesic g, using the terms length and angle in the senses of Riemannian Geometry. The component h_n on B_{n-1} is thereby proved continuous, and is accordingly less in absolute value than some positive constant M. We shall now prove the following lemma:

LEMMA. *It is possible to extend the definition of the field H_n beyond the boundary B_{n-1} of C_n to a geodesic manifold $x_n = e$, where e is a sufficiently small positive constant, in such a manner that the vectors H_n of the extended field are all directed interiorly on the new boundary, and the extended field has no new singularities other than isolated singularities Q_k of index $-v_k$, respectively, on the geodesics g_k exteriorly normal to B_{n-1} at the points P_k.*

Let e be an arbitrarily small positive constant. Our field H_n will now be extended as follows. For the points $(x_1, \cdots, x_n)$ for which

$$o < x_n \leqq e \tag{1}$$

the first $(n-1)$ components of H_n will be taken as equal to those of H_n at $(x_1, \cdots, x_{n-1}, 0)$, while for the same points we set

$$h_n(x_1, \cdots, x_n) = h_n(x_1, \cdots, x_{n-1}\ 0) - x_n(M/e). \tag{2}$$

The extended field H_n thereby defined is seen to have, beside the original singularities of H_n, an isolated singular point, Q_k, lying between B_{n-1} and the new boundary $x_n = e$, on the geodesic g_k normal to B_{n-1} at P_k.

It remains to show that the index of Q_k is $-v_k$.

To determine this index of H_n at Q_k we note that Q_k is interior to the region satisfying (1), and the following condition,

$$(x_1 - a_1)^2 + \cdots + (x_{n-1} - a_{n-1})^2 \leqq r^2, \tag{3}$$

where $(a_1, \cdots, a_{n-1}, 0)$ represents the point P_k and where r is so small a positive constant that h_n is positive on B_{n-1} throughout (3). Let Z_{n-1} be the boundary of this region. The index of Q_k is the degree v of that transformation F of Z_{n-1} on a unit $(n-1)$ sphere R_{n-1} which is defined by the vectors H_n on Z_{n-1}. This degree v will be unaltered if we subject H_n on Z_{n-1} to the two following continuous deformations, D_1 and D_2, thought of as made in the space (x).

where E_n is the characteristic of C_n, and I_{n-1} is the index-sum of the projection on B_{n-1} of the exteriorly directed vectors of H_n.

The theorem holds for $n=1$, if we agree that a vector at a boundary point a_0 of a one-dimensional manifold is normal to a_0, and projects on a_0 into a null vector of index 1.

On B_{n-1} let C_{n-1} be any regular complex with boundary B_{n-2}, that contains in its interior all of the points at which H_n is exteriorly normal to B_{n-1}, and contains none of points at which H_n is interiorly normal to B_{n-1}. Let H_{n-1} be the projection of H_n on C_{n-1}. Let E_{n-1} be the characteristic of C_{n-1}. We come now to the following theorem.

THEOREM A_1. *If the fields H_n and H_{n-1}, respectively, are exteriorly normal to B_{n-1} and B_{n-2} at most at isolated points, then*

$$I_n = (-1)^n (E_n - E_{n-1}) + I_{n-2}, \tag{5}$$

where I_{n-2} is the index-sum of the projection on B_{n-2} of the exteriorly directed vectors of H_{n-1}.

To prove this theorem we note first from (4) that

$$I_{n-1} = (-1)^{n-1} E_{n-1} + I_{n-2}, \tag{6}$$

and then use (4) and (6) to get (5).

The definitions preceding Theorem A_1 correspond to the case $i=1$ of the following definitions, given successively for $i=1, 2, \cdots, n$.

On B_{n-i} let C_{n-i} be a regular complex, with boundary B_{n-i-1} (possibly null), that contains (in its interior if $n \neq i$) all of the points at which H_{n-i+1} is exteriorly normal to B_{n-i}, and contains none of the points at which H_{n-i+1} is interiorly normal to B_{n-i}. Let H_{n-i} be the projection of H_{n-i+1} on C_{n-i}. Let E_{n-i} be the characteristic of C_{n-i}.

We come now to a succession of theorems of which Theorem A_1 is the first. Here $i = 1, 2, \cdots, n-1$

THEOREM A_i. *If the fields $H_n \cdots H_{n-i}$ are exteriorly normal to $B_{n-1} \cdots B_{n-i-1}$, respectively, at most at isolated points, then*

$$I_n = (-1)^n [E_n - E_{n-1} + \cdots + (-1)^n E_{n-i}] + I_{n-i-1},$$

where I_{n-i-1} is the index-sum of the projection on B_{n-i-1} of the exteriorly directed vectors of H_{n-i}.

If in particular H_{n-k} is never exteriorly normal to B_{n-k-1}, then I_{n-k-1} is zero, and C_j may be taken as a null complex for each $j < n-k$. Each Theorem A_i for which $i \geqq k$ will then give

$$I_n = (-1)^n \, [E_n - E_{n-1} + \cdots + (-1)^k E_{n-k}]$$

In Theorem A_{n-1} we note that I_0 must reduce to E_0 according to our conventions. We have then the following theorem.

THEOREM 2. *If the fields $H_n \cdots H_1$ are exteriorly normal respectively to $B_{n-1} \cdots B_0$ at most at isolated points, then*

$$I_n = E_0 - E_1 + \cdots + (-1)^n E_n,$$

where E_i is the characteristic of C_i, and I_n is the index-sum of H_n.

6. *General Boundary conditions.* The preceding hypothesis that the various fields of projections have isolated singularities will now be removed.

When a field H_r is defined over a complex C_r, H_r and C_r will be called a *corresponding* field and complex. In the preceding section the complexes C_r and fields H_r have the following properties.

They form a sequence of regular complexes

$$C_n C_{n-1} \cdots C_0, \tag{1}$$

and corresponding fields,

$$H_n H_{n-1} \cdots H_0, \tag{2}$$

of which each complex is on the boundary of its predecessor, and each field H_r is the projection of the preceding field on the complex C_r corresponding to H_r, while each complex is so chosen on the boundary of the preceding complex as to contain none of the points at which the preceding field is interiorly normal, and to contain in its interior all of the points on the boundary of the preceding complex at which the preceding field is exteriorly normal.

A sequence of complexes (1) with the properties of the preceding paragraph will be called a sequence of complexes characterizing H_n.

There are certain special cases that may arise in the determination of a sequence characterizing H_n.

If the field H_r is exteriorly normal over all of the boundary of C_r, C_{r-1} must necessarily be chosen as the complete boundary of C_r, and hence be itself without boundary. If C_{r-1} be for any reason without boundary, C_{r-2} must necessarily be chosen as a null complex. Or again if H_r is never exteriorly normal to the boundary of C_r, C_{r-1} may be chosen as a null complex. If for any reason C_r is chosen as a null complex we take all of the complexes

$$C_r \cdots C_0$$

as null complexes, and regard the corresponding fields as empty.

A sequence characterizing H_n always exists when the singularities of each of the fields H_r are isolated. We shall prove in § 8 that a sequence characterizing H_n always exists provided that C_n and H_n be given analytically, regardless of whether the singularities of the fields H_r are isolated or not.

We shall now prove a very general theorem in which the fields on the sequence of complexes characterizing H_n will no longer be assumed to have isolated singularities, the field H_n for the present alone excepted.

THEOREM 3. *If $C_0 C_1 \cdots C_n$ be a sequence of complexes characterizing H_n, and if the singularities of H_n are isolated, then*

$$I_n = E_0 - E_1 + \cdots + (-1)^n E_n, \qquad (n > 0)$$

where I_n is the index-sum of H_n, and *E_i is the characteristic of C_i.*

The method of proof will be that of mathematical induction.

If one recalls that E_0 is the number of points at which the field H_1 is directed toward the exterior of C_1 on the boundary of C_1, it readily appears that the theorem holds for $n = 1$.

Consider now the field H_{n-1} on C_{n-1}. In the special case where C_{n-1} is null Theorem 3 reduces to Theorem 2. If, on the other hand, C_{n-1} is not null, refer $\sum_n$ in the neighborhood of any point of B_{n-1} to geodesic coördinates (x), as in § 5. Let C_{n-1} be regularly subdivided so finely that the set $\bar{C}_{n-1}$ of $(n-1)$-cells of C_{n-1} that contain as interior or boundary points, points at which H_{n-1} has singularities will contain only points at which the n-th component h_n of H_n is positive. According to Lemma A § 4, the field H_{n-1} on B_{n-1} can be continuously deformed without change of H_{n-1} except on $\bar{C}_{n-1}$, into a new field which has within $\bar{C}_{n-1}$ at most a finite set π of singularities. In this deformation let t be the parameter, and vary from 0 to 1 inclusive. At the point $(x_1, \cdots, x_{n-1}, 0)$ of B_{n-1} let

$$\phi_i(x_1, \cdots, x_{n-1}, t) \qquad (i = 1, 2, \cdots, n-1)$$

be the components of the field at each instant t during the deformation.

We now return to H_n. Let e be an arbitrarily small positive constant. We shall extend the definition of H_n so as to include the points (x) for which

$$(1) \qquad 0 < x_n \leqq e$$

To define H_n for the points (1) we set

[illegible]

$$(3) \qquad h_n(x_1, \cdots, x_n) = h_n[x_1, \cdots, x_{n-1}, 0].$$

Denote the new boundary, $x_n = e$, by B'_{n-1}. The geodesics normal to B_{n-1} establish a homeomorphism F between B_{n-1} and B'_{n-1} in which points on the same geodesic correspond. Under F let C_j on B_{n-1} correspond to C_j' on B'_{n-1}. $j = 0, 1, 2, \cdots, n-1$.

Because h_n is positive on $\bar{C}_{n-1}$, and because of (3), the extension of H_n has been one in which no new singularities have been introduced. The points on B'_{n-1} at which H_n is exteriorly normal to B'_{n-1} are simply the images under F of the points of the set π.

Let H'_{n-1} be the projection of H_n on B'_{n-1}. The complexes

$$C'_{n-1}C'_{n-2} \cdots C_0', \tag{4}$$

and the field H'_{n-1} satisfy the hypotheses of Theorem 3 for the case where n is replaced by n-1. That is, the singularities of H'_{n-1} are isolated, for they occur only at the points of π, and the complexes (4) form a characterizing sequence for H'_{n-1}. According to the principle of mathematical induction we may use Theorem 3 to infer that

$$I'_{n-1} = E_0 - E_1 + \cdots + (-1)^{n-1}E_{n-1}, \tag{5}$$

where I'_{n-1} is the index-sum of H'_{n-1} on C'_{n-1}, and where E_i is the characteristic of C_i and hence of C_i's image C'_i.

But since H_n is exteriorly normal to B_{n-1} at most at isolated points, Theorem A_0 applies and gives

$$I_n = (-1)^n E_n + I'_{n-1} \, . \tag{6}$$

Theorem 3 now follows from (5) and (6).

If there be given any sequence of complexes

$$C_0C_1 \cdots C_n \tag{7}$$

it will be convenient to term the corresponding sum

$$E_0 - E_1 + \cdots + (-1)^n E_n \tag{8}$$

the alternating characteristic of the sequence (7).

7. *The generalized index of infinite sets of singularities.* The index of an isolated singularity as defined in § 4 will be here called an *elementary index.* The extension of the notion of index to infinite sets of singularities of H_n will be in two directions.

First consider any set s of singularities of H_n which are all interior to some regular element S_n with regular coördinates (x). In the space of the coördinates (x) suppose the points of s are all interior to a Jordan manifold

J_{n-1} which, together with its interior D_n, lies on S_n. Let R_{n-1} be a unit $(n-1)$-sphere in the space (x). We orient R_{n-1} and J_{n-1} similarly. The *generalized index* v of the singularities of s will be taken as the degree of that transformation F of J_{n-1} onto R_{n-1} which is defined by H_n on J_{n-1}. The proper use of this index requires a proof of the following.

The index v is independent of the choice of J_{n-1} among Jordan manifolds chosen as above.

For suppose J'_{n-1} and D'_n were, respectively, a second choice of a Jordan manifold and its interior domain. Let C_n be regularly subdivided so finely that the complex $\bar{C}_n$ which is made up of all n-cells that contain points of s on their interiors or boundaries, lies wholly interior to both D_n and D'_n. According to Lemma A, H_n can be replaced by a new field identical with H_n at the points exterior to $\bar{C}_n$ but having a finite set of singularities s' within $\bar{C}_n$.

In D_n one can chose a set C'_n of n-cells of a sufficiently fine regular division of C_n in such a manner that C'_n contains the points of s' as interior points and is connected with respect to n-cells. According to Lemma B the new field can now be replaced by a field K_n identical with the previous one except within C'_n where K_n will have just one singularity P, and that with an elementary index equal to the elementary index-sum, say v', of s'.

But in all these changes of the given field we have not altered the field on J_{n-1} so that the degree of the transformation F of J_{n-1} on R_{n-1} is the same for the original field as for the altered field. But for the altered field with its isolated singularity P this degree, by the very definition of an elementary index, is the elementary index of P, and must then equal v'. Thus the generalized index of s as determined from J_{n-1} equals v'. Similarly the generalized index of s as determined from J'_{n-1} will also equal v'. Thus the generalized index determined from J_{n-1} equals that determined from J'_{n-1}, and the proof is complete.

We can now extend the definition of the index beyond that of sets of singularities contained in a regular element. We suppose simply that the set s is interior to a regular sub-complex $\bar{C}_n$ of C_n upon which there are no other singularities than those of s, and that the field H_n on $\bar{C}_n$ possesses a characterizing sequence of complexes. *This sequence of complexes will also be said to characterize s.*

We now define the index of s to be the alternating characteristic of any [illegible] *complexes characterizing s.*

Practically this index is easy to determine, for example, for the case of a closed curve or surface of singularities in an analytic problem. One can in general determine the index of such sets of singularities if one knows the nature of nearby vectors. In the case of a closed curve in space, for example, one can enclose the curve in a torus-like surface, and by considering the vectors on this torus-like surface obtain a characterizing sequence, and hence an index. Theoretically one needs the justification of the following lemma.

The generalized index of a set of singularities is independent of the particular sequence of complexes used to characterize it.

Suppose C_j' and C_j'' $(j = 0, 1, \cdots, n)$ are two sequences of complexes characterizing fields H_n' and H_n'', both of which contain a set of singularities s, and no others.

As in the proof of the preceding lemma we can here replace the given vector field by a new one which does not differ from the old either on the boundary of C_n' or of C_n'', but which has on C_n' and C_n'' a finite set of singularities s^0. By virtue of Theorem 3 the alternating characteristics of our two characterizing sequences will both equal the elementary index-sum I_n of s^0, and thus be equal to each other. Thus the statement in italics is proved.

In case an index can be assigned to a set of singularities in more than one of the preceding ways, it is readily seen by means of the methods of the preceding proofs that the resulting index, or index-sum will always be the same.

If there be given on C_n any field which possesses a characterizing sequence, its singularities can always be grouped into a finite number of distinct sets each of which has a generalized index. The sum of these indices will be called the *generalized index-sum*. Our final theorem is the following.

THEOREM 4. *If there be given on C_n a vector field H_n which possesses a characterizing sequence, the generalized index-sum of the field equals the alternating characteristic of the characterizing sequence.*

To prove the theorem we consider separately each set s to which an index v has been assigned, and replace the given field by a new field which differs from the old only in the neighborhood of the points of s, and which in that neighborhood has at most isolated singularities. The latter will have an elementary index-sum equal to v. The application of Theorem 3 to the field thereby resulting will give the present theorem.

Corollary. If the alternating characteristic is not zero the field must have at least one singular point.

8. *Proof of the existence of sequences characterizing an analytic field.* We shall now prove the following.

If C_n and H_n are analytic a particular choice of a sequence of complexes characterizing H_n may always be made.

The cosine of the angle between H_n and the exterior normal to the boundary B_{n-1} of C_n will be an analytic function f of any regular analytic parameters of B_{n-1}. The domain on B_{n-1} in which f is positive will, in general, be bounded, if at all, by analytic manifolds without singularities, and in that case we can choose C_{n-1} of the characterizing sequence as the domain $f \geqq 0$. To proceed consider the projection H_{n-1} of H_n on C_{n-1}. Using H_{n-1} and C_{n-1} as we have just used H_n and C_n we arrive at a definition of C_{n-2}, and so on.

In the special case where the locus $f = 0$ is not free from singularities, the locus $f = e$ will be free from singularities if e be a sufficiently small positive constant. We distinguish between the cases $f \equiv 0$ and $f \not\equiv 0$.

If $f \equiv 0$ the vectors of H_n will all be tangent to B_{n-1}, and C_{n-1} may properly be chosen as a null complex. This is consistent with the fact that $f \geqq e$ is here never satisfied.

Suppose $f \not\equiv 0$. The points at which the locus $f = 0$ is singular satisfy

$$(1) \qquad f_{x_1} = f_{x_2} = \cdot \cdot \cdot = f_{x_n} = 0.$$

We regard $(x) = (x_1, \cdot \cdot \cdot x_n)$ for the moment as complex variables, with values neighboring real values. The points at which (1) is satisfied in the neighborhood of a real initial solution (x^0) of (1) will be made up, according to Weierstrass's second law[1] of the implicit fnuction theory, either of the point (x^0) alone, or of one or more connected "Gebilde" g. Each g has the property that any two points on it can be joined by a regular curve h along which the solution (x) of (1) is an analytic function of the independent variables of g except possibly at the end points of h. If (x^0) satisfies $f = 0$ we can infer from (1) that $f \equiv 0$ on each g.

The real points at which $f = e$ can never satisfy (1) if e be sufficiently small. For if such were the case one could use the Weierstrass-Bolzano cluster point theorem to prove the existence of a point on B_{n-1} at which f would be zero, and at which (1) would be satisfied, and in each of whose neighborhoods there would be points at which (1) would be satisfied and $f \neq 0$, contrary to the nature of the domains g just described.

Thus for a sufficiently small positive constant e, $f = e$ will be free from [illegible]ties, and $f \geqq e$ will be an admissible choice of C_{n-1}. We proceed [illegible] to the choice of C_{n-2}, C_{n-3}, etc.

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[1] Osgood, *Lehrbuch der Funktionentheorie I*.

9. *Examples.* Examples of the simplest sort will now be given to illustrate the application of Theorem 4. Each field is supposed null only at interior points of its domain.

Example 1. Let there be given a field interior to a circle, with the field directed exteriorly on one boundary semicircle C_1, but directed interiorly on the other. Suppose the projection of the field on the boundary is directed exteriorly to C_1 at C_1's end points. Then the index sum I_2 of the field is given as follows:

$$I_2 = E_0 - E_1 + E_2 = 2 - 1 + 1 = 2.$$

Example 2. Let the field be given within a sphere, and on the sphere be interiorly directed except within a ring-shaped region. Let the projection of the field on the sphere be directed into the ring on the boundary of the ring. Then

$$I_2 = E_0 - E_1 + E_2 - E_3 = 0 - 0 + 0 - 1 = -1.$$

Examples of greater complexity in higher dimensions can equally well be given.

The application of the results of this paper to the theory of fixed points of deformations, or to coincidences of transformations of the same class are sufficiently obvious to require no further elucidation.

A Complete Solution of LaPlace's Equation by an Infinite Hypervariable.

By P. W. KETCHUM.

1. *Introduction:* In a former paper* the writer has shown that most of the important theorems pertaining to analytic functions of a complex variable also hold for analytic functions of hypervariables provided the commutative and associative laws are satisfied. Also, while there is no variable of three units over the field of real numbers which is harmonic, i. e., such that every analytic function satisfies Laplace's equation, there are many restricted variables of more than three units which have that property. But in all cases the functions of hypervariables possess line singularities so that no hypervariable furnishes the totality of harmonic functions.

In that paper, however, the hypervariables were assumed to have a *finite* number of units. In the present paper a restricted hypervariable will be studied which possesses an infinity of units, and whose analytic functions form the totality of harmonic functions.

$$
\begin{array}{ccc|c|ccc}
 & & & \vdots & & & \\
\tfrac{1}{2}e_{-6} & \tfrac{1}{2}(-e_{-1}+e_{-5}) & \tfrac{1}{2}(e_{-2}+e_{-4}) & e_3 & \tfrac{1}{2}(-e_2+e_4) & \tfrac{1}{2}(e_1+e_5) & \tfrac{1}{2}(-e_0+e_6) \\
\tfrac{1}{2}(e_{-1}+e_{-5}) & \tfrac{1}{2}e_{-4} & \tfrac{1}{2}(e_{-1}+e_{-3}) & e_2 & \tfrac{1}{2}(-e_1+e_3) & \tfrac{1}{2}(e_0+e_4) & \tfrac{1}{2}(e_1+e_5) \\
\tfrac{1}{2}(-e_{-2}+e_{-4}) & \tfrac{1}{2}(-e_{-1}+e_{-3}) & \tfrac{1}{2}e_{-2} & e_1 & \tfrac{1}{2}(-e_0+e_2) & \tfrac{1}{2}(-e_1+e_3) & \tfrac{1}{2}(-e_2+e_4) \\
\hline
\cdot\ e_{-3}\ \cdot & e_{-2} & e_{-1} & e_0 & e_1 & e_2 & e_3\ \cdots \\
\hline
\tfrac{1}{2}(-e_2-e_4) & \tfrac{1}{2}(-e_1-e_3) & \tfrac{1}{2}(-e_0-e_2) & e_{-1} & \tfrac{1}{2}e_{-2} & \tfrac{1}{2}(e_{-1}+e_{-3}) & \tfrac{1}{2}(e_{-2}+e_{-4}) \\
\tfrac{1}{2}(e_1-e_5) & \tfrac{1}{2}(e_0-e_4) & \tfrac{1}{2}(-e_1-e_3) & e_{-2} & \tfrac{1}{2}(-e_{-1}+e_{-3}) & \tfrac{1}{2}e_{-4} & \tfrac{1}{2}(-e_{-1}+e_{-5}) \\
\tfrac{1}{2}(-e_0-e_6) & \tfrac{1}{2}(e_1-e_5) & \tfrac{1}{2}(-e_2-e_4) & e_{-3} & \tfrac{1}{2}(-e_{-2}+e_{-4}) & \tfrac{1}{2}(e_{-1}+e_{-5}) & \tfrac{1}{2}e_{-6} \\
 & & & \vdots & & &
\end{array}
$$

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* "Analytic Functions of Hypercomplex Variables," *Transactions of the American Mathematical Society*, Vol. 30 (1928) pp. 641-667. References and the meaning of the [illegible] are found in that paper. In Eq. 1 of that paper, the second semicolon should be a subscript *j*. On pp. 660 and 661, the equation $x_1 = x_1$ should read $x_i = x_i$. The ϵ_k's in (4) are defined by the relation $\epsilon = \sum \epsilon_k e_k$.

2. *Definition of the Variable:* The variable to be studied is

$$w = ze_{-1} + xe_0 + ye_1, \tag{1}$$

where e_{-1}, e_0 and e_1 are three of an infinite set of units whose laws of multiplication are given in the accompanying table, (e. g., $e_{-1}e_2 = \frac{1}{2}(e_{-1} + e_{-3})$, etc.), and e_0 is the modulus of the algebra. The table shows that, disregarding the signs, the subscripts of the units of the product of two units are the sum and difference of the subscripts of the factors.

A direct computation shows that the products in the table are obtained if we put*

$$e_n = i^n \cos n\alpha, \qquad e_{-n} = i^n \sin n\alpha, \qquad [i = (-1)^{\frac{1}{2}}, \quad n = 1, 2, \cdots], \tag{2}$$

where the functions on the right can be combined functionally in any way, but are not to be regarded as a collection of function values (that is, $\cos(m - n)\alpha$ cannot be identified with $\cos(n - m)\alpha$, etc.).

Let $a = \sum a_i e_i$ and $b = \sum b_i e_i$, be any two numbers of the algebra and let $c = \sum c_i e_i$ be the product ab. Then,

$$c_0 = \tfrac{1}{2}a_0 b_0 + \tfrac{1}{2}\sum(-1)^n a_n b_n.$$

From this relation it is evident that values of a and b exist for which c_0 is infinite, so that *the product of two numbers may not exist.*

If $\sum | a_i |$ converges, then a is said to be absolutely convergent. Consider two absolutely convergent numbers a and b. If $c = ab$, all the terms of c_k will be of the form $\gamma_{ijk} a_i b_j$ where the γ's are 0, 1, or $\pm \frac{1}{2}$. If each such term is replaced by its absolute value, the sum, d_k, will be greater than or equal to $| c_k |$ (if d_k and $| c_k |$ exist). Suppose the terms of the series $\sum d_k$ are formally rearranged so as to form the double series $\sum \lambda_{ij} | a_i b_j |$, where $\lambda_{ij} = \sum | \gamma_{ijk} |$. Then for i and j not zero and $i \neq -j$, $\lambda_{ij} = 1$; if either i or j vanishes, $\lambda_{ij} = 1$; and if $i = -j \neq 0$, $\lambda_{ij} = \frac{1}{2}$. Hence this double series is term by term less than or equal to the absolutely convergent series $\sum | a_i b_j |$, and must therefore be itself absolutely convergent. A valid rearrangement of the terms may now be carried out so as to change the double series back to the form $\sum d_k$, showing that this series converges absolutely. But this series dominates the series $\sum | c_k |$ which must as a result also converge absolutely, thus showing that the number c is absolutely convergent. Hence, *the product of two absolutely convergent numbers is absolutely convergent.*

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[*] The writer is much indebted to Prof. J. B. Shaw for these remarkable relations.

Any number having components only in a finite number of directions is absolutely convergent, so the product of any such number with any absolutely convergent number is absolutely convergent. Hence w^n *is absolutely convergent* for n any positive integer and for all values of w. Also, *if either a or b has components in only a finite number of directions, the product ab always exists,* since the expression for any of the components of ab will contain only a finite number of terms.

The multiplication table shows immediately that for any two numbers a and b, $ab = ba$ provided one of the products is absolutely convergent. It is also obvious when the units are expressed as in (2) that for any three numbers, a b and c, $(ab)c = a(bc)$ provided one member is absolutely convergent.

If division is defined in the ordinary manner *the quotient of two numbers is not unique.*

In the following, a three dimensional region or volume will be represented by τ or V, a two dimensional region or surface by S, and a one dimensional region or curve by C. Only curves and surfaces which are sufficiently smooth will be considered, and only those volumes which are bounded by such surfaces. Also, all volumes and surfaces will be assumed to be simply connected. Any multiply connected regions can be made simply connected by the introduction of cuts in the usual manner. The following discussion is limited to single valued functions, but most of the results will hold for multiple valued functions by first making the function single valued by the introduction of Riemann spaces in a manner analogous to the treatment of multiple valued functions in complex variables. For simplicity, all real (conjugate) functions will be taken as analytic, although most of the results undoubtedly could be stated for more general classes of functions.

3. *Differentiation:* Since division is not unique it might be expected that differentiation in the ordinary sense would fail, but such is not the case. If, for a function $f(w) = \sum f_i(x, y, z)$, we define the *derivative* (at a point w where the conjugate functions are analytic) by the relation $df = f'dw$, and require that *f' be independent of the value of dw,* there is obtained, by a method precisely the same as is used in obtaining the Sheffer's equations for a finite variable, the following system of equations:

$$(3A) \qquad \partial f_n / \partial x = f'_n ; \qquad (n = -\infty, \cdots, \infty)$$

$$(3B) \qquad \begin{cases} \partial f_{\pm n} / \partial y = \tfrac{1}{2}(f'_{\pm n-1} - f'_{\pm n+1}) ; \\ \qquad\qquad (n = 2, \cdots, \infty) \\ \partial f_{\pm 1} / \partial y = f'_0 - \tfrac{1}{2} f'_{\pm 2} ; \end{cases} \qquad \begin{matrix} \partial f_0 / \partial y = -\tfrac{1}{2} f'_1 ; \\ \\ \partial f_0 / \partial y = -\tfrac{1}{2} f'_{-1} . \end{matrix}$$

$$
\text{(3C)}\quad\begin{cases} \partial f_{\pm n}/\partial z = \pm \tfrac{1}{2}\,(f'_{\mp n\pm 1} + f'_{\mp n\mp 1})\,; & \partial f_0/\partial z = -\tfrac{1}{2}f'_{-1}\,; \\ \qquad\qquad (n = 2,\cdots,\infty) & \\ \partial f_1/\partial z = -\tfrac{1}{2}f'_{-2}\,; & \partial f_{-1}/\partial z = f'_0 + \tfrac{1}{2}f'_2. \end{cases}
$$

These are necessary and sufficient conditions that f' be independent of dw.

The functions f'_n can be eliminated from these equations in several ways, giving equations analogous to the Cauchy-Riemann or Sheffer's equations. The simplest method of elimination is the substitution of (3A) in (3B) and (3C), giving

$$
\text{(4A)}\quad\begin{cases} \partial f_{\pm n}/\partial y = \tfrac{1}{2}[(\partial f_{\pm n\mp 1}/\partial x) - (\partial f_{\pm n\pm 1}/\partial x)]\,; & \partial f_0/\partial y = -\tfrac{1}{2}(\partial f_1/\partial x)\,; \\ \qquad\qquad (n = 2,\cdots,\infty) & \\ \partial f_1/\partial y = \partial f_0/\partial x - \tfrac{1}{2}\partial f_2/\partial x\,; & \partial f_{-1}/\partial y = -\tfrac{1}{2}\partial f_{-2}/\partial x\,; \end{cases}
$$

$$
\text{(4B)}\quad\begin{cases} \partial f_{\pm n}/\partial z = \mp\tfrac{1}{2}[(\partial f_{\mp n\pm 1}/\partial x) + (\partial f_{\mp n\mp 1}/\partial x)]\,; & \partial f_0/\partial z = -\tfrac{1}{2}(\partial f_{-1}/\partial x)\,; \\ \qquad\qquad (n = 2,\cdots,\infty) & \\ \partial f_1/\partial z = -\tfrac{1}{2}(\partial f_{-2}/\partial x)\,; & \partial f_{-1}/\partial z = (\partial f_0/\partial x) + \tfrac{1}{2}(\partial f_2/\partial x). \end{cases}
$$

A function will be said to be *monogenic* if it possesses a derivative for all values of w in some region τ and if its conjugate functions are analytic in τ. In order for a function to be monogenic, it is necessary that its conjugate functions satisfy (4). These conditions, together with the analyticity of the conjugate functions, are also sufficient, since, if we put $f'_n = \partial f_n/\partial x$, (4) reduce to (3). Hence, *for any given set of real analytic functions $f_i(x, y, z)$, in order for the function $f(w) = \sum f_i$ to be monogenic, it is necessary and sufficient that the functions f_i satisfy Eqs. 4 for all values of x, y, and z, in some region τ.*

From (4), the following *necessary conditions for monogenicity* are obtained, by elimination of the x derivatives:

$$
\begin{aligned} &\partial f_{-1}/\partial y = \partial f_1/\partial z\,; \qquad \tfrac{1}{2}(\partial f_{-2}/\partial y) = \tfrac{1}{2}(\partial f_2/\partial z) - (\partial f_0/\partial z)\,; \\ \text{(5)}\quad &(\partial f_0/\partial y) + \tfrac{1}{2}(\partial f_2/\partial y) = -\tfrac{1}{2}(\partial f_{-2}/\partial z)\,; \\ &(\partial f_{\mp n\pm 1}/\partial y) + (\partial f_{\mp n\mp 1}/\partial y) = (\partial f_{\pm n+1}/\partial z) - (\partial f_{\pm n-1}/\partial z)\,; \quad (n = 2,\cdots,\infty). \end{aligned}
$$

and also the equations:

$$
\begin{aligned} &(\partial f_0/\partial x) - \tfrac{1}{2}(\partial f_1/\partial y) - \tfrac{1}{2}(\partial f_{-1}/\partial z) = 0\,; \\ &(\partial f_1/\partial x) + (\partial f_0/\partial y) - \tfrac{1}{2}(\partial f_2/\partial y) - \tfrac{1}{2}(\partial f_{-2}/\partial z) = 0\,; \\ \text{(6)}\quad &(\partial f_{-1}/\partial x) - \tfrac{1}{2}(\partial f_{-2}/\partial y) + (\partial f_0/\partial z) + \tfrac{1}{2}(\partial f_2/\partial z) = 0\,; \\ &(\partial f_{\pm n}/\partial x) + \tfrac{1}{2}(\partial f_{\pm n\mp 1}/\partial y) - \tfrac{1}{2}(\partial f_{\pm n\pm 1}/\partial y) \mp \tfrac{1}{2}(\partial f_{\mp n\pm 1}/\partial z) \\ &\qquad\qquad \mp (\partial f_{\mp n\mp 1}/\partial z) = 0. \qquad (n = 2,\cdots,\infty). \end{aligned}
$$

[illegible] 4A and the second equation of 4B, there is obtained step by step [illegible] functions f_i all defined in terms of f_0, as follows:

$$\begin{aligned}
&f_1 = 2\int_a^x (\partial f_0/\partial y)\,dx + \phi_1(y, z); && f_{-1} = -2\int_a^x (\partial f_0/\partial z)\,dx + \phi_{-1}[illegible]\\
(13)\quad &f_2 = 2f_0 - 2\int_a^x (\partial f_1/\partial y)\,dx + \phi_2; && f_{-2} = -2\int_a^x (\partial f_{-1}/\partial y)\,dx [illegible]\\
&f_{\pm n} = f_{\pm n \mp 2} - 2\int_a^x (\partial f_{\pm n \mp 1}/\partial y)\,dx + \phi_{\pm n}; && (n = 3 \cdots \infty).
\end{aligned}$$

[illegible] α is any point in τ and x any point in τ which can be reached fr[illegible] varying x only, and not passing outside of τ. If from the functions defined in (13), there is formed the function $f(w) = \sum f_n e_n$, in order for [illegible] to be a monogenic function it is sufficient that the functions f satisfy [illegible] 4 in a subregion of τ. By their definitions it is obvious that they must [illegible]fy Eqs. 4A. Substitution shows that they also satisfy Eqs. 4B if, [illegible] if, the functions ϕ_i satisfy the following conditions:

$$\begin{aligned}
&\partial\phi_1/\partial z = \partial\phi_{-1}/\partial y, && (\partial\phi_1/\partial y) + (\partial\phi_{-1}/\partial z) = 2(\partial f[illegible])_a;\\
(14)\quad &\partial\phi_2/\partial z = \partial\phi_{-2}/\partial y, && (\partial\phi_2/\partial y) + (\partial\phi_{-2}/\partial z) = 2(\partial f[illegible]);\\
&(\partial\phi_{\pm n}/\partial y) + (\partial\phi_{\mp n}/\partial z) = -2(\partial f_{\pm n \mp 2}/\partial y)_a && (n = 3 \cdots \infty).
\end{aligned}$$

[illegible] this set of equations, for every pair of functions ϕ_n and ϕ_{-n} there is a p[illegible] equations. Proceeding step by step it follows from the existence theorem[illegible] solutions of simultaneous first order partial differential equations* th[illegible] always exist functions ϕ_n which are analytic and satisfy (14) i[illegible] Functions f_n defined by Eqs. 13, therefore, always exist which satisfy Eqs. [illegible] therefore form a function which is monogenic in some subregion o[illegible] It follows that *the totality of conjugate functions of monogenic functions* [illegible] *include all harmonic functions which are analytic.* Hence, we h[illegible] obtained a complete solution of Laplace's equation in three variables in the same sense that the analytic functions of a complex variable form a complete solution of Laplace's equation in two variables.

6. *Expansion in series:* If

$$\lim_{w \to \alpha} f_i(x, y, z) = A_i,$$

for every value of i, then $f(w)$ will be said to have the limit A as w approaches α. A function $f(w)$ will be said to be continuous at a point α if its conjuga[illegible] functions are continuous at that point. A monogenic function is evidently continuous in its region of monogenicity. A series will be said to be convergent, absolutely convergent, or uniformly convergent if its conjugate series are all convergent, absolutely convergent or uniformly convergent, respectively. Also, the sum of a convergent series is defined by the equation,

---

* Goursat-Hedrick, *Mathematical Analysis*, Vol. 2, Pt. 2, p. 57.

$$f(w)=\sum_{j=0}^{\infty} U_j(w)=\sum_{i=-\infty}^{\infty}\ \sum_{j=0}^{\infty} U_{ji}(x,y,z)e_i.$$

From these definitions the following theorems are easily obtained:

1. A series which is uniformly convergent in a region and whose terms are analytic in τ defines a function which is continuous in τ.

2. If $f(w)=\sum\limits_{0}^{\infty} U_j(w)$, and if this series is uniformly convergent along a curve C, and if the terms are analytic on C, then

$$\int_C f(w)\,dw=\sum_{0}^{\infty}\int_C U_j(w)\,dw.$$

3. If, in a formal manner, the series $\sum U'_j(w)$ is formed, and if this derived series converges uniformly in a region τ, then for values of w in τ,

$$f'(w)=\sum_{0}^{\infty} U'_j(w).$$

4. A series which converges uniformly in a region τ, and whose terms are analytic in τ, defines a function which is monogenic in τ. This theorem can be proved with the aid of the analogue of Morera's theorem.

As a corollary to this theorem it follows that if a power series in $w-\alpha$ converges at every point of a region τ, which includes α in its interior, then $f(w)$ is monogenic in τ.

A function will be said to be *analytic* in a region τ if it can be expanded in a power series about any point in the interior of τ. A function will be called an analytic function if it is analytic in some region τ. From the above corollary it follows that *every function which is analytic in a region* τ *is monogenic within* τ; and every analytic function is also a monogenic function.

It can also be shown by a method entirely analogous to that used in finite hypervariables (l. c. Sec. 7) that if a function is monogenic in the neighborhood of a point α, it can be expanded in a power series in $w-\alpha$ which conerges in some region about α. Hence, *every function which is monogenic ithin a region is also analytic within that region*; and every monogenic unction is also an analytic function.

From the last theorem there follows at once by the method of real ariables the *Taylor expansion* of a function which is monogenic in the eighborhood of α, namely,

$$f(w)=\sum_{0}^{\infty}(e_0/j!)f^{(j)}(\alpha)(w-\alpha)^j; \tag{15}$$

he series converging for some region about α.

7. *Connection with spherical harmonics:* Let $(w^m)_n$ be the nth conjugate function of the function w^m. It can be shown by induction that these onjugate functions, when expressed in spherical coördinates (ρ, θ, ϕ) satisfy he following equations:

$$\begin{aligned} (w^m)_0 &= \rho^m P_m(\cos\theta), \\ (w^m)_{\pm n} &= \rho^m \frac{2m!}{(m+n)!} \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi \quad P_m{}^n(\cos\theta), \quad (n\neq 0), \end{aligned} \tag{16}$$

where the functions $P_m{}^n$ are the associated Legendre functions. The proof makes use of recurrence relations existing between the conjugate functions, which are easily obtained from the multiplication table; and the following ecurrence relations between the Legendre functions and their derivatives:

$$\begin{aligned} 2mP_m{}^{(i)}(\mu) = {}& (\mu^2-1)P_{m-1}{}^{(i+1)}(\mu)+2(m+i)\mu P_{m-1}{}^{(i)}(\mu) \\ & + (m+i)(m+i-1)\,P_{m-1}{}^{(i-1)}(\mu), \qquad (i\neq 0), \end{aligned}$$

where the superscripts indicate number of differentiations. This formula can be proved by applying Leibnitz' formula for the nth derivative of a product to Rodrigue's formula for Legendre functions.

Now consider a power series in w of the form

$$S(w)=\sum_{0}^{\infty}\alpha^{(m)}w^m.$$

The function $S_0(w)$, which may properly be called the real part of $S(w)$ is,

$$S_0(w)=\sum_{0}^{\infty}S_0{}^{(m)}(w),$$

where the functions $S_0^{(m)}(w)$ are homogeneous and of degree m in x, y and z. By direct computation,

$$S_0^{(m)}(w) = \tfrac{1}{2}\alpha_0^{(m)}(w^m)_0 + \tfrac{1}{2}\sum_{n=-m}^{m}(-1)^n\alpha_n^{(m)}(w^m)_n.$$

Substituting values from Eq. 16, it follows that $S_0^{(m)}(w)$ is equivalent to $\rho^m Y_m(\theta, \phi)$ where Y_m is the surface spherical harmonic of degree m.

Hence
$$S_0(w) = \sum_0^{\infty} \rho^m Y_m(\theta, \phi).$$

It has been shown that any harmonic function which is analytic about the origin can be expressed as a particular function $S_0(w)$. Hence, we have an independent proof of the theorem that any function which is harmonic and analytic in the neighborhood of the origin may be expanded in a series of spherical harmonics of the form $\sum_0^{\infty} \rho^m Y_m(\theta, \phi)$.

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Functions with Assigned Initial Values.

BY W. J. TRJITZINSKY.

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1. *Introduction.* By initial values of a function are meant the values of the function and of its successive derivatives at a point. The problem of constructing indefinitely differentiable functions of a real variable with assigned initial values at a point has been undertaken by de la Vallée Poussin in connection with the study of quasi-analytic functions; he gives the representation in the form $\sum_{j=1}^{\infty} \alpha_j \cos m_j x$. I have extended his method, introducing the representation of the form $\sum_{j=1}^{\infty} \alpha_j f(m_j x)$ ‡ where $f(x)$ is an even power series restricted in a certain way. In this paper three distinct methods are used for the solution of the problem in question; the first method [§ 2] giving an extension of both of the mentioned representations, and the third [§ [illegible]] having an interest of its own in the study of a type of essentially singular points. An asymptotic representation will be given in connection with the second method [§ 3].

2. *Representation in a Series of Power Series.* We introduce a definition.

DEFINITION. $f(x) = \sum_{k=0}^{\infty} b_k x^k$ *is of class* (A) *if for a fixed* h, $p = 1, 2, 3, \ldots$, *and all real values of* x,

$$|f(x)| \text{ and } |f^{(p)}(x)| < h, \; f(0) \neq 0 \text{ and } f^{(p)}(0) \neq 0.$$

Examples of functions of class (A) are $\cos x + \sin x$, e^{ix}. If $f_1(x)$ is a function of class (A), indefinitely many functions of that class can be constructed by taking

$$f(x) = \sum_{j=1}^{\infty} d_j f_1(p_j x), \tag{1}$$

where $\sum_{j=1}^{\infty} |d_j|$ converges, and the p_j are real numbers and $|p_j| \leq 1$ ($j = 1,$

C. de la Vallée Poussin, *The Rice Institute Pamphlet*, Vol. 12 (1925), No. 2, pp. 164-172.

[‡] W. J. Trjitzinsky, "Representation of Functions which are determined by initial Values," *Annals of Mathematics*, Vol. 29 (December, 1927), pp. 77-78.

$2, \cdots$); this is easily verified by taking the nth derivative of (1) and observing that it is bounded for all values of x and that its modulus is less than a constant independent of n.

We shall now determine the constants d_j so that the sum function $F(x) = \sum_{k=1}^{\infty} \alpha_k f(m_k x)$ has an assigned set of initial values $F(0)$, $F^{(n)}(0)$, at $x = 0$. Let $f(x)$ be of class (A), and let m_k be positive numbers, not necessarily integers, such that $m_{k+1}/m_k > \lambda > 1$, $(k = 1, 2, \cdots)$; also $F(0)$, $F^{(n)}(0)$ will be taken so that $| F(0)/f(0) |$, $| F^{(n)}(0)/f^{(n)}(0) | < G$, $(n = 1, 2, \cdots)$.

Consider

$$F_n(x) = a_1 f(m_1 x) + a_2 f(m_2 x) + \cdots + a_n f(m_n x). \tag{2}$$

Determine $a_1, a_2, \cdots, a_n$ by the equations:

$$\begin{aligned} F(0)/f(0) &= a_1 + a_2 + \cdots + a_n, \\ F^{(1)}(0)/f^{(1)}(0) &= m_1 a_1 + m_2 a_2 + \cdots + m_n a_n, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ F^{(n-1)}(0)/f^{(n-1)}(0) &= m_1^{n-1} a_1 + m_2^{n-1} a_2 + \cdots + m_n^{n-1} a_n. \end{aligned} \tag{3}$$

Solving this system by determinants we observe that a solution of equations of the form (3) has been already given by Vallée-Poussin *; we need only to replace $(-1)^j C_{2j}$, wherever they occur in his proof, by $F(0)/f(0)$, $F^{(j)}(0)/f^{(j)}(0)$, and m_j^2 by m_j; we have then

$$a_k = (1/2\pi i) \int_C \frac{\phi_n(z^2) \cdot \psi(z)\, dz}{(z^2 - m_k)\, \phi_n^{(1)}(m_k)}, \quad (k = 1, 2, \cdots, n), \tag{4}$$

where

$$\phi_n(z) = \prod_{j=1}^{n} [1 - (z/m_j)],$$

$$\psi(z) = \sum_{j=0}^{\infty} [F^{(j)}(0)/f^{(j)}(0)] \cdot 1/z^{2j+1}$$

and C is a circle having $z = 0$ for center and a radius $R \geqq G$. $(F^{(0)}(0) = F(0)$, $f^{(0)}(0) = f(0))$.

When $n \rightarrow \infty$, $\phi_n(z^2) \rightarrow \phi(z^2) = \prod_{j=1}^{\infty} [1 - (z^2/m_j)]$, which converges; hence, as $n \rightarrow \infty$,

$$\lim a_k = \alpha_k = (1/2\pi i) \int_C \frac{\phi(z^2)\psi(z)\, dz}{(z^2 - m_k)\phi^{(1)}(m_k)} \tag{5}$$

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* De la Vallée Poussin, *loc. cit.*, pp. 166-167.

and exists. Further, we find that

$$(6) \qquad |a_k| \text{ and } |\alpha_k| < h'\left(\frac{m_1 m_2 \cdots m_{k-1}}{m_k^{k-1}}\right),$$

where h' is a positive constant independent of k and n.*

The inequalities (6), together with the boundedness of $f(x)$ and of its successive derivatives for all real values of x insure absolute and uniform convergence of the sum function $\sum_{j=1}^{n} a_j f(m_j x)$ and of its successive derivatives for all values of n, and also that of the limiting series $\sum_{j=1}^{\infty} \alpha_j f(m_j x)$ and its successive derivatives. We are now enabled to formulate the following theorem:

THEOREM I. *Let m_j be a sequence of positive numbers such that for values of n*

$$m_{n+1}/m_n > \lambda > 1.$$

Let $f(x)$ be a function of class (A) and let $F(0)$, $F^{(j)}(0)$ $(j = 1, 2, \cdots)$ be numbers such that $|F(0)/f(0)|$ and $|F^{(j)}(0)/f^{(j)}(0)| < G$, a positive number. Coefficients α_j can then be so determined that the series

$$(7) \qquad F(x) = \alpha_1 f(m_1 x) + \alpha_2 f(m_2 x) + \cdots + \alpha_n f(m_n x) + \cdots$$

and all of its derivatives will converge, and the sum function will have $F(0)$, $F^{(1)}(0)$, $F^{(2)}(0), \cdots$ for initial values at $x = 0$. The coefficients α_j can be given in the form of complex integrals (5).

We shall state now that when

$$(8) \qquad \lim_{n\to\infty}\left(\frac{m_1 m_2 \cdots m_n}{m_{n+1}}\right) = 0$$

the above representation is unique; that is, having assigned a function $f(x)$ of class (A) it is impossible to have two distinct functions of the form $F(x) = \sum_{j=1}^{\infty} \alpha_j f(m_j x)$, with an assigned set of initial values $F(0)$, $F^{(j)}(0)$ $(j = 1, 2, \cdots)$, $|F(0)/f(0)|$, $|F^{(j)}(0)/f^{(j)}(0)| < G$, $|\alpha_n| < h'\left(\frac{m_1 \cdots m_{n-1}}{m_n}\right)$ (h' = constant). This can be proved by a method which is an immediate extension of the proof of a similar theorem concerning trigonometric series.†

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* De la Vallée Poussin, *loc. cit.*, pp. 168-169.

† De la Vallée Poussin, *loc. cit.*, pp. 170-171.

DEFINITION. *$F(x)=\sum_{j=1}^{\infty}\alpha_j f(m_j x)$ where $f(x)$ is a* *class (A), $|\alpha_n| < h\left(\frac{m_1 m_2 \cdots m_{n+1}}{m_n^{n-1}}\right)$ and $\lim_{n\to\infty}\left(\frac{m_1 n}{}\right.$* *be said to be of class (B).*

As the difference of two functions of class (B) is of to prove that, provided all the initial values are zero, a f vanishes identically. The coefficients $a_1, a_2, \cdots, a_n$ mu of equations:

$$
\begin{aligned}
R_0 &= -(a_{n+1} + a_{n+2} + \cdots) = a_1 + a_2 + \cdot \\
R_1 &= -(m_{n+1} \cdot a_{n+1} + m_{n+2} \cdot a_{n+2} + \cdots) \\
&\qquad = m_1 a_1 + m \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
R_{n-1} &= -(m^{n-1}{}_{n+1}\, a_{n+1} + m^{n-1}{}_{n+2}\, a_{n+2} + \cdots \\
&\qquad = m_1^{n-1} a_1 + m_2^{n-1} a_2
\end{aligned} \tag{9}
$$

We have, consequently

$$
(10)\quad a_k = (1/2\pi i)\int_C \frac{\phi_n(z^2)\cdot\sum_{j=1}^{\infty} R_j\cdot(1/z^{2j+1})}{(z^2 - m_k)\,\phi_n^{(1)}(m_k)}dz, \qquad (k
$$

The first term in R_{n-1} [that is, $(-m^{n-1}{}_{n+1}\, a_{n+1})$] appr which follows from the inequalities satisfied by a_j, α_j; on inequalities we have further that $\lim_{n\to\infty} R_{n-1} = 0$; it is ob $R_j \to 0$, as $n \to \infty$. Hence, it follows from (10) that $\lim_{n\to\infty}$ of k. This demonstrates that a function of class (B) wi is identically zero.

We formulate the following theorem:

THEOREM II. *Functions of class (B) are determin* *initial values at $x = 0$.*[*]

Let us return to Theorem I; the restriction on the $F^{(j)}(0)$ can be removed.[†] Suppose that $F(0), F^{(j)}(0)$ is if $|F(0)/f(0)|$, $|F^{(j)}(0)/f^{(j)}(0)|$ are bounded, Theor if not, then a function $G(x)$ can be constructed as follo

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[*] These functions are neither analytic nor quasi-anal trigonometric case.

[†] Compare with the method used on p. 164 of the Pan

$= \sum a_n f(m_n x)/m^n_n$. Differentiating in succession and letting $x = 0$ we [illegible] a set of formulas:

$$
(11)\quad
\begin{aligned}
&G(0)/f(0) - F(0)/f(0) = \sum_{n=1}^{\infty} a_n/m_n^n = M_0,\\
&G^{(1)}(0)/f^{(1)}(0) - F^{(1)}(0)/f^{(1)}(0) = \sum_{n=1}^{\infty} a_n/m_n^{n-1} = M_1,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&G^{(j)}(0)/f^{(j)}(0) - F^{(j)}(0)/f^{(j)}(0) = \sum_{n=j}^{\infty} a_n/m_n^{n-j} = M_j,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}
$$

provided that a_j satisfy the following set of equations:

$$
(12)\quad
\begin{aligned}
&m_1 a_1 = F^{(2)}(0)/f^{(2)}(0),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&m_1^{j-1}a_1 + m_2^{j-2}a_2 + \cdots + m_{j-1} \cdot a_{j-1} = F^{(j)}(0)/f^{(j)}(0),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}
$$

then m_j are chosen increasing so rapidly that $|a_j| \leq 1$. We observe then that M_0, M_1, $M_2, \cdots$ form a bounded sequence of numbers. Letting $M_j = X^{(j)}(0)/f^{(j)}(0)$, Theorem I can be applied for the construction of a function $X(x) = \sum_{j=1}^{\infty} \alpha_j f(m_j x)$ having $X^{(j)}(0)$ for initial values at $x = 0$. ($X^{(j)}(0) = M_j f^{(j)}(0)$). A function $F(x)$ with any assigned set of initial values $F(0)$, $F^{(1)}(0), \cdots$ can be represented by

$$
(13)\quad
\begin{aligned}
F(x) = G(x) - X(x) &= F(0) + \sum_{n=1}^{\infty} [(a_n/m_n^n) - \alpha_n]\, f(m_n x)\\
&= F(0) + \sum_{n=1}^{\infty} c_n f(m_n x).
\end{aligned}
$$

Thus the restriction on the initial values involved in Theorem I has been removed.

We shall consider now the order of approximation furnished by the representation (13); this will include the representation of Theorem I as a particular case. It will be noted that $|a_n| \leq 1$ and that m_j should and can be taken so that $m^{n+1}/m_n > \lambda > 1$. Since $|\alpha_n| < h(m_1 \cdots m_{n-1}/m_n^{n-1})$ it will then follow that

$$
(14)\qquad |C_n| < h'\left(\frac{m_1 \cdots m_{n-1}}{m_n^{n-1}}\right).
$$

Letting $R_n(x)$ denote the remainder after $(n+1)$ terms in (13) and using (14), we have

$$| R_n(x) | < h'M \left\{ \left(\frac{m_1 m_2 \cdots m_n}{m^n{}_{n+1}}\right) + \left(\frac{m_1 m_2 \cdots m_{n+1}}{m_{n+2}{}^{n+1}}\right) + \cdots \left(\frac{m_1 m_2 \cdots m_{n+j}}{m_{n+j+1}{}^{n+j}}\right) + \cdots \right\},$$

where $| f(x) | < M$ (for all values of x). Since $m_j/m_k < 1/\lambda < 1$ if $j < k$,

$$| R_n(x) | < h'M \{ (1/\lambda^n) + (1/\lambda^{n+1}) + \cdots + (1/\lambda^{n+j}) + \cdots \} < h/\lambda^n, \quad (h = \text{constant}).$$

We have therefore the following theorem:

THEOREM III. *The approximation furnished by the representation of Theorem I and, more generally, by the representation* (13), *is of the order of* $1/\lambda^n$, where $r > 1$.

3. *Representation in a Series of Infinite Integrals.* Let $g(m_j x) = \int_0^\infty \frac{e^{-t}dt}{1 + m_j x t}$, $x \geq 0$, $m_j > 0$; differentiation under the sign of integration with respect to x is possible, so that

$$(15) \qquad (d^n/dx^n)\, g(m_j x) = (-1)^n m_j{}^n n! \int_0^\infty \frac{t^n e^{-t} dt}{(1 + m_j x t)^{n+1}} \cdot$$

Letting $x = 0$, we derive

$$(d^n/dx^n)\, g(m_j x) = (-1)^n m_j{}^n n! \int_0^\infty t^n e^{-t} dt.$$

Since

$$\int_0^\infty t^n e^{-t} \underset{x=0}{dt} = \Gamma(n+1) = n!$$

$$(16) \qquad (d^n/dx^n)\, g(m_j x)\Big|_{x=0} = (-1)^n m_j{}^n (n!)^2 \text{ and } g^{(n)}(0) = (-1)^n (n!)^2.$$

From (16) it follows that the formal Taylor's expansion of $g(m_j x)$ at $x = 0$ diverges for all x. Also, we have

$$(17) \qquad | (d^n/dx^n)\, g(m_j x) | \leq | (d^n/dx^n)\, g(m_j x) |_{x=0} = m_j{}^n (n!)^2 \quad (x \geq 0)$$
$$\text{and } | g^{(n)}(x) | \underset{x=0}{\leq} (n!)^2.$$

Let m_k be positive numbers such that $m_{k+1}/m_k > \lambda > 1$, $(k = 1, 2, \cdots)$ and $F(0)$, $F^{(n)}(0)$ such that $| F(0) |$, $| F^{(n)}(0)/(n!)^2 | < G$, $(n = 1, 2, \cdots)$.

Consider

$$(18) \qquad F_n(x) = a_1 g(m_1 x) + a_2 g(m_2 x) + \cdots + a_n g(m_n x),$$

where $g(x) = \int_0^\infty \frac{e^{-t}dt}{1+xt}$; determine $a_1, a_2, \cdots, a_n$ by the equations:

$$
\begin{aligned}
&F(0) = a_1 + a_2 + \cdots + a_n,\\
&(-1)F^{(1)}(0)/(1!)^2 = m_1a_1 + m_2a_2 + \cdots + m_na_n,\\
(19)\quad &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&(-1)^{n-1}F^{(n-1)}(0)/[(n-1!)]^2 = m_1^{n-1}a_1 + \cdots + m_n^{n-1}a_n.
\end{aligned}
$$

These equations differ from the equations (3) in having their right members different; but, as in the other case, the left members are bounded as a set: therefore, coefficients a_k can be obtained, replacing the quantities $F(0)/f(0)$, $F^{(j)}(0)/f^{(j)}(0)$ in the formula (4) by $F(0)$, $(-1)^jF^{(j)}(0)/(j!)^2$: then

$$
(20)\qquad a_k = (1/2\pi i)\int_C \frac{\phi_n(z^2)\cdot\psi(z)\,dz}{(z^2-m_k)\phi_n^{(1)}(m_k)},
$$

where $\phi_n(z) = \prod_{j=1}^{n}[1-(z/m_j)]$, $\psi(z) = \sum_{j=0}^{\infty}\frac{(-1)^jF^{(j)}(0)}{(j!)^2}\frac{1}{z^{2j+1}}$ and C is a circle having $z=0$ for the center and a radius $R \geq G$; further,

$$
(21)\qquad \lim_{n\to\infty} a_k = \alpha_k = (1/2\pi i)\int_C \frac{\phi(z^2)\psi(z)\,dz}{(z^2-m_k)\phi^{(1)}(m_k)}
$$

where $\phi(z^2) = \lim_{n\to\infty}\phi_n(z^2) = \prod_{j=1}^{\infty}[1-(z^2/m_j)]$ and converges, which accounts for the existence of α_k. When $n\to\infty$, $F_n(x)\to F(x)$ and (18) will become

$$
(22)\qquad F(x) = \alpha_1 g(m_1x) + \alpha_2 g(m_2x) + \cdots + \alpha_n g(m_nx) + \cdots.
$$

This process can be justified as follows:

Since the numbers a_k and α_k satisfy the inequalities (6), it follows that (18) and its successive derivatives converge absolutely and uniformly for all values of n; the same is true of the limiting series (22) (x is restricted to the values $x \geq 0$). Here, however, the function $g(x)$, though possessing the property that $g(x)$, $g^{(n)}(x)$, $(n = 1, 2, \cdots)$, are each bounded when $x \geq 0$, does not possess the property of the functions of class (A) (used in the representation of Theorem I) of having all of its successive derivatives bounded as a set. A theorem similar to Theorem I can now be formulated.

THEOREM IV. *Let m_j be a sequence of positive numbers such that for all n,*

$$
m_{n+1}/m_n > \lambda > 1.
$$

Let $g(x) = \int_0^\infty \frac{e^{-t}dt}{1+xt}$ and let $F(0)$, $F^{(j)}(0)$, $(j = 1, 2, \cdots)$, be num-

bers such that $|F(0)|$, $|F^{(j)}(0)/(j!)^2| < G$, *a positive number. Coefficients* α_j *can then be so determined that the series*

$$F(x) = \alpha_1 g(m_1 x) + \alpha_2 g(m_2 x) + \cdots + \alpha_n g(m_n x) + \cdots \tag{23}$$

and all its derivatives will converge when $x \geq 0$, *and the sum function will have* $F(0)$, $F^{(1)}(0), \cdots$ *for initial values at* $x = 0$. *The coefficients* α_j *can be given by* (21).

DEFINITION. $F(x) = \sum_{j=1}^{\infty} \alpha_j g(m_j x)$ *where* $|\alpha_j| < h\left(\frac{m_1 m_2 \cdots m_{j-1}}{m_j{}^{j-1}}\right)$ *and* $\lim_{n\to\infty}\left(\frac{m_1 m_2 \cdots m_n}{m_{n+1}}\right) = 0$ *will be said to be of class* (C).

The difference of two functions of class (C) is of class (C); in proving that a function of class (C) with zero initial values vanishes identically, we arrive at the set of equations (9); whence, the proof is exactly the same as that by which Theorem II was derived. Hence, we have a theorem:

THEOREM V. *Functions of class* (C) *are determined uniquely by their initial values at* $x = 0$.

In removing the restriction on the initial values $F(0)$, $F^{(j)}(0)$ in Theorem IV, we let $F(0)$, $F^{(j)}(0)$ be any set of numbers and construct a function

$$G(x) = F(0) + \sum_{n=1}^{\infty} a_n\, g(m_n x)/m_n{}^n;$$

differentiating in succession and letting $x = 0$ we have two sets of equations differing from (11) and (12) only in having $F(0)$, $(-1)^j F^{(j)}(0)/(j!)^2$ and $G(0)$, $(-1)^j G^{(j)}(0)/(j!)^2$ in place of $F(0)/f(0)$, $F^{(j)}(0)/f^{(j)}(0)$ and $G(0)/f(0)$, $G^{(j)}(0)/f^{(j)}(0)$, respectively. Letting $M_j = (-1)^j N^{(j)}(0)/(j!)^2$ and having M_j bounded, as a set, we construct by the aid of Theorem IV a function $N(x) = \sum_{j=1}^{\infty} \alpha_j g(m_j x)$ having $N(0)$, $N^{(j)}(0)$ for its initial values at $x = 0$. Therefore, a function $F(x)$ with any assigned set of initial values $F(0)$, $F^{(j)}(0)$ can be given by

$$\begin{aligned} F(x) = G(x) - N(x) &= F(0) + \sum_{j=1}^{\infty} (a_j/m^j{}_j - \alpha_j) g(m_j x) \\ &= F(0) + \sum_{n=1}^{\infty} c_n g\ (m_n x). \end{aligned} \tag{24}$$

This removes the restriction on initial values in Theorem IV.

We have

$$|c_n| < h'\left(\frac{m_1 \cdots m_{n-1}}{m_n^{n-1}}\right). \tag{25}$$

Letting $R_n(x)$ denote the remainder after $(n+1)$ terms in (23) or (24), we have

$$|R_n(x)| < h'\left\{\left(\frac{m_1 m_2 \cdots m_n}{m^n_{n+1}}\right) + \cdots\right\}, \text{ since } |g(x)| \leqq 1, \ (x \geqq \ ).$$

Hence

$$|R_n(x)| < h/\lambda^n, \qquad (\lambda > 1, x \geqq 0).$$

Therefore we have

THEOREM VI. *The approximation furnished by the representations in Theorem* IV *and* (24) *is of the order of* $1/\lambda^n$ $(\lambda > 1)$.

Note. As in Theorem III the approximation can be made closer by making the constants m_j increase more rapidly, for then λ is greater.

We have

$$|g^{(n)}(x)| = n!\int_0^\infty \frac{t^n e^{-t}dt}{(1+xt)^{n+1}} < n!\left(\int_0^\infty t^n e^{-t}dt\right)\cdot\left(\int_0^\infty dt/(1+xt\right.$$

$$=(n!)^2 \int_0^\infty dt/(1+xt)^{n-1}.$$

Letting $1 + xt = v$ and $x > x_0 > 0$,

$$|g^{(n)}(x)| < (n!)^2 \int_1^\infty \frac{dv}{xv^{n+1}} = \frac{n!(n-1)!}{x} \tag{26}$$

$$\text{and } |g^{(n)}(m_j x)| < \frac{n!(n-1)!}{m_j x}$$

Hence,

$$|F^{(n)}(x)| = \left|\sum_{j=1}^\infty m_j^n \alpha_j g^{(n)}(m_j x)\right| < \sum_{j=1}^\infty |m_j \alpha_j g^{(n)}(m_j x)| \tag{27}$$

$$< \frac{n!(n-1)!}{x} \sum_{j=1}^\infty m_j^{n-1} |\alpha_j| = P_n/x.$$

The constants P_n are bounded for every finite n on account of the inequalities satisfied by α_j; but as a set they are not necessarily bounded.

From (27) it follows that $\lim_{n\to\infty} F^{(n)}(x) = 0$ (not necessarily uniform with respect to n).

Consequently, it is shown that *it is possible to construct functions of a real variable with any assigned set of initial values at* $x = 0$, *such that all* *successive derivatives of finite orders are zero at infinity.* All the representations of section (3) of this paper possess this property.

With m_j restricted as before and α_j satisfying inequalities (6), functions of the form

$$F(x)=\sum_{j=1}^{\infty}\alpha_j\int_0^{\infty}\frac{e^{-t}dt}{1+m_jxt} \qquad (x\geqq 0) \tag{28}$$

are such that $F(1/x)$ can be represented asymptotically for large values of x; i. e., functions of this section can be so represented.

We observe that

$$\int_0^{\infty}e^{-t}\left(\sum_{j=0}^{\infty}\frac{\alpha_j}{1+m_jxt}\right)\ dt=\sum_{j=1}^{n}\alpha_j\int_0^{\infty}\frac{e^{-t}dt}{1+m_jxt}$$
$$+\int_0^{\infty}e^{-t}\left(\sum_{j=n+1}^{\infty}\frac{\alpha_j}{1+m_jxt}\right)\ dt.$$

Thus

$$\left|\int_0^{\infty}e^{-t}\left(\sum_{j=1}^{\infty}\frac{\alpha_j}{1+m_jxt}\right)dt-\sum_{j=1}^{n}\alpha_j\int_0^{\infty}\frac{e^{-t}dt}{1+m_jxt}\right|$$
$$=\left|\int_0^{\infty}e^{-t}\left(\sum_{j=n+1}^{\infty}\frac{\alpha_j}{1+m_jxt}\right)dt\right.$$
$$\leqq\{\,|\alpha_{n+1}|+|\alpha_{n+2}|+\cdots\}=\epsilon_n,$$

where ϵ_n is independent of x and can be made arbitrarily small and $x\geqq 0$.

Hence

$$\lim_{n\to\infty}\sum_{j=1}^{n}\alpha_j\int_0^{\infty}\frac{e^{-t}dt}{1+m_jxt}=F(x)=\int_0^{\infty}e^{-t}\left(\sum_{j=1}^{\infty}\frac{\alpha_j}{1+m_jxt}\right)dt. \tag{29}$$

Using this expression for $F(x)$ we integrate by parts n times and obtain

$$F(x)=L(0)+\mathrm{L}^{(1)}(0)\cdot x+\cdots+L^{(n)}(0)\cdot x^n \tag{30}$$
$$+\,x^{n+1}\int_0^{\infty}e^{-t}L^{(n+1)}(xt)\,dt,$$

where $L(xt)=\sum_{j=1}^{\infty}\frac{\alpha_j}{1+m_jxt}$ and hence $L^{(n)}(0)=(-1)^n n!\sum_{j=1}^{\infty}\alpha_j m_j{}^n$.

Further, it can be verified that

$$|\,L^{(n+1)}(xt)\,|\leqq(n+1)\,!\sum_{j=1}^{\infty}|\,\alpha_j\,|\,m_j{}^{n+1}=G_{n+1},$$

so that

$$|\int_0^{\infty}e^{-t}L^{(n+1)}(xt)\,dt\,|\leqq G_{n+1}. \tag{31}$$

On account of the inequalities satisfied by α_j, it follows that the quantities $L(0)$, $L^{(j)}(0)$ are bounded for all finite values of n, but are not necessarily bounded as a set. (If α_j are all positive the constants $L(0)$, $L^{(n)}(0)$ are certainly not bounded as a set). The G_n are unbounded as a set.

Using (30) and (31) we have

$$(32)\quad |F(x)-\{L(0)+L^{(1)}(0)\cdot x+\cdots+L^{(n)}(0)\cdot x^n\}|\leq x^{n+1}\cdot G_{n+1}.$$

Replacing x by $1/x$ we have

$$(33)\quad x^n\,|F(1/x)-\{L(0)+L^{(1)}(0)\cdot 1/x+\cdots+L^{(n)}(0)\cdot 1/x^n\}|\leq G_{n+1}$$

Therefore for large values of x

$$(34)\quad F(1/x)\sim\sum_{n=0}^{\infty}L^{(n)}(0)/x^n=\sum_{n=0}^{\infty}\frac{(-1)^n n!\sum_{j=0}^{\infty}\alpha_j m_j^{\,n}}{x^n},\quad (L^{(0)}(0)=L(0)),$$

asymptotically;[*] equation (34) amounts to the statement that for small values of x the function $F(x)$ can be computed from a finite number of terms of the series $\sum_{n=1}^{\infty} L^{(n)}(0)\cdot x^n$.

4. *Representation at an Essentially Singular Point.* We shall have $z=0$ as an essentially singular point; let it be a limiting point of a sequence of poles situated on the positive half of the imaginary axis. The problem will be to construct a function

$$(35)\qquad F(x)=\sum_{j=1}^{\infty}[B_j/(iy_j-x)],$$

where y_j are positive, real and such that $y_n/y_{n+1}>\lambda>1$ for all values of n, with an assigned set of initial values $F(0)$, $F^{(j)}(0)$ such that

$$|F(0)|,\ |F^{(j)}(0)/j!|<G,\qquad (j=1,2,\cdots).$$

Assume $F_n(x)=\sum_{j=1}^{n}[b_j/(iy_j-x)]$ and determine the constants b_j by a set of n equations:

$$(36)\quad \begin{aligned} C_0&=b_1+b_2+\cdots+b_n,\\ iF(0)&=(b_1/y_1)+(b_2/y_2)+\cdots+(b_n/y_n),\\ i_2F^{(1)}(0)/1!&=(b_1/y_1^{\,2})+(b_2/y_2^{\,2})+\cdots+(b_n/y_n^{\,2}),\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ i^{n-1}F^{(n-2)}(0)/(n-2)!&=(b_1/y_1^{\,n-1})+(b_2/y_2^{\,n-1})+\cdots+(b_n/y_n^{\,n-1}),\end{aligned}$$

---

[*] A definition of an asymptotic expansion is given in Whittaker and Watson, Modern Analysis, third edition, Cambridge, 1920, p. 151.

these equations are suggested by those that are derived when in $F(x) = \sum_{j=1}^{\infty} [B_j/(iy_j - x)]$ and its successive derivatives x is put equal to zero; equations (36) reduce to the latter set when $n \rightarrow \infty$. The first of the equations (36) is introduced in order to have (36) of the same form as (19) and (3), since then it is known how to solve those sets; C_0 is an arbitrary constant such that $|C_0| < G$. A solution of (36) is given by

$$(37) \qquad b_k = (1/2\pi i) \int_C \frac{\phi_n(z^2)\psi(z)\,dz}{[z^2 - (1/y_k)]\phi_n^{(1)}(1/y_k)},$$

where $\phi_n(z) = \prod_{j=1}^{\infty} (1 - y_j z)$,

$$\psi(z) = C_0(1/z) + iF(0)(1/z^3) + [i^2F^{(1)}(0)/1!] \cdot (1/z^5) + [i^3F^{(2)}(0)/2!] \cdot (1/z^7) + \cdots,$$

and C is a circle having $z = 0$ for the center and a radius $R \geq G$; further,

$$(38) \qquad \lim_{n\to\infty} b_k = B_k = (1/2\pi i) \int_C \frac{\phi(z^2)\,\psi(z)\,dz}{[z^2 - (1/y_k)] \cdot \phi^{(1)}(1/y_k)},$$

where $\phi(z^2) = \lim_{n\to\infty} \phi_n(z^2) = \prod_{j=1}^{\infty} (1 - y_j z^2)$ and converges. When $n \rightarrow \infty$, $F_n(x)$ approaches $F(x) = \sum_{j=1}^{\infty} [B_j/(iy_j - x)]$. This process is justified since b_k, B_k satisfy the inequalities (6) with m_j replaced by $(1/y_j)$ and therefore $F_n(x) = \sum_{j=1}^{n} [b_j/(iy_j - x)]$ and its successive derivatives converge uniformly for all values of n, the same holding true for $F(x) = \sum_{j=1}^{\infty} [B_j/(iy_j - x)]$. We formulate then the following theorem:

THEOREM VII. *Let y_j be a sequence of positive numbers such that for all values of n*

$$y_n/y_{n+1} > \lambda > 1.$$

Let $F(0)$, $F^{(j)}(0)$ $(j = 1, 2, \cdots)$ be numbers such that $|F(0)|$, $|F^{(j)}(0)/j!| < G$, a positive number. Coefficients B_j can then be so determined that the series

$$(39) \qquad F(x) = [B_1/(iy_1 - x)] + [B_2/(iy_2 - x)] + [(B_3/(iy_3 - x)] + \cdots + [B_j/(iy_j - x)] + \cdots$$

(for which $x = 0$ is an essentially singular point, being a limiting point of a sequence of poles) and all its derivatives will converge for all real values of x and the sum function will have $F(0)$, $F^{(1)}(0)$, $\cdots$ for its initial values at $x = 0$. The coefficients B_j are given by (38).

Take two functions with the same initial values, but in their representation by means of Theorem VII let C_0 have different values in each of them; let them have the same poles. The difference of such two functions

$$A(x) = [\gamma_1/(iy_1 - x)] + [\gamma_2/(iy_2 - x)] + \cdots \tag{40}$$

has zero initial values; making use of (38) we derive

$$\gamma_k = (s/2\pi i) \int_C \frac{\phi(z^2)\,dz}{z[z^2 - (1/y_k)]\,\phi^{(1)}(1/y_k)} \tag{41}$$

where s is a constant and $\phi(z) = \prod_{j=1}^{\infty} (1 - y_j z)$.

Thus, it has been shown, that *since y_j can be chosen in an indefinite number of ways, there are infinitely many functions of this class with zero initial values at an essentially singular point.*

Let now $F(0)$, $F^{(j)}(0)\,(j = 1, 2, \cdots)$ be any set of initial values. We construct a function

$$G(x) = F(0) + \sum_{n=1}^{\infty} [a_n y_n^n/(iy_n - x)]; \tag{42}$$

differentiating successively, and letting $x = 0$, we have two sets of equations differing from (11) and (12) by having $iG(0)$, $iF(0)$ and $i^j G^{(j-1)}(0)$ $(j-1)!$, $F^{(j-1)}(0)/(j-1)!$ and $1/y_j$ in place of $G^{(1)}(0)/f^{(1)}(0)$, $F^{(1)}(0)/f^{(1)}(0)$, $G^{(j)}(0)/f^{(j)}(0)$, $F^{(j)}(0)/f^{(j)}(0)$ and m_j, respectively; letting

$$M_j = i^j N^{(j-1)}(0)/(j-1)!, \qquad (j = 1, 2, \cdots)$$

by Theorem VII we can construct a function $N(x) = \sum_{j=1}^{\infty} [B_j/(iy_j - x)]$ with $N^{(0)}(0) = N(0)$, $N^{(j)}(0)$ for initial values. The function

$$F(x) = G(x) - N(x) \tag{43}$$

will have $F(0)$, $F^{(1)}(0)$, $F^{(2)}(0), \cdots$ for initial values at $x = 0$. In the preceding, the first of the formulas (11) is omitted. We have shown therefore that *functions can be constructed having any assigned set of constants for initial values at an essentially singular point.* In the general case y_j must be diminishing rapidly enough in order that the modified set of equations (12) should define the constants a_n of (42) so that, for all values of n, $|a_n| \leq 1$.

The method of this section can be readily generalized. Let $z = 0$ be the limiting point of a sequence of poles γ_j, all situated on a straight line through the origin making an angle $\theta (\neq 0, \neq \pi)$ with the positive half of the real axis; such poles can be given by $\gamma_j = (ctg\theta + i)y_j$; we shall suppose that y_j satisfy the same conditions as before. We consider functions of the form

$$F(x)=\sum_{j=1}^{\infty} \alpha_j/[(ctg\theta+i)y_j-x]; \tag{44}$$

all of the preceding part of this section will hold with respect to functions of the form (44), provided $i^{n-1}F^{n-2}(0)/(n-2)!$ is everywhere replaced by $(ctg\theta+i)^{n-1}F^{n-2}(0)/(n-2)!$ Then in Theorem VII the condition will be introduced that

$$|(ctg\theta+i)^{n-1}\cdot F^{n-2}(0)/(n-2)!| < G, \qquad (n=1,2,\cdots). \tag{45}$$

From (45) it follows that $F(0)$, $F^{(j)}(0)$ are more restricted for values of θ nearer to 0 or π. The restriction on the $F(0)$, $F^{(j)}(0)$ is least in the particular cases when $\theta=\pm(\pi/2)$.

By initial values in the above representations are meant the limits attained by the function and its successive derivatives when the variable approaches the essentially singular point (of the type considered) along a fixed direction. Cauchy's example

$$e^{-1/z^2}$$

has a different type of an essentially singular point at $z=0$, but the function and its successive derivatives possess definite limits when the variable approaches $z=0$ along the real axis.

5. *An Example.* The coefficients α_n of the representations considered in Theorems I, IV and VII are of the same general form. Let us consider the case, which could be used in connection with any of those theorems, when m_r or $1/y_r=\lambda^r$ $(\lambda>1)$; denote the coefficients of $\psi(z)$ by G_j; they will be assumed such that $\psi(z)=\sum_{j=0}^{\infty}(G_j/z^{2j+1})$ is an entire function. It is known how to expand a function $\phi(z)=\prod_{j=1}^{\infty}[1-(z/\lambda^j)]$ in a power series in z; taking successive derivatives of $\phi(z/\lambda)\cdot[1-(z/\lambda)]=\phi(z)$ and letting $z=0$ we derive

$$\phi^{(r)}(0)=(-1)^r r!/(\lambda-1)(\lambda^2-1)\cdots(\lambda^r-1); \qquad \text{hence}$$

$$\phi(z)=\sum_{r=0}^{\infty} c_r z^r, \qquad c_0=1,$$

$$c_r=(-1)^r/(\lambda-1)(\lambda^2-1)\cdots(\lambda^r-1), \qquad (r=1,2,\cdots).$$

Further

$$\phi(z^2)/(z^2-\lambda^n)=\sum_{r=0}^{\infty}[c_r z^{2r}/(z^2-\lambda^n)]=\sum_{r=0}^{\infty} {}_n d_r\cdot z^{2r};$$

$$\sum_{r=0}^{\infty} c_r z^{2r}=(z^2-\lambda^n)\sum_{r=0}^{\infty} {}_n d_r\cdot z^{2r}$$

$$=-{}_n d_0\lambda^n+\sum_{r=1}^{\infty}({}_n d_{r-1}-{}_n d_r\lambda^n)z^{2r}.$$

Comparing coefficients, have

$$ {}_nd_{r-1} - {}_nd_r\,\lambda^n = c_r, \quad -{}_nd_0\,\lambda^n = 1. \qquad (r = 1, 2, \cdots), $$

$$ {}_nd_r = -(c_r/\lambda^n) - (c_{r-1}/\lambda^{2n}) - \cdots - (1/\lambda^{(r+1)n}) \qquad (r = 0, 1, 2, \cdots). $$

Consider

$$ \phi^{(1)}(\lambda^n)\cdot\alpha_n = (1/2\pi i)\int_C \frac{\phi(z^2)}{(z^2-\lambda^n)}\cdot\psi(z)\,dz = $$

$$ = (1/2\pi i)\int_C \left(\sum_{r=0}^{\infty} {}_nd_r\, z^{2r}\right)\left[\sum_{j=0}^{\infty}(G_j/^{2j+1})\right]dz; $$

both of the power series of the integrand are entire functions and multiplication termwise is justifiable. Since the coefficient of $1/z$ in the integrand is $\sum_{r-0}^{\infty} G_r\,{}_nd_r$, and $\phi^{(1)}(\lambda^n) = \sum_{r=1}^{\infty} r\,c_r\,\lambda^{n(r-1)}$, we obtain finally,

$$ \alpha_n = \left(\sum_{r=0}^{\infty} G_r\cdot{}_nd_r \Big/ \sum_{r=1}^{\infty} r\,c_r\lambda^{n(r-1)}\right), \qquad (n = 1, 2, 3, \cdots), \tag{46} $$

where

$$ c_r = (-1)^r/(\lambda-1)(\lambda^2-1)\cdots(\lambda^r-1), \qquad c_0 = 1, $$

$$ {}_nd_r = -(c_r/\lambda^n) - (c_{r-1}/\lambda^{2n}) - \cdots - (1/\lambda^{(r+1)n}). $$

In the numerator of (46) terms can be so rearranged as to have a power series in $1/\lambda^n$ with coefficients independent of n; this would facilitate actual computation.

6. *Application of Darboux's Formula.*[*] Suppose that $F(x)$ is a function of a real variable, indefinitely differentiable, and such that the function and its successive derivatives of all finite orders are finite and continuous on the interval $(0, x)$, then, even though $F(x)$ is not necessarily analytic, Darboux's formula holds; this becomes evident when we follow the steps by means of which that formula is proved.

Let $\phi(t)$ be a polynomial of degree n, $(0 \leq t \leq 1)$. Consider the identity

$$ (d/dt)\sum_{m=1}^{\infty}(-1)^m\cdot x^m\phi^{(n-m)}(t)\cdot F^{(m)}(xt) $$

$$ = -x\phi^{(n)}(t)F^{(1)}(xt) + (-1)^n x^{n+1}\phi(t)F^{(n+1)}(x\cdot t); $$

integration with respect to t between the limits 0 and 1 gives the formula

$$ \phi^{(n)}(0)\left[\mathrm{F}(x) - F(0)\right] $$

$$ = \sum_{m=1}^{n}(-1)^{m-1}x^m\left[\phi^{(n-m)}(1)F^{(m)}(x) - \phi^{(n-m)}(0)F^{(m)}(0)\right] $$

$$ + (-1)^n x^{n+1}\int_0^1 \phi(t)F^{(n+1)}(xt)\,dt. $$

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[*] *Journal de Mathématiques*, Ser. 3, Vol. 2 (1876), p. 271; also see Whittaker and Watson, *Modern Analysis*, third edition, Cambridge, 1920, p. 125.

In particular, if $\phi(t) = (t-1)^n$ *we have*

$$(47)\qquad F(x) = [\sum_{m=0}^{n} (x^m F^{(m)}(0)/m!)] \\ + [(-1)^n x^{n+1}/n!] \int_0^1 (t-1)^n F^{(n+1)}(xt)\,dt.$$

When $\lim_{n\to\infty} (1/n!) \int_0^1 (t-1)^n F^{(n+1)}(xt)\,dt = 0$, $F(x)$ is analytic. With the hypothesis made concerning $F(x)$, it cannot be asserted that $\lim_{n\to\infty} (1/n!) \int_0^1 (t-1)^n F^{(n+1)}(xt)\,dt = 0$, even if $\sum_{m=0}^{\infty} x^m F^{(m)}(0)/m!$ converges.

Replacing in (47) x by $1/x$ we have:

$$(48)\qquad | x^m \{F(1/x) - \sum_{m=0}^{n} [(F^{(m)}(0)/m!)\cdot(1/x^m)]\} | \\ = | (1/xn!) \int_0^1 (t-1)^n \cdot F^{(n+1)}(t/x)\,dt |$$

$\leqq [P_{n+1}/x(n+1)!]$ ($P_n \geqq | F^{(n)}(x) |$ for all real x and $n = 1, 2, \cdots$).

Hence the following theorem follows:

THEOREM VIII. *Let $F(x)$ be an indefinitely differentiable, non- analytic function such that $F(x)$, $F^{(1)}(x)$, $F^{(2)}(x), \cdots$ are continuous and finite in the interval $(0, x)$; let $F(0)$, $F^{(1)}(0)$, $F^{(2)}(0), \cdots$ be the initial values at $x = 0$; then for large values of x*

$$F(1/x) \sim \sum_{m=0}^{\infty} (F^{(m)}(0)/m!) \cdot 1/x^m \text{ asymptotically.}$$

COROLLARY. *If a function $F(x)$ with zero initial values at $x = 0$ satisfiies the conditions of Theorem VIII then for large values of x*

$$F(1/x) \sim 0 \textit{ asymptotically.}$$

These results can be applied to the representations of section (2).

7. *General Observations.* The methods of sections 3 and 4 have an advantage over that of section 2 in having the initial values less restricted (Theorems IV and VII), which advantage would be of importance when m_j (such that $m_{j+1}/m_j > \lambda > 1$) or y_j (such that $y_j/y_{j+1} > \lambda > 1$) are fixed, since when the restrictions on the initial values are removed it may be necessary to have m_j increasing more rapidly, or y_j decreasing more slowly (to satisfy equations (12) or the modified sets of equations (12)). The method of section 2 offers a greater multiplicity of representations, since there are infinitely many functions of class (A).

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The Abstract Identity of Modular Systems and Ideals.

BY DUNCAN HARKIN.

Introduction. The object of this paper is to show the abstract identity of two theories: one, the Theory of Ideals, first developed by Richard Dedekind in 1871; the other, Leopold Kronecker's Modular Systems, made known in the famous "Festschrift" of 1882. The method followed is one of abstraction and transformation by formal equivalence,* in the spirit of E. H. Moore's "General Analysis".

The terminology of the paper agrees closely with that of Kronecker. The theorems, however, follow Dedekind for the most part. By this dual scheme, where the notation and external form is apparently that of Kronecker but where the root idea and fundamental meaning is taken almost bodily from Dedekind, the two theories are seen to be two instances of one and the same general theory.

1. A field $\mathfrak{F}$ is defined as usual.† Given a set of numbers $[a]$ and two fundamental operations S and P (called addition and multiplication), such that either operating on a pair of numbers produces an unique number of the set, and if O stands for either S or P (disjunctively), the field properties may be briefly characterized by the four postulates 1.01-1.04.

1.01 $$O(a_1, a_2) = a = O(a_2, a_1)$$

The operation is commutative.

1.02 $$O(O(a_1, a_2), a_3) = O(a_1, O(a_2, a_3))$$

The operation is associative.

$$P(S(a_1, a_2), a_3) = S(P(a_1, a_3), P(a_2, a_3))$$

Multiplication is distributive with respect to addition.

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* E. T. Bell, *Algebraic Arithmetic*, American Mathematical Society Colloquium Publications, Vol. 7 (1927), Chap. V, §§ 1-2.

† E. T. Bell, *Algebraic Arithmetic*, pp. 6-7; L. E. Dickson, *Algebras and their Arithmetics*, Chicago, 1923, p. 201; G. Scorza, *Corpi Numerici e Algebre*, Palermo, 1921, pp. 3-13.

1.03. There exists in the set an identity element i such that, for any number a of the set,

$$O(a, i) = a.$$

1.04. For every number a of the set there exists in the set an inverse r, such that

$$O(a, r) = i.$$

1.1. From these postulates the uniqueness of the identity and inverse elements may be proved. The unique identity element under S is written 0, and, under P, 1. It is also easily shown that cancellation of like components from the two sides of an equality is permissible, i. e. if $O(a_1, a_2) = O(a_1, a_3)$, then $a_2 = a_3$.

1.2. A special variety* is produced by modification of the postulates. Whole numbers are defined by so restricting postulate 1.04 that under multiplication the only number having an inverse is the identity 1 itself. A further restriction of 1.04, so that there is no inverse under either addition or multiplication, except the identity, defines the set of positive whole numbers.

1.3. The indeterminate or arbitrary number u, due to Gauss but first taking important place in the work of Kronecker, is defined as *any* number, in the sense that its having once stood for a particular number a_1 does not prevent it from standing for another number, distinct or the same. This definition of indeterminates leads immediately to the theorem:

1.31. *In an equation in n indeterminataes (or powers of the same indeterminate), coefficients of like indeterminates (or like powers of the same indeterminate) are equal.*

By hypothesis, we have

$$\sum_i^{1\ldots n} a_{1i}u_i = \sum_i^{1\ldots n} a_{2i}u_i.$$

Since the u_i are arbitrary, this equation must hold when $u_1 \neq 0$ and $u_i = 0$ $(i = 2, 3, \cdots, n)$; therefore

$$a_{11} = a_{21}.$$

Similarly, for $u_2 \neq 0$, $u_i = 0$ $(i = 1, 3, \cdots, n)$,

$$a_{12} = a_{22};$$

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* E. T. Bell, *loc. cit.*, p. 5.

and so on. In general,

$$a_{1i} = a_{2i} \qquad (i = 1, 2, \cdots, n).$$

The alternate form of the theorem is proved in like manner.

1.4. Further elements l_i are adjoined, subject to no other restriction than that of combining in the usual way under S and P. An integrity-domain $\mathfrak{J}$ is then defined as the set $[\psi(l_i; a_j)]$ of all rational integral functions of these elements with whole-number coefficients. That $\mathfrak{J}$ is properly called an "integrity-domain" is seen by observing that the ψ_i satisfy all the postulates for whole numbers. A rationality-domain $\mathfrak{R}$ consists of the set $[\phi]$ of all rational functions of the elements l_i with whole-number coefficients. Consequently, every ϕ can be written as a quotient ψ_1/ψ_2, and the set satisfies the postulates 1.01-1.04.

1.5. Following the nomenclature of Kronecker* and, before him, Gauss, who first introduced into mathematics the terminology of the natural sciences, we define class, genus, and species. A set of n algebraic quantities (definition) λ_i $(i = 1, 2, \cdots, n)$ are said to constitute a class $\mathfrak{C}$ of algebraic quantities if their elementary symmetric functions are equal to quantities of a given rationality-domain $\mathfrak{R}$. If the domain is restricted to an integrity-domain $\mathfrak{J}$, then the class $\mathfrak{C}$ consists of algebraic integers. The n algebraic quantities λ (or integers α) are called conjugate algebraic quantities (or integers).

1.51. *If a class* $\mathfrak{C}$ *of* n *conjugate algebraic integers* α_i $(i = 1, 2, \cdots, n)$ *satisfy the system of equations*

$$(1) \qquad \sum \alpha_1 = \mathfrak{f}_1, \quad \sum \alpha_1\alpha_2 = \mathfrak{f}_2, \quad \cdots, \quad \alpha_1\alpha_2 \cdots \alpha_n = \mathfrak{f}_n,$$

where the $\mathfrak{f}_i$ $(i = 1, 2, \cdots, n)$ *belong to a given integrity-domain* $\mathfrak{J}$, *then any algebraic integer* $\alpha = \alpha_i$ $(i = 1, 2, \cdots, n)$ *of the class satisfies the equation*

$$(2) \qquad \alpha^n - \mathfrak{f}_1\alpha^{n-1} + \mathfrak{f}_2\alpha^{n-2} - \cdots \pm \mathfrak{f}_n = 0$$

or

$$(2') \qquad \alpha^n - \alpha^{n-1}\sum \alpha_1 + \alpha^{n-2}\sum \alpha_1\alpha_2 - \cdots \pm \alpha_1\alpha_2 \cdots \alpha_n = 0;$$

and this last is equivalent to the equation

$$(3) \qquad \prod_{i}^{1 \cdots n} (\alpha - \alpha_i) = 0.$$

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* L. Kronecker, "Grundzüge einer arithmetischen Theorie der algebraischen Grössen", *Crelle's Journal für Mathematik*, Vol. 92 (1882), pp. 1-122; *Werke*, II, 237-387.

We proceed first with the demonstration of the first part of the theorem. By hypothesis,

$$\sum_{i}^{1\ldots n} \alpha_i - \mathfrak{f}_1 = 0.$$

Multiplying this by α_1, we get

$$\alpha_1 \sum_{i}^{1\ldots n} \alpha_i - \mathfrak{f}_1\alpha_1 = 0,$$

which can also be written

$$\alpha_1^2 - \mathfrak{f}_1\alpha_1 + \sum_{i}^{2\ldots n} \alpha_1\alpha_i = 0.$$

Since

$$\mathfrak{f}_2 = \alpha_1\alpha_2 + \sum_{i,j}^{2\ldots n} \alpha_i\alpha_j$$

this last equation can finally be put in the form

$$\alpha_1^2 - \mathfrak{f}_1\alpha_1 + \mathfrak{f}_2 - \sum_{i,j}^{2\ldots n} \alpha_i\alpha_j = 0.$$

Again, multiplication by α_1 gives

$$\alpha_1^3 - \mathfrak{f}_1\alpha_1^2 + \mathfrak{f}_2\alpha_1 - \sum_{i,j}^{2\ldots n} \alpha_1\alpha_i\alpha_j = 0,$$

which, as before, can be put

$$\alpha_1^3 - \mathfrak{f}_1\alpha_1^2 + \mathfrak{f}_2\alpha_1 - \mathfrak{f}_3 + \sum_{i,j,k}^{2\ldots n} \alpha_i\alpha_j\alpha_k = 0.$$

. . . .

This easily leads to the empirical formula

$$\sum_{i}^{1\ldots r} (-1)^i \mathfrak{f}_{r-i}\alpha_1^i + (-1)^r \sum_{i,\ldots,k}^{2\ldots n} \alpha_1\alpha_i \cdots \alpha_k = 0,$$

which is proved by mathematical induction up to, as the notation indicates, $r = n$. For $r = n$,

$$\sum_{i}^{1\ldots n} (-1)^i \mathfrak{f}_{n-i}\alpha_1^i + (-1)^n \sum_{i,\ldots,k}^{2\ldots n} \alpha_1\alpha_i \cdots \alpha_k = 0.$$

This is obviously the same as

$$\sum_{i}^{1\ldots n} (-1)^i \mathfrak{f}_{n-i}\alpha_1^i + (-1)^n \alpha_1\alpha_2 \cdots \alpha_n = 0$$

or

$$\sum_{i}^{1\ldots n} (-1)^i \mathfrak{f}_{n-i}\alpha_1^i + (-1)^n \mathfrak{f}_n = 0,$$

which can be written

$$\sum_{i}^{0\ldots n} (-1)^i \mathfrak{f}_{n-i}\alpha_1^i = 0.$$

By parity,

$$\sum_{i}^{0\,\ldots\,n}(-1)^{i}\,\mathfrak{f}_{n-i}\alpha_j{}^{i}=0 \qquad (j=1,2,\cdots,n)$$

or, if $\alpha=\alpha_j$ $(j=1,2,\cdots,n)$,

$$\sum_{i}^{0\,\ldots\,n}(-1)^{i}\,\mathfrak{f}_{n-i}\alpha^{i}=0,$$

as was to be proved.

The last part of the theorem is proved by expanding the product

$$\prod_{i}^{1\,\ldots\,n}(\alpha-\alpha_i)=0.$$

This gives the equation

$$\alpha^{n}-\alpha^{n-1}\sum\alpha_i+\alpha^{n-2}\sum\alpha_i\alpha_j-\cdots\pm\alpha_1\alpha_2\cdots\alpha_n=0.$$

By means of the Galois theory of equations, equations (1) and (2) may be shown equivalent.

1.52. *The sum, difference, or product of any two algebraic integers* α_1 *and* α_2 *of a class* $\mathfrak{C}$ *is an algebraic integer of the same class.*

1.6. A genus $\mathfrak{G}$ is composed of all rational functions of the algebraic quantities of a given class $\mathfrak{C}$ under a rationality-domain $\mathfrak{R}$. Under a given integrity-domain, $\mathfrak{J}$, all the rational integral functions of the algebraic integers belonging to a class $\mathfrak{C}$ constitute a species $\mathfrak{S}$. Given an integrity domain and, under it, a class $\mathfrak{C}$ of algebraic integers and a genus $\mathfrak{G}$, then a principal species $\mathfrak{S}^*$ is defined as the totality of all the rational integral functions of the integral algebraic numbers which (functions) also belong to the genus $\mathfrak{G}$.

1.61. *For every principal species* $\mathfrak{S}^*$ *there exists a finite basis* $\mathfrak{B}(\omega_1,\omega_2,\cdots,\omega_n)$ *of* $\mathfrak{S}^*$ *such that any quantity of the principal species* $\mathfrak{S}^*$ *may be expressed linearly in terms of the elements* ω_i *of the basis with integral quantities* ψ_i *of the integrity domain* $\mathfrak{J}$ *for coefficients.*

1.7. Conjugate algebraic units are algebraic integers whose product is unity. This is equivalent to saying that the norm of an algebraic unit ϵ is 1, or that an algebraic unit is an algebraic integer satisfying an equation whose first and last coefficients are both 1.

2. *Forms and their Properties.*

2.1. A form F is a linear combination of algebraic integers, belonging to a given species and genus, with indeterminate coefficients. More generally, F may be regarded as a form in the indeterminates with integral algebraic coefficients. Thus

$$F = \sum u_i \alpha_i = \sum \alpha_i u_i \qquad (i = 1, 2, \cdots, n).$$

A form F is said to contain an algebraic integer α if α can be expressed linearly in terms of the elements of the form. More generally, a form F_1 is said to contain a form F_2 if each element of the second form can be expressed linearly in terms of the elements of the first form.

2.11. *A form* $F = \sum u_i \alpha_i$ $(i = 1, 2, \cdots, n)$ *contains each of its elements* α_i $(i = 1, 2, \cdots, n)$.

2.12. *Every form is equivalent to itself.*

2.121. *A form remains unaltered by the addition to its elements of any quantity contained in the form.*

By definition,

$$F = u_1\alpha_1 + u_2\alpha_2 + \cdots + u_n\alpha_n.$$

Consider

$$F' = u_1'\alpha_1 + u_1'\psi\alpha_2 + u_2'\alpha_2 + \cdots + u_n'\alpha_n.$$

This may be written

$$F' = u_1'\alpha_1 + (u_1'\psi + u_2')\alpha_2 + \cdots + u_n'\alpha_n$$

or

$$F' = u_1\alpha_1 + u_2\alpha_2 + \cdots + u_n\alpha_n,$$

where

$$u_2 = u_1'\psi + u_2' \quad \text{and} \quad u_i = u_i' \qquad (i = 1, 3, 4, \cdots, n).$$

2.13. *The sum of two forms is a form.*

By definition,

$$F_1 = \sum_i^{1 \ldots r} u_i\alpha_i' \quad \text{and} \quad F_2 = \sum_i^{1 \ldots s} v_i\alpha_i''$$

$$\therefore \quad F_1 + F_2 = \sum_i^{1 \ldots n} w_i\alpha_i,$$

where $n = r + s$, and $w_i\alpha_i = u_i\alpha_i'$ $(i = 1, \cdots, r)$ or $w_i\alpha_i = v_i\alpha_i''$ $(i = r + 1, \cdots, s)$.

2.14. *The product of two forms is a form.*

Defining F_1 and F_2 as in (2.13), we have

$$F_1F_2 = \sum w_i\alpha_i, \qquad (i = 1, \cdots, n)$$

where $n = rs$, and $w_i\alpha_i = u_hv_k\alpha_h'\alpha_k''$ $(h = 1, \cdots, r;\ k = 1, \cdots, s)$. This substitution is permissible by 1.52 and the fact that the product of two indeterminates is again indeterminate.

2.15. *The sum of two forms contains either form.*

2.16. *The product of two forms is contained in either of the components.*

F_1 and F_2 being defined as in 2.13 and their product F_1F_2 as in 2.14, it is sufficient to show that F_1 contains F_1F_2. To show this, every element $\alpha_i'\alpha''$ of F_1F_2 must be linearly expressible in terms of the elements α_i' of F_1, i. e. $\alpha_i'\alpha'' = \sum u_i'\alpha_i'$ $(i = 1, \cdots, r)$. This is possible by putting $u_i' = \alpha''$.

2.17. *Given two forms F_1 and F_2, then, if there exists a form F such that the product F_2F is equivalent to F_1, it follows that F_2 contains F_1.*

By 2.16, F_2 contains F_2F which, by hypothesis, is equivalent to F_1.

The converse of this theorem, namely: *If F_1 is contained in F_2, there exists an unique form F such that F_1 is equivalent to F_2F,* is not evident. Its validity will be the aim and the conclusion of the present section.

2.18. *If a form F is contained in a quantity ψ of the original domain, then there exists a form F' such that $F = \psi F'$.*

By hypothesis, ψ contains $F = \sum u_i\alpha_i$ and therefore contains each α_i, i. e. $\alpha_i = \psi\alpha_i'$. Since $F = \sum u_i\alpha_i = \psi \sum u_i\alpha_i'$, it is evident that $F' = \sum u_i\alpha_i'$.

2.19. *If $\psi F_1 = \psi F_2$, then $F_1 = F_2$.*

2.2. The norm Γ of a form F is the product of F and all forms conjugate to F.

2.21. *The elements of the norm Γ of a form F are rational integral quantities.*

By definition,

$$NF = \Gamma = \prod_{j}^{1 \cdots n} (u_1\alpha_{1j} + u_2\alpha_{2j} + \cdots + u_n\alpha_{nj}).$$

Since this product expanded involves only symmetric functions of the α_{ij}, it follows by 1.51 that the resulting coefficients will be quantities of the original field.

The reduced norm T of a form F is the quotient of the norm Γ of the form divided by the G. C. D. d of its rational integral coefficients, i. e. $T=\Gamma/d$.

2.22. *The norm of the product of two forms is equivalent to the product of the norms of the forms.*

Write $NF_1=F_1\bar{F}_1$, where $\bar{F}_1=F_{12}F_{13}\cdots F_{1n}$, the F_{1j} $(j=1, 2, \cdots, n)$ being conjugate forms. Then

$$NF_1NF_2=F_1\bar{F}_1F_2\bar{F}_2=F_1F_2\bar{F}_1\bar{F}_2=F\bar{F},$$

where $\quad F=F_{11}F_{21}$ and $\bar{F}=F_{12}F_{21}F_{13}F_{23}\cdot\cdot\cdot F_{1n}F_{2n}$.

2.3. A primitive form E (Einheitsform) is a form whose norm is equivalent to 1.

2.31. *Unity* 1 *contains every form.*

2.32. *A primitive form E contains every form F.*

By 2.16, E contains its norm NE which, by definition, is equivalent to 1. But 1, by 2.31, contains any form F. $\therefore E$ contains F.

2.33. *Every primitive form E is equivalent to* 1.

By the proof of 2.32, E contains 1 and, by 2.31, 1 contains E.

2.34. *The sum of two primitive forms E_1 and E_2 is a primitive form.*

By 2.13, E_1+E_2 is equivalent to some form F and is therefore contained in F. But E_1+E_2 contains E_1, by 2.15. Hence F contains E_1 and, by 2.22, NF contains $NE_1=1$. Since NF is also contained in 1, by 2.31, $NF=1$ and F is primitive.

2.35. *The product of two primitive forms is equivalent to a primitive form.*

By 2.14, E_1E_2 is equivalent to some form, say F; and, by 2.22, $NE_1NE_2=NF$. By definition, $NE_1=1$ and $NE_2=1$, therefore $NF=1$ and so F is primitive.

2.36. *The sum of a primitive form E and of any form F is equivalent to the primitive form E.*

The sum $E+F$ contains E, by 2.15. It remains now to show that E contains $E+F$. But E contains itself and, by 2.32, E contains F: therefore E contains $E+F$.

2.37. *The product of a primitive form E and any form F is equivalent to F.*

F contains EF, by 2.16. It now remains to show that EF contains F. This follows from 2.12 and 2.33.

These theorems justify Dedekind's definition of an ideal and bear witness to his "remarkably sharp-witted method of deduction",* for in the two properties which he makes definitive lies the germ of the whole development. Dedekind's definition † can be summarily put

$$\text{I} \quad S(\mathfrak{a}, \mathfrak{a}) = \mathfrak{a}$$
$$\text{II} \quad P(\mathfrak{o}, \mathfrak{a}) = \mathfrak{a}$$

or, in the notation here used,

$$S(F, F) = F$$
$$P(E, F) = F.$$

2.38. *If $F_1 + F_2$ is equivalent to a primitive form E and if F_1 contains the product F_2F_3, then F_1 contains F_3.*

By hypothesis, F_1 contains F_2F_3, and, by 2.16, F_1 contains F_1F_3: therefore F_1 contains $F_1F_3 + F_2F_3 = F_3(F_1 + F_2)$ which, by hypothesis, equals F_3E. But, by 2.37, $F_3E = F_3$, whence the theorem.

2.39. *The set of all the forms which can contain a given form is finite.*

By 2.21, the norm of the given form F is a rational quantity ψ, which, by 2.16, is contained in F. Let $F' = \sum u_i\alpha_i$ $(i = 1, 2, \cdots, n)$ be any form containing F. By 1.81,

$$F' = u_1(\psi_{11}\omega_1 + \psi_{12}\omega_2 + \cdots + \psi_{1n}) + \cdots + u_n(\psi_{n1}\omega_1 + \cdots + \psi_{nn}\omega_n),$$

where $(\omega_1, \omega_2, \cdots, \omega_n)$ is a basis. By 2.121, since F' contains ψ, we may put

$$\begin{aligned} F' &= u_1'(\psi_{11}\omega_1 + \cdots + \psi_{1n}\omega_n) + \cdots + u_n' \ (\psi_{n1}\omega_1 + \cdots + \psi_{nn}\omega_n) + \psi \\ &= u_1'(\psi_{11}'\omega_1 + \cdots + \psi_{1n}'\omega_n) + \cdots + u_n' \ (\psi_{n1}'\omega_1 + \cdots + \psi_{nn}'\omega_n) + \psi. \end{aligned}$$

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[*] Kronecker, *Grundzüge*, § 21.

[†] R. Dedekind, "Sur la théorie des nombres entiers algébriques", *Bulletin des Sciences Mathématiques et Astronomiques*, t. 9 (1876), pp. 278-288.

where ψ_{ij}' are the residues (mod. ψ) of the ψ_{ij}. But, since the ψ_{ij} are rational quantities, the number of residues (mod. ψ) is finite. Hence the theorem.

2.4. A prime form P is a non-primitive form such that, if any form F' contains P, then F' is equivalent (disjunctively) either to P or to a primitive form E.

2.41. *The sum of two prime forms P_1 and P_2 is equivalent to a primitive form E or else (disjunctive) the two prime forms are equivalent to each other.*

The sum $P_1 + P_2$ is equivalent, by 2.13, to some form F which therefore contains $P_1 + P_2$ and, consequently, by 2.15, each of P_1 and P_2 separately. By the definition of a prime form, this leaves two alternatives: either F is equivalent to both P_1 and P_2 (hence P_1 equivalent to P_2) or F is a primitive form E.

2.42. *Every non-primitive form F is contained in some prime form P.*

Either F is already a prime form (and the condition immediately satisfied) or, by 2.39, F is contained in a finite set of forms. But among this finite set there is some form, say F_h, of smallest norm. Hence the prime form P sought is F_h.

2.43. *For every form F there exists a form F' such that their product FF' is a quantity of the original field.*

This is satisfied by choosing $F' = \bar{F}$, where $F\bar{F} = NF$.

This theorem is the crux of the theory. Its importance is pointed out by both Dedekind and Kronecker, as well as by subsequent writers.*

2.44. *If $F_1 = F_1F_2$, then F_2 is primitive.*

By hypothesis, $F_1 = F_1F_2$; but, by 2.43, there exists an F' such that $F_1F' = \psi$. Therefore $\psi = \psi F_2$, and so $F_2 = E$, by 2.19 and 2.33.

2.45. *If $F_1F_2 = F_1F_3$, then $F_2 = F_3$.*

By hypothesis, $F_1F_2 = F_1F_3$ and, by 2.43, there is a form F'_1 so that $F'_1F_1F_2 = F'_1F_1F_3$ gives $\psi F_2 = \psi F_3$ which, by 2.19, implies $F_2 = F_3$.

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* See, in particular, A. Hurwitz, "Ueber die Theorie der Ideale," *Göttinger Nachrichten* (1894), pp. 291-298.

2.46. *If a form F_1 is contained in a prime form P, then there exist an unique form F_2 such that $F_1 = F_2P$.*

By hypothesis, F_1 is contained in P, and so F_1F' in PF'. But, by 2.43 F' can be chosen so that $PF' = \psi$, and 2.18 implies an F_2 so that $F_1F' = \psi F$. From this, $F_1F'P = \psi F_2P$ and therefore $\psi F_1 = \psi PF_2$ or, by 2.19, $F_1 = PF_2$.

To show the uniqueness of F_2, assume another F'', so that $F_1 = PF_2 = PF''$, or $F'PF_2 = F'PF''$. By 2.43 and 2.19, this gives $\psi F_2 = \psi F''$ and $F_2 = F''$.

2.47. *Every form F is uniquely expressible as a product of prime forms.*

By 2.42, F is contained in some prime form, say P_1, and, by 2.46 $F = P_1F_1$. By parity, $F_1 = P_2F_2, \cdots, F_{r-1} = P_rE$. Thus $F = P_1P_2 \cdots P_r$.

To show the decomposition unique, assume another decomposition, so that

$$F = P_1P_2 \cdots P_r = P_1'P_2' \cdots P_s'.$$

By 2.16 and 2.14, P_1 contains $P_1P_2 \cdots P_r$ and also $P_1'P_2' \cdots P_s'$. From this it follows, by 2.38 and 2.41, that $P_1 = P_1'$. Hence $P_2P_3 \cdots P_r = P_2'P_3' \cdots P_s'$. By parity, $P_2 = P_2'$, $P_3 = P_3', \cdots, P_r = P_r'$ ($r \leq s$). So by 2.41, $P_{r+1}' \cdots P_s' = E$ and, finally,

$$P_1'P_2' \cdots P_r' \cdots P_s' = P_1'P_2' \cdots P_r'E = P_1'P_2' \cdots P_r'.$$

2.48. *If one form F_1 is contained in another form F_2, then there exists a unique form F_3 such that $F_1 = F_2F_3$.*

By hypothesis F_1 is contained in F_2, and so F_1F' is contained in F_2F'. By 2.43, F' can be chosen so that $F_1F' = \psi$, a quantity in the original domain. Thus F_2F' is contained in ψ or, by 2.18, there exists a form F_3 such that $F_1F' = \psi F_3$. From this, $F_1F_2F' = \psi F_2F_3$ or $\psi F_1 = \psi F_2F_3$. The theorem follows by 2.19.

To show F_3 unique, assume an alternative F'''. But from $F_1 = F_2F_3 = F_2F'''$ it follows, by 2.45, that $F_3 = F'''$.

This completes the restoration of the arithmetic properties of unique factorization and decomposition. Thus the ordinary laws of divisibility again hold.

In order to facilitate the translation from the present theory (*) to that of Dedekind and of Kronecker, a dictionary of equivalent symbols and terms follows:

*	*D*	*K*
F	Ideal $\mathfrak{a}$	Form
E	Order $\mathfrak{o}$	Primitive form
P	Prime ideal $\mathfrak{p}$	Prime form
Γ	Norm of ideal $N(\mathfrak{a})$	Norm of form

3. By giving various interpretations to the symbols and relations, we obtain particular instances of the same abstract general theory. The development of section 2 made fundamental the composition of forms, as in the work of Kummer. However, this entire section could have been based on the Gaussian transformation of forms. In fact, the two ideas form two possible interpretations of the same general concept. Dedekind's Führer or Bearer furnishes another possible viewpoint.

Weber, in his Algebra, develops a theory of functionals, wherein integral associate functionals correspond to ideals; the product of two functionals to the product of two corresponding ideals; an integral number to a principal ideal; and units to the ideal or order $\mathfrak{o}$ of Dedekind.

Geometrically, an ideal may be made to correspond to a system of points on a fixed fundamental curve; and a principal ideal to the complete intersection system of the fundamental curve and another algebraic curve. Brill and Noether translate the idea of equivalence into coresiduality. Hensel and Landsberg have developed an extensive application of modular-systems to the theory of algebraic functions of curves on a surface.

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Concerning the Cut Points of a Continuous Curve when the Arc Curve, AB, Contains Exactly N Independent Arcs.[*]

BY NORMAN E. RUTT.

In 1925 Menger[†] introduced the term "curve" for those compact connected spaces which have the property that for every point of the set and every $\epsilon > 0$ there exists a neighborhood of diameter less than ϵ which contains the point and has a totally disconnected[‡] boundary. A point of such a set is said to be ***regular*** if it is possible to choose these neighborhoods in such a manner that its boundary is finite, while it is *of order n* if for every ϵ we can find the neighborhood of diameter less than ϵ whose boundary consists of exactly n points. In a subsequent paper he shows that in a regular curve, *i. e.*, a curse containing only regular points, there is corresponding to every point of order n a set of n arcs having the point in common and otherwise distinct.[§] It should be noted that every regular curve of Menger is also a continuous curve,[¶] that is, it is connected im kleinen at every point. Of course there are continuous curves which are neither regular nor even curves in the sense defined above.

In this paper is considered a plane continuous curve in which there are two points a and b which have the property that *there are in the continuous curve exactly n arcs from a to b which have no points other than a and b in common while it is impossible to find a similar set of $n + 1$ arcs from a to b.* Under these hypotheses it is proved that there are exactly n points in the curve whose omission separates a from b in the curve. This theorem provides a sort of converse to the theorem of Menger mentioned. The precise converse of the theorem of this paper is an easy consequence of the theorem itself, and thus furnishes a proof of Menger's theorem for more general *plane continuous curves.* In the course of the argument very important use is made of the arc-curve $E(ab)$ of the continuous curve E with respect to the distinct points a and b, i. e., the set of all points $[X]$ *of* E such that for every point x of the set there is an arc of E from a to b which contains x.*

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[*] Presented to the American Mathematical Society, May 7, 1927.

[†] Cf. K. Menger, "Grundzüge einer Theorie der Kurven," *Mathematische Annalen*, Vol. 95 (1927), p. 277.

[‡] A set of points is said to be *totally disconnected* if it contains no connected subsets other than single points.

[§] See "Zur allgemeine Kurventheorie," *Fundamenta Mathematicae*, Vol. 10, (1927), p. 98.

[¶] For fundamental definitions and properties of continuous curves see R. L. Moore, "Report on Continuous Curves from the Viewpoint of Analysis Situs," *Bulletin of the American Mathematical Society*, Vol. 29 (1923), pp. 289-302.

The statement of the principal theorem of this paper is due to Professor J. R. Kline, and it has been with his advice and direction that the proof has been constructed.

1. GENERAL THEOREM. *If there exists an integer n such that for two given points a and b of a continuous curve E there are arcs ax_ib $(i = 1, 2, \cdots, n)$ of E no two of which have any points except a and b in common, and such that by no means is it possible to construct $n+1$ arcs like these from a to b, then there exists a set of n distinct points $p_1, p_2, \cdots, p_n$ of E such that $E - (p_1 + p_2 + \cdots + p_n)$ is the sum of two mutually separated sets E_a and E_b, E_a containing a and E_b containing b.*

Notation and definitions. Let F be the arc curve of the point set $a + b$ with respect to E, that is, let F be the set of all arcs that can be drawn in E from a to b. It is a continuous curve every complementary domain of which has a simple closed curve as its boundary.† Let a particular set of n arcs ax_ib satisfying the hypotheses of the theorem be $M_1, M_2, \cdots, M_n$ (such a set will hereafter be referred to as *independent*) and let these be ordered in such a way that the simple closed curve formed by M_{i-1} and M_{i+1} contains in its interior (M_i), that is $M_i - (a + b)$, but not (M_j), where $j = 1, 2, \cdots, n$ and $i = 2, 3, \cdots, (n-1)$ and $j \neq i$. Let M_{ij} be the simple closed curve $M_i + M_j$, and let its interior be I_{ij}. Let F_{ij} represent $M_{ij} + F \times I_{ij}$. F_{ji} is a continuous curve; ‡ the arc-curve of $(a + b)$ with respect to F_{ij} is F_{ij}.§ Let any complementary domain of F_{jj+1} whose boundary has at least one point on each of the arcs (M_j) and (M_{j+1}) be called a *cross-domain* or *c-domain* of F.

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* This notion has been introduced by W. L. Ayers in a paper which has been accepted for publication by the *Transactions of the American Mathematical Society*. Ayres proves that the arc curve $E(ab)$ is itself a continuous curve. For an abstract of this paper see the *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 410. As this paper is as yet unpublished the numerous references to it throughout these pages are somewhat incomplete.

† W. L. Ayres, "Concerning the Arc-curves and Basic Sets of a Continuous Curve"; see remarks above in introduction.

‡ H. M. Gehman, "Concerning the Subsets of a Plane Continuous Curve," *Annals of Mathematics*, Vol. 27, 1925, No. 1, p. 34.

§ W. L. Ayres, "Concerning the Arc-curves and Basic Sets of a Continuous Curve."

In what follows ordinarily c-domains will be represented by small Greek letters γ, δ, their boundaries by the corresponding capitals Γ, Δ.

I(a) THEOREM. *At least one of the complementary domains of F_{12} is a cross-domain of F.*

Proof: Consider the set $F_{12} - M_{12}$. If this is vacuous the theorem is obvious as then I_{12} is a c-domain. Since F_{12} and M_{12} are both continuous curves and $M_{12} \subset F_{12}$, if $F_{12} - M_{12}$ is not vacuous, it consists of a countable number of maximal connected parts. Suppose that one of these, say K, has limit points both upon (M_1) and upon (M_2). If both a and b are also limit points of K there must in K be an arc from a to b.* Either this is an $n+1$st arc from a to b contradicting hypothesis, or it is one of the arcs M_j, $j > 2$, contrary to the convention used in ordering $M_1, \cdots, M_n$. Suppose then that b is not a limit point of K. As the limit points of K form a closed set with representative points both upon (M_1) and upon (M_2) there is a first k_1 on one and a first k_2 on the other. Now $M_{12} - (k_1 + k_2)$ is disconnected into (k_1ak_2) and (k_1bk_2). The arcs (k_1ak_2) and (k_1bk_2) will then occur in the same maximal connected subset of $F_{12} - (k_1 + k_2)$ only if some maximal connected subset of $F_{12} - M_{12}$ has limit points on both (k_1ak_2) and (k_1bk_2). Suppose one of these, say L, has one, l_a, in (k_1ak_2) and one, l_b, in (k_1bk_2). Draw an arc (k_1xk_2) in K and one (l_ayl_b) in L, as these must intersect exterior to I_{12} and thus K be identically L, a contradiction arises which shows that $F_{12} - (k_1 + k_2)$ is disconnected. But then k_1 and k_2 are on the boundary of a certain bounded complementary domain of F_{12}.†

Suppose then that no maximal connected subset of $F_{12} - M_{12}$ has a limit point on (M_1) and another on (M_2). As before $F_{12} - (a + b)$ is disconnected and so a and b lie on the boundary of some bounded complementary domain D of F_{12}. The boundary of this domain is a subset of F_{12} and is a simple closed curve which is divided into two arcs (ayb) and (azb) by the points a and b. As neither of these arcs has any points exterior to M_{12} it is clearly impossible for them both to have points on either (M_1) or (M_2), or for one to have points on both (M_1) and (M_2). But if (ayb) has no point on (M_1) it must have at least one on (M_2) as otherwise it is an $(n + 1)$st arc from a to b in F. Then similarly (azb) has a point on (M_1). Thus as one of the two arcs has a point on (M_1) and the other one on (M_2) the domain which they bound is a cross-domain of F. The same result of course follows when (ayb) has a point on (M_1).

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* R. L. Wilder, "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol 7, p. 342.

† G. T. Whyburn, Abstract of an unpublished paper in the *Bulletin of the American Mathematical Society*, Vol. 33, p. 388, No. 10.

The theorem is thus true for F_{12}. But as the argument used in the proof involved only F_{12} itself it evidently applies immediately to each set F_{jj+1} $(j = 2, 3, \cdots, n-1)$.

Notation and definitions. Let δ be any c-domain of F_{jj+1}, that is of F in I_{jj+1}. Its boundary Δ has a first point a_j on M_j and also a last b_j, likewise it has a first and last a_{j+1} and b_{j+1} on M_{j+1}. It may happen, (1) that $a_j \equiv a \equiv a_{j+1}$, (2) that $b_j \equiv b \equiv b_{j+1}$, (3) that $a_j \equiv a \equiv a_{j+1}$ and $b_j \equiv b \equiv b_{j+1}$. In these cases it will be convenient still to speak of these coincident pairs as if they were distinct points.

The symbols $a_j x a_{j+1}$ and $b_j y b_{j+1}$ each represent two arcs or points of Δ; of these one from the first pair and one from the second have at most end points in common, and these two will be called the *a- and b-arcs of* δ, Δ, even although they may reduce to points. If γ and δ are c-domains of F_{jj+1} and γ precedes δ these two will be said to be *adjacent* when the b-arc of Γ, the a-arc of Δ, and the subarcs of M_j and M_{j+1} which their ends intercept either fail to form a simple closed curve or form one whose interior contains no c-domain of F.

I(b) THEOREM. *If γ and δ are adjacent c-domains of F_{jj+1}, γ preceding δ, then from a point of the b-arc of Γ in I_{jj+1} to a point of the a-arc of Δ in I_{jj+1} there is an arc of $F_{jj+1} - M_{jj+1}$ having only an end point in common with either Γ or Δ.*

Proof: The argument will be given only for F_{12}. Let the *a- and b-arcs* in question be $b_1 y b_2$ and $a_1 z a_2$. If $(b_1 y b_2) \times (a_1 z a_2) \neq 0$ any point of the intersection will be considered to satisfy the theorem. If this product is zero the two arcs involved in it form with the arcs of (M_1) and (M_2) which their ends intercept a simple closed curve C with interior I. If I contains no constituent of F with limits on both $(b_1 y b_2)$ and $(a_1 z a_2)$ by an argument like that in I(a) I contains a c-domain of F. Hence the theorem. Clearly the theorem is similarly true in the special cases $\gamma + \Gamma = a$, and $\delta + \Delta = b$.

I(c) THEOREM. *If the c-domains of F_{jj+1} form an infinite set this set must have at least one limit point not already a limit point of some finite subset of the set, and only a or b can be such a limit point.*

Proof: That one such limit must exist is evident from the fact that the domains in question form a bounded infinite set. Suppose c is one distinct from a and b. At least one of the distances c to (M_j), c to (M_{j+1}) is not zero; suppose the first is ϵ. Let c be the center of a circle of radius $\epsilon/2$. Within circle an infinite number of the c-domains in question must have points and as each of these must also have a boundary point on (M_j) each c-domain of this infinite number must have diameter greater than $\epsilon/2$, an evident contradiction.

It may also be noted that if a is one of the limits of the c-domains of F_{jj+1} it can not be a boundary point of any c-domain of F_{jj+1}, for if it ... point of Δ and if p_j and p_{j+1} are points of Δ on (M_j) and (M_{j+1}), ... are of δ, p_jyb one on M_j and $p_{j+1}zb$ one of M_{j+1}, then a is exterior to the simple closed curve $p_jxp_{j+1}zbyp_j$ while all but δ of the set of c-domains of F_{jj+1} of which a is a set limit are interior to this curve. It is clear that the theorem may be stated equally truly for the boundaries of the c-domains.

Notations and definitions. Owing to I(c) the c-domains of F_{jj+1} may be represented as $\cdots, \delta_{-m}, \cdots, \delta_{-2}, \delta_{-1}, \delta_0, \delta_1, \delta_2, \cdots, \delta_m, \cdots$, δ_0 having been selected at random. The i c-domains $\gamma_{jj+1}, \cdots, \gamma_{j+i-1j+i}$ will be said to form a *chain of F_{jj+i}* from M_j to M_{j+i} if (1) one and only one of them is a complementary domain of each of the curves $F_{jj+1}, \cdots, F_{j+i-1j+i}$; and (2) Γ_{q-1q} has at least one point of (M_q) in common with Γ_{qq+1} $(q=j+1, \cdots, j+i-1)$. $\Gamma_{jj+1}, \cdots, \Gamma_{j+i-1j+i}$ will be referred to as *links* of the chain, and the points of Γ_{jj+1} on (M_j) and of $\Gamma_{j+i-1j+i}$ on (M_{j+i}) as *end points* of the chain.

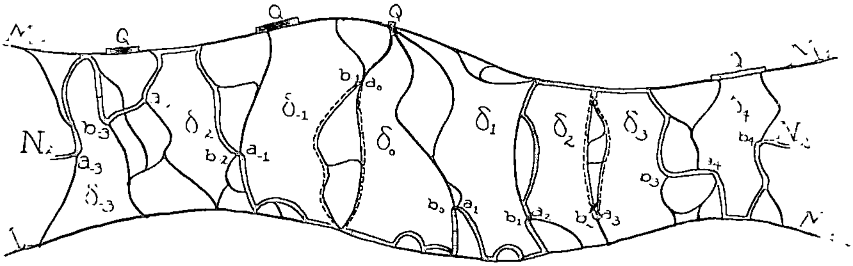

FIG. 1.

The heavy line is N_2. From the sum have been omitted the dotted loops.

I(d) THEOREM. *If Q is any subset of (M_j) there are in F_{jj+i} i independent arcs $N_{j+1}, N_{j+2}, \cdots, N_{j+i}$ from a to b, F_{t-1t} containing N_t $(t = j+1, \cdots, j+i)$; where $N_{j+1} \times Q = 0$, and $N_t \times (M_t)$ is a subset of the end points on M_t of the chains of F_{jt} which have at least one opposite end point among the points of Q.*

For the sake of simplifying the subscripts this theorem without loss of generality will be proved for F_{11+i}. The proof will be by induction. During the first part refer to fig. 1.

Proof: Let the set of all the c-domains of F in F_{12} be represented by $\sum_i \delta_i = \cdots + \delta_{-s} + \cdots + \delta_0 + \cdots + \delta_t + \cdots$ and the set of their boundaries by $\sum_i \Delta_i = \cdots + \Delta_{-s} + \cdots + \Delta_0 + \cdots + \Delta_t + \cdots$. Let $\{\Delta\}$ be the subset of $\sum_i \Delta_i$ consisting of all and only those for which $\Delta_i \times Q \neq 0$. If a is not a limit of the set $\sum_i \delta_i$ it will be convenient to consider it the first of the set $\sum_i \Delta_i$, under similar circumstances to consider b the last; under this convention $\sum_i \Delta_i$ will necessarily contain three or more members. If Δ_{i-1} and Δ_i are adjacent elements of $\sum_i \Delta_i$ let $b_{j-1}a_j$ be the arc of I(b) (of course possibly $b_{j-1}a_j$ represents a point, even maybe a or b) for every j, and let similarly a_jb_j be that arc of Δ_j for which $a_jb_j \times M_1 = 0$ if $\Delta_j \times Q \neq 0$ otherwise for which $a_jb_j \times M_1 \neq 0$. Now the sum $a_{j-1}b_{j-1} + b_{j-1}a_j + a_jb_j$ is either an arc from a_{j-1} to b_j or fails to be one only because $(a_{j-1}b_{j-1}) \times (a_jb_j) \neq 0$ In the latter case, however, a proper subset of $a_{j-1}b_{j-1} + a_jb_j$ is an arc from a_{j-1} to b_j and the sum above will be then supposed to represent this arc; the same meaning is to be assigned to the more extensive sums below. Thus $b_{-r-1}a_{-r-1} + a_{-r-1}b_{-r} + b_{-r}a_{-r+1} + \cdots + b_{-2}a_{-1} + a_{-1}b_{-1} + b_{-1}a_0 + a_0b_0 + b_0a_1 + a_1b_1 + b_1a_2 + \cdots + b_{s-1}a_s + a_sb_s + b_sa_{s+1}$ is a simple continuous arc from b_{-r-1} to a_{s+1} say N_2 for which $N_2 \times Q = 0$ and $N_2 \times (M_2) = \{\Delta\} \times (M_2)$ by reason of the choice of a_jb_j. If a finite r and s exist for which $\Delta_{-r-1} \equiv a$ and $\Delta_{s+1} \equiv b$ the first step in the induction is now established. If one or both subscripts fail to exist continuing the process indefinitely toward a or b gives rise to a set which becomes an arc upon the addition of its limits which by I(c) are at most a or b. In all cases then is the first step complete.

Assuming now that in F_{1k} from a to b there are $k - 1$ independent arcs $N_2, \cdots, N_k$ where $1 < k < i$, $N_2 \times Q = 0$, and $N_k \times (M_k)$ is a subset of the ends on (M_k) of those chains of F_{1k} among the ends on (M_1) of each of which there is at least one point of Q, the same thing for one greater subscript is to be proved. Let $\sum_i \gamma_i$ represent the set of c-domains of F_{kk+1}, and let $Q_k = N_k \times (M_k)$. By the first part of the proof there is in K_{kk+1} from a to b an arc N_{k+1} for which $N_{k+1} \times Q_k = 0$, hence an arc independent of $N_2, \cdots, N_k$, and for which $N_{k+1} \times (M_{k+1})$ is a subset of $\{\Gamma_{kk+1}\} \times (M_{k+1})$ where $\{\Gamma_{kk+1}\}$ represents the subset of all the boundaries Γ_i of domains of $\sum_i \gamma_i$ for which $\Gamma_i \times Q_k \neq 0$. But if $\Gamma_i \times Q_k \neq 0$ then $\Gamma_i \times (M_k)$ includes an end of some chain from (M_1) to (M_k) and γ_i is a last domain of a chain from (M_1) to (M_{k+1}) so $\Gamma_i \times (M_{k+1})$ consists of ends on (M_{k+1}) of chains of F_{1k+1} whose ends on (M_1) include at least one point of Q for each chain, since the points of Q_k are the ends of similar chains of F_{1k}. It is to be noted that this construction makes $N_t \times (M_t)$ not a proper subset of the end points on M_t of the chains specified. The induction, however, is complete.

I(e) COROLLARY. *If* $Q_{t+1} = (M_{t+1}) \times \{\Gamma_{tt+1}\}$ $(t = 1, 2, \cdots, [illegible] - 1)$; [illegible] $Q_{t+1} = (M_{t+1}) \times N_{t+1}$ *and if* $\{\overline{\Gamma}_{tt+1}\} = \sum_i \Gamma_i - \{\Gamma_{tt+1}\}$ *then*

$$N_{t+1} \times (M_t) = \{\overline{\Gamma}_{tt+1}\} \times (M_t) \subset (M_t) - Q_t.$$

The construction used above demonstrates this.

I(f) COROLLARY. *In* F_{1n} *there is at least one chain from* (M_1) *to* (M_n). For suppose not and apply I(d) to F_{1n} with $Q = (M_1)$. Then if I [illegible] are $n + 1$ independent arcs from a to b, namely $M_1, N_2, N_3, \cdots, N$ [illegible] N [illegible], and M_n as clearly $(M_1) \times N_2 = 0$ and $N_n \times (M_n) = 0$.

Remarks: The general theorem is now proved, except for the final step [illegible] § 5, in case both a and b are on the simple closed curve which forms the outer boundary of F; for in this case M_1 and M_n may always be chosen as the two arcs into which a and b divide this curve and the statements I(a) [illegible] I [illegible] inclusive still hold. The unbounded complementary domain of F together with any chain of F_{1n} existing by I(f) constitute the set of n domains employed in § 5. In §§ 2, 3, 4 it will accordingly be assumed that either [illegible] is not on the outer boundary of F. By transformations of simple sorts some of the cases in which this is true can be reduced to the case already treated, but as these are special it seems not worth while to treat them separately.

The points of F_{1n} which are either end points of chains of F_{1n} or are points common to two links of some such chain are of special importance and will be called *invariant.* As already mentioned, if $F_{1n} = F$ the argument of § 5 applies without any further discussion with the consequence that the set of invariant points of F_{1n} is a subset of every set of n independent arcs of F from a to b.

2. An analysis of the non-vacuous set $F - F_{1n}$ must now be made. Let it be G; it consists of a set of maximal connected sets of which at most a finite number are of diameter greater than any assigned positive number ϵ,[*] and each of which has at least two limit points in M_{1n}.[†] Let $G = H + J$ [illegible] $+ K$ where, H consists of those constituents of G all whose boundary points are in either (M_1) or (M_n) but not both, and none of which contains an arc which with an arc of (M_1) (or of (M_n)) forms a simple closed curve having I_{1n} in its interior (the sets making up H are at most countably infinite); J consists of sets at most finite in number differing in no respect from those in H except that a simple closed curve as described in connection

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W. L. Ayres, "Concerning Continuous Curves and Correspondences," *Annals of Mathematics*, Second Series, Vol. 28, No. 3, 1927, July, p. 396.

[†] W. L. Ayres, "Concerning Arc Curves and Basic Sets of a Continuous Curve."

with H now exists containing I_{1n} in its interior; I is made up of all those constituents of G among whose boundary points are a or b but obviously not both, it consists of subsets A, all whose maximal connected parts have a as limit, and B, similarly defined; K is composed of all those constituents of G not already included in I and having at least one limit on each of (M_1) and (M_n).

II(a) THEOREM. *The set $F_{1n} + H$ possesses all the properties of F_{1n} that were derived in section* 1.

Let any maximal connected part of H be H_i. Now $\overline{H}_i$, the set plus its limit points, is a continuous curve.* As F_{1n} is a continuous curve and as for every i, $\overline{H}_i \times F_{1n} \neq 0$, $F_{1n} + \sum_i \overline{H}_i$ is a continuous curve.† But since $\sum_i \overline{H}_i - H \subset F_{1n}$, $F_{1n} + \sum_i \overline{H} = F_{1n} + H$ and so $F_{1n} + H$ is also a continuous curve. The arc curve of $a + b$ with respect to $F_{1n} + H$ is $F_{1n} + H$, so the outer boundary of $F_{1n} + H$ is a simple closed curve. This simple closed curve must have a point on each of (M_1) and (M_n) as otherwise H must have contained constituents properly belonging to J or K. Let now $H + M_{1n}$ be inverted with center of inversion a point of I_{1n}, and M_{1n} as invariant simple closed curve. The unbounded complementary domain of $F_{1n} + H$ becomes a bounded cross-domain of $\Pi(M_{1n} + H)$ and as H had no constituent with limit points on both (M_1) and (M_n), $\Pi(M_{1n} + H) - M_{1n}$ has no constituent with limit points in both (M_1) and (M_n). Thus as in I(a) a and b are on the boundary of a bounded complementary domain of $\Pi(M_{1n} + H)$ which is also a c-domain of it. When a c-domain of this type exists it is clearly the only possible c-domain and so must be the inverse of the unbounded complementary domain of $F_{1n} + H$ which accordingly has a boundary containing a and b. The two arcs into which a and b divide this boundary may be taken as M_1 and M_n. The theorem is thus proved. Hereafter F will represent $F_{1n} + H$ so that $G = I + J + K$.

If $A \neq 0$, it has last points i_{1a} and i_{na} on M_1 and M_n respectively perhaps not both different from a but clearly both different from b. With regard to B there are similar points.

II(b). *A contains arcs (axi_{1a}), (axi_{na}) and B contains similar arcs.*

For a is a limit of every constituent of A and if i_{1a} were not a limit of some particular one there would be a contradiction.

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* See second footnote referred to in section 1 and in the theorem there used read "exterior" as well as "interior."

† H. M. Gehman, "Some Conditions under which a Continuum is a Continuous Curve," *Annals of Mathematics*, Second Series, Vol. 27, p. 381.

II(c). *All the limits of J occur on an arc of* (M_1) [*or* (M)] *which except perhaps at its ends contain no limits of I or K.*

II(d). *The only limit points, not already limits of a particular constituent, which K can have are a and b.*

The set K needs to be discussed in detail. Obviously the limits of any constituent of it on (M_1) or (M_n) occupy an interval whose interior points are none of them limits of any other constituent. In every K_j draw an arc from a limit on (M_1) to one on (M_n). This arc forms with an arc of F going through a a simple closed curve whose interior will be called I_j. It may happen for some arcs in K_j that I_j contains I_{1n} and for some not; clearly at most one constitutent of K_j can have this property, if such a constituent exists it will be represented by K_0. For each of the remaining constituents K_j either I_j never contains I_{1n} in which case j will be considered negative, or always contains I_{1n} in which case j will be positive. If i and j are negative and $I_i \subset I_j$ then let i and j be integers such that $i < j$, whereas if i and j are positive and $I_i \supset I_j$ let i and j be integers and $i < j$. In view of II(c) and obvious facts, this representation is possible and may be taken as K: $\cdots, K_{-2}, K_{-1}, K_0, K_1, K_2, \cdots, K_l, \cdots$, where of course K_0 may be vacuous. Precedence among the constituents of K will hereafter be determined by referring to their subscripts.

The constituent K_j has on (M_1) and (M_n) first limit points a_{1j} and a_{nj} and last limit points b_{1j} and b_{nj}. On (M_1) the interval $b_{1j}a_{1j+1}$ will be called C_j and on (M_n) the interval $b_{nj}a_{nj+1}$ will be called D_j, and C_j. D_j will be referred to as a pair of *corresponding intervals.* If $K_0 = 0$ C_0 and C_{-1} will represent the same interval, D_0 and D_{-1} likewise. If $A \neq 0$ there is a first constituent of K say K_{-l} and a first pair of corresponding intervals $C_{-l-1} = i_{1a}a_{1,-l}$ and $D_{-l-1} = i_{na}a_{n,-l}$, one but not both of which may begin with a. Even when A is vacuous there may of course be a first K_{-l} with both C_{-l-1} and D_{-l-1} beginning at a. Relations between K and B of course present no features not possible between K and A. Obviously C_j and D_j may reduce to points but will nevertheless be referred to as intervals. When a (or b) is end of an interval the interval will be assumed to lack this point.

II(e). *Any pair of corresponding intervals is on the boundary of a common complementary domain of F.*

Assuming K and J vacuous the theorem is obvious as then $i_{1a}i_{1b} = C_0$ and $i_{na}i_{nb} = D_0$ are the only corresponding intervals and are on the outer boundary of F. If $J = 0$ but $K \neq 0$ invert $F - F \times I_{1n}$ from any point of I_{1n} leaving M_{1n} invariant and getting a set $\bar{F}$. By the same argument as that of the first case of I(a) any point c of C_j and any point d of D_j are on the boundary of a common bounded complementary domain of $\bar{F}$, and any other point c' of C_j can fail to be on the same boundary only if a constituent of $\bar{F} - M_{1n}$ has a limit point interior to C_j, which is ridiculous; similarly any point d' of D is on the boundary of this same domain. Hence C_j and D_j are on the boundary of a common complementary domain of F. If the arc of II(c) occurs on one of a pair of corresponding intervals as it may if $J \neq 0$ it can occur only on C_0 or D_0 and on one of these only if $K_0 = 0$. In this case it again separates $C_0 = C_{-1}$ (or $D_0 = D_{-1}$) into two distinct sets both of which are easily proved to be on the boundary of the same domain as $D_0 = D_{-1}$ (or $C_0 = C_{-1}$).

The relation of the chains of F_{1n} to the pairs of corresponding intervals is of critical importance. Let C and D be any corresponding pair. If at least one chain of F_{1n} has ends on both C and D, these two are said to be *joined.* If no chain has ends on C (or D) then C (or D) is said to be *free.* If the only chains having ends on C (or D) have opposite ends all preceding D (or C) then C (or D) is said to be *a-joined*; if all following, C (or D) is said to be *b-joined*; if both preceding and following, C (or D) is said to be *ab-joined.* The following is a complete list of the possible relations of C and D to the chains of F_{1n}:

1 C free, D free
2 C free, D *a*-joined
3 C free, D *b*-joined
4 C free, D *ab*-joined
5 C *a*-joined, D free
6 C *b*-joined, D free
7 C *ab*-joined, D free
8 C *a*-joined, D *b*-joined
9 C *b*-joined, D *a*-joined
10 C *a*-joined, D *a*-joined
11 C *a*-joined, D *ab*-joined
12 C *b*-joined, D *b*-joined
13 C *b*-joined, D *ab*-joined
14 C *ab*-joined, D *a*-joined
15 C *ab*-joined, D *b*-joined
16 C *ab*-joined, D *ab*-joined
17 C and D joined.

The object of the next two sections is to show that the fundamental assumptions of the theorem imply that at least one pair of corresponding intervals is joined (relation 17). Section 3 is devoted to some preliminary considerations.

3. In any chain of F_{1n} an arc may be drawn from an end of the chain on (M_1) to one on (M_n) by way of domains of the chain with the exception of precisely n points one on each of the arcs (M_j) $(j = 1, \cdots, n)$ necessarily invariant. Such an arc will be called an *arc of the chain.* Two important arcs may also be drawn on the boundaries of domains of the chain.

Let an arc be drawn from the last end of the chain on M_1 to the last end on M_n so as to contain the b-arc of every link of the chain, this will be called the *b-boundary arc* of the chain. It may reduce to point b. The *a-boundary arc* is similarly defined. Special cases of no significance arise if more than one domain of the chain has a or b as limit.

III(a) THEOREM. *If on M_1 there is the order $a\,c_1 d_1 b$ and on M_n $a\,d_n c_n\,b$ and c_1, c_n are ends of a chain $\gamma_1, \cdots, \gamma_{n-1}$, d_1, d_n of a chain $\delta_1, \cdots, \delta_{n-1}$: then c_1, d_n are ends of a chain, also c_n, d_1.*

Proof: Construct an arc $c_1 c_n$ of $\gamma_1, \cdots, \gamma_{n-1}$ and one $d_1 d_n$ of $\delta_1, \cdots, \delta_{n-1}$. Owing to the given order of their end points these two arcs intersect in I. Counting from c_1 let p be the first common point. If p is interior to some curve $M_{j,j+1}$, $(j = 1, 2, \cdots, (n-1))$ then γ_j and δ_j coincide as p is interior to both; accordingly $\gamma_1, \cdots, \gamma_{j-1}, \gamma_j \equiv \delta_j, \delta_{j+1}, \cdots, \delta_{n-1}$ and $\delta_1, \cdots, \delta_{j-1}$, $\delta_j \equiv \gamma_j, \gamma_{j+1}, \cdots, \gamma_{n-1}$ are the chains fulfilling the requirements. If p is on M_j then p is in each of the sets Δ_{j-1}, Δ_j, Γ_{j-1}, and Γ_j. But then $\gamma_1, \cdots, \gamma_{j-1}$, $\delta_j, \cdots, \delta_{n-1}$ and $\delta_1, \cdots, \delta_{j-1}, \gamma_j, \cdots, \gamma_{n-1}$ answer the theorem.

This theorem shows that cases 10 to 16, inclusive, at the end of section 2 all reduce to 17 and from now on will be omitted. A common feature of cases 1 to 7 inclusive is the occurrence of at least one free interval. Let us consider the situation.

Any free interval must be a subset of some maximum free segment at least one end of which (but not necessarily both, as one may be a or b) is an end of a chain. Let $(l_1 f_1)$ be a maximum free segment with $f_1 = b$. Let γ_1 be the last c-domain of F_{12} which is a link of a chain having l_1 on Γ_1. If Γ_1 contains b complete the chain $\gamma_1, \cdots, \gamma_{n-1}$ in any convenient way. If not let γ_2 be the last c-domain of F_{23} which is part of a chain from $al_1 \subset M_1$ to (M_n) for which $\Gamma_1 \times \Gamma_2 \neq 0$. Either Γ_2 contains b in which case complete the chain $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$ in any convenient way, or it does not. In this case let γ_3 be the last c-domain of F_{34} which is part of a chain from al_1 to (M_n) with $\Gamma_2 \times \Gamma_3 \neq 0$. If b is on Γ_3 complete the chain at random, if not repeat the process above, and so on until a link of a chain is thus found with b on its boundary, or a complete chain consisting of last possible links is found from l_1 to an extreme point l_n on (M_n). Draw the b-boundary arc of the chain above until it reaches b (call it then $l_1 l b$) or until it reaches (M_n) (call it then $l_1 l l_n$).

If l_1 is a, then a similar construction working toward a results either in an arc $f_1 f a$ or an arc $f_1 f f_n$. If $a \neq l_1$ and $f_1 \neq b$ then either l_1 and f_1 are on the boundary of some common c-domain of F_{12} or there is from l_1 an arc which is b-boundary arc of a chain consisting of last possible domains to an extreme point l_n on (M_n), and from f_1 to f_n (an extreme point of (M_n) toward a) an arc which is a-boundary arc of a chain of first possible domains.

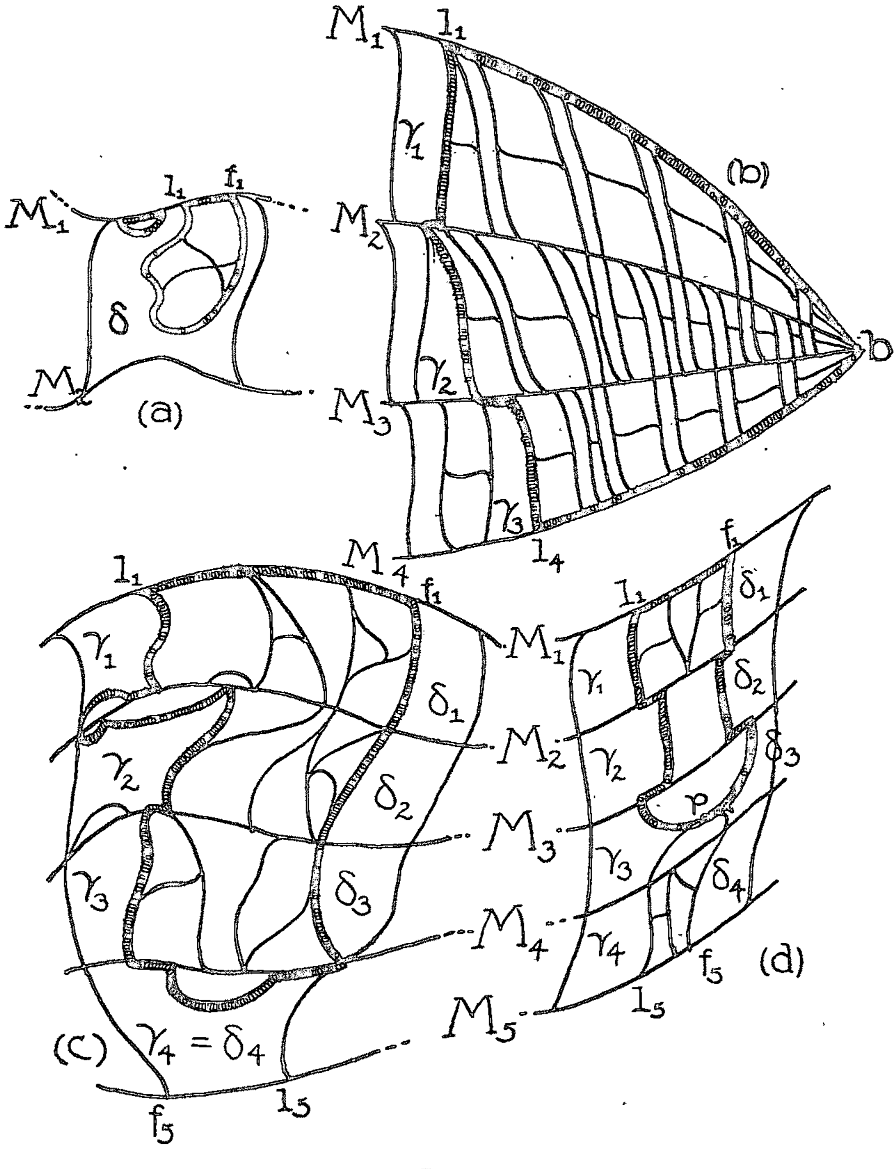

Fig. 2.
Pockets are enclosed by heavy line.

The arcs $l_1 l_n$ and $f_1 f_n$ may not intersect but if they do proceeding from l_1 call the first common point p and in the sum of the arcs draw the arc $l_1 p f_1$, where possibly $l_n = p = f_n$.

Now by means of the minor[*] arcs of M_{1n} complete the following simple closed curves; (1) if l_1 and f_1 are on the same Δ join them along the minor arc[*] of Δ and the minor arc of M_{1n}, fig. 2(a); (2) if there is only one chain and arc, l_1lbl_1, $l_1ll_nbl_1$, f_1faf_1, $f_1ff_naf_1$, fig. 2(b); (3) if there are two arcs intersecting, $l_1pf_1l_1$, fig. 2(c); (4) if there are two arcs not intersecting, $l_1l_nf_nf_1l_1$. These curves and the subsets of F_{1n} which they enclose will be called *pocket subsets* or *pockets* of F_{1n} determined by (l_1f_1). Clearly maximum free segments on M_n as say (l_nf_n) will determine similar sets in connection with it. Evidently the boundary of any pocket of F_{1n} with respect to F_{1n} consists of a finite number of invariant points unless the point p happens to exist and be not invairant, fig. 2(d).

III(b) THEOREM. *In the pocket subset of* (l_1f_1) (*or* (l_nf_n)) *the arcs* $M_1, M_2, \cdots$, (*or* $M_n, M_{n-1}, \cdots$) *may be diverted in such a way that* $M_1 \times (l_1f_1) = 0$ (*or* $M_n \times (l_nf_n) = 0$) *and yet the set remains independent.*

Proof: The theorem is obvious in case (1) above as there M_1 may merely follow the minor arc of Δ, $M_2, \cdots, M_n$ being undisturbed. The other cases will be handled by I(d). Let $Q = (l_1f_1)$, and construct the arcs N_2, N_3, $N_4, \cdots$. Suppose $l_2, l_3, \cdots$, are the last points common to successive links of $\gamma_1, \cdots, \gamma_{n-1}$ the chain described in connection with the pocket above, and $f_2, f_3, \cdots$, the first points common to the successive links of $\delta_1, \cdots, \delta_{n-1}$ the chain of first links also used. As seen in the discussion of pocket subsets ? above one or other of these chains may fail to exist so that the corresponding set of points just mentioned reduces either to a or to b; but as this circumstance has the effect of simplifying matters special treatment of such a case is not necessary. Now if not vacuous $Q_2 = N_2 \times M_2 = M_2 \times \{\Delta_{12}\}$† and so, l_2f_2 being an arc of M_2, $Q_2 \subset l_2f_2 - l_2f_2 \times (\Gamma_2 + \Delta_2)$, as otherwise, referring to III(a) and the supposition that (l_1f_1) is free, a contradiction arises. Therefore evidently l_2 precedes Q_2, and Q_2 precedes f_2. Likewise if $l_3f_3 \subset M_3$ and $Q_3 \neq 0$, then $Q_3 \subset l_3f_3 - l_3f_3 \times (\Gamma_3 + \Delta_3)$, and so on. But by supposition (l_1f_1) is free and so $Q_n = 0$, hence some last Q, say $Q_k (k < n)$, is non-vacuous. Here also if arc $l_kf_k \subset M_k$, then $Q_k \subset l_kf_k - l_kf_k \times (\Gamma_k + \Delta_k)$, evidently with l_k preceding Q_k and Q_k preceding f_k. As the boundary of the pocket with respect to F_{1n} is a subset of $\sum_{i=1}^{n} l_i + \sum_{i=1}^{n} f_i + p$ and as by I(e) for $2 \leqq j \leqq k \; l_{j-1} + f_{j-1} \subset N_j \times M_{j-1}$ it is now clear that between l_{j-1} and f_{j-1} N_j is interior to the pocket. Moreover, since $Q_{k+1} = 0$, N_{k+1} skirts all the c-domains of F_{kk+1} by way of M_k, and thus clearly does not leave the pocket between l_k and f_k. Between l_i and f_i let M_i coincide with N_{i+1} and be unchanged in position elsewhere $(1 \leq i \leq k)$. The arc M_1 thus avoids $(l_1 f_1)$; $M_2, \cdots, M_k$ remain independent of each other, are unchanged in position outside of the pocket, and are independent of $M_{k+1}, \cdots, M_n$ which have remained unaltered. It is to be noticed that M_k may possibly now pass through the perhaps not invariant point p, fig. 2(d).

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[*] By minor arc of M_{1n} will here be meant only that which does not contain both a and b; by minor arc of Δ is meant that arc of it which with the minor arc of M_{1n} forms a simple closed curve not containing δ. These definitions can cause no confusion in this connection as a, l_1, f_1, b are all on M_1.

† The symbol $\{\Delta_{12}\}$ has a meaning similar to $\{\Gamma_{kk+1}\}$ in the proof of I(d).

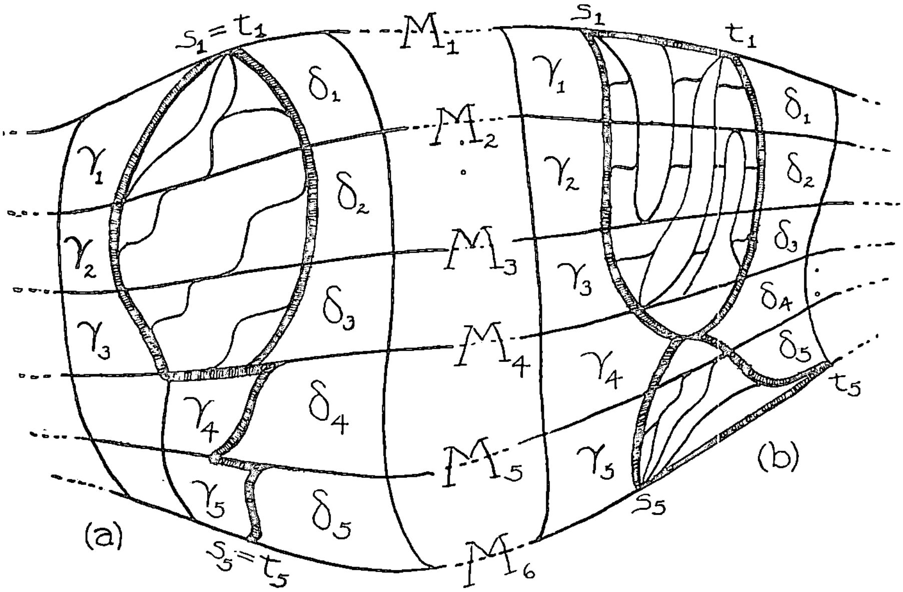

FIG. 3.
Switches $/s_n t_1/$ in heavy outline.

Cases 8 and 9 when they occur imply a condition of different significance. This will now be discussed.

Let C and D be a pair of corresponding intervals of type 8. There are on C end points of one or more chains of F_{1n} but all the opposite ends of these precede D on (M_n), the last such point preceding being say s_n. On the other hand there is on (M_1) at least one end of a chain whose ends on (M_n) follow s_n for D is not free. Let t_1 be the first point of (M_1) which is an end of a chain among whose opposite ends are points following s_n. That is t_1 is the first end of some chain $\delta_1, \cdots, \delta_{n-1}$ with a-boundary arc $t_1t_2 \cdots t_{n-1}t_n$, t_n not preceding s_n but perhaps coinciding with it, whose links are the first ones possible satisfying the other requirements.

Two possibilities arise with regard to t_1. (1) Perhaps t_1 is a last but not only end of a chain of last domains $\gamma_1, \cdots, \gamma_{n-1}$ ($\gamma_1 \neq \delta_1$ evidently) with b-boundary arc $s_1s_2 \cdots s_{n-1}s_n$ ($t_1 = s_1$).* Here s_1 is not on C but other ends of this chain may be. This chain may have ends on (M_n) preceding s_n but none following. (See fig. 3(a)). (2) Otherwise t_1 is an only end or is not an end of chains whose first links precede δ_1 and whose last opposite end is s_n. Here as t_1 is not on C it must be preceded by some last end s_1 of a chain of last links $\gamma_1, \cdots, \gamma_{n-1}$ and b-boundary arc $s_1s_2, \cdots, s_{n-1}s_n$ ($s_1 \neq t_1$), fig. 3(b). The segment (s_1t_1) is free. The supposition that t_1 is a limit of preceding end points of chains reduces easily to (1).

With regard to s_n there are also two possibilities as follows: (1) $\Delta_{n-1} \times s_nb \supset \Delta_{n-1} \times M_n \supset s_n$ and $\neq s_n$ where s_nb is an arc of M_n; and (2) s_n precedes $\Delta_{n-1} \times M_n$ hence (s_nt_n) is free, t_n being first point of $\Delta_{n-1} \times M_n$. The chains γ and δ can neither cross nor have a common link. The subset of F composed of s_1s_n (b-boundary arc of γ), s_nt_n on M_n, t_nt_1 (a-boundary arc of δ), t_1s_1 on M_1, and the subset of F_{1n}, if any, which is included between them will be called a *switch subset* or *switch* of F_{1n}. The notation for it will be $/s_nt_1/$, s_1 and t_n being omitted altogether as of less importance even when distinct from s_n and t_1. The bar / indicates that C is a-joined, D b-joined, and makes the property of III(c) below easier to remember.

The discussion above was referred not only to the corresponding pair $C\,D$ but also to M_1 and point a. A similar discussion for $C\,D$ referring to point b and M_n determines a similar set but not necessarily the same one. If, however, the sets are not identical a particular pair of intervals (no longer corresponding) may be chosen instead of $C\,D$ which will when discussed from M_n and b give the identical set $/s_nt_1/$. Similarly 9 on $C\,D$ either from a and M_n or b and M_1 may be used to determine switches, the same or not, for which the symbol will be $\diagdown s_1t_n \diagdown$, as above s_i preceding t_i ($i = 1, n$).

The most important property of the switch is the following:

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[*] Although s_n itself is a point of the b-boundary arc of a chain with ends on C, $s_1s_2 \cdots s_{n-1}s_n$ is not necessarily such an arc; the chain determining it may have all its ends on (M_1) following C. It is a convenience however to introduce the symbol s_n at once, before indicating which of the b-boundary arcs of which it may be an end is the most important.

III(c) THEOREM. *In the Switch* $/s_nt_1/$ *the arcs* $M_2, M_3, \cdots, M_n$ *can be shifted from the points* $s_2, s_3, \cdots, s_n$ *to the points* $t_1, t_2, \cdots, t_{n-1}$ *without losing their independence, and in such a way that* M_2 *reaches* M_1 *first at* t_1 *and* M_n *leaves its original position for good at* s_n. *In the switch* $\diagdown s_1t_n \diagdown$ *the arcs* $M_1, M_2, \cdots, M_{n-1}$ *can be shifted from* $s_1, s_2, \cdots, s_{n-1}$ *to* $t_2, t_3, \cdots, t_n$ *without losing their independence, and in such a way that* M_1 *leaves its original position for good at* s_1 *and* M_{n-1} *reaches* M_n *at* t_n *and not before.*

Only one of these statements will be proved as the procedure for proving the other is similar.

Proof: Refer to I(d) and I(e), and to prove the first let Q be the arc (at_1) of M_1. Construct the arcs $N_2, \cdots, N_n$. For either supposition concerning t_1 clearly the γ chain is one having an end in Q whereas no domain of the δ chain can be a member of a chain having an end in Q. Thus $N_2, \cdots, N_n$ coincide with $M_2, \cdots, M_n$ at the points $s_2, \cdots, s_n$, but with $M_1, \cdots, M_{n-1}$ at $t_1, \cdots, t_{n-1}$. Let $M_2, \cdots, M_n$ follow $N_2, \cdots, N_n$ from $s_2, \cdots, s_n$ to $t_1, \cdots, t_{n-1}$. M_2 hence meets M_1 first at t_1 as t_1 is the first point of M_1 on the boundary of a c-domain of F_{12} which has no point of Q on its boundary. As s_n is the last end of any chain having an opposite end in Q, N_n and with it M_n as diverted leaves the original position of M_n at s_n. $N_2, \cdots, N_n$ having been independent so are $M_2, \cdots, M_n$ as now diverted.

In the next section of this proof it will be necessary to make various, even an infinite number of, switch and pocket constructions simultaneously. In what ways these constructions may affect each other is accordingly an essential problem. Various observations follow.

On Switches. (1) Any switch and its associated chains occupy on both (M_1) and (M_n) more than one point, for in $/s_nt_1/$ if $s_1 = t_1$ and $(as_1) \subset M_1$ then $\Gamma_1 \times (as_1) \neq 0$, and if $s_n = t_n$ and $(t_nb) \subset M_n$ then $\Delta_{n-1} \times (t_nb) \neq 0$.*

(2) Given $/s_nt_1/$, no point of (at_1) arc of M_1 is joined by any chain of F_{1n} to a point of (s_nb) arc of M_n. Also given $\diagdown s_1t_n \diagdown$ no point of (at_n) is so joined to one of (s_1b). These follow from the nature of the sets.

(3) If among the point pairs s_1, s_1', t_1, t_1', s_n, s_n', t_n, t_n' one pair is non-coincident then $/s_nt_1/$ and $/s_n't_1'/$ have at most n points and these all invariant in common. This is the argument for one case. Suppose t_1' follows t_1 on (M_1); then s_1' does not precede t_1 as $(s_1't_1')$ is free. If s_n' were to precede t_n, then a point, t_1, preceding t_1' would be joined by a chain to a point,

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* Notation here is as in the discussion immediately preceding III(c).

t_n, following s_n' which contradicts (2). But then the links of the chain γ' joining s_1' to s_n' follow or at most coincide with those of δ joining t_1 to t_n. Hence the switches have in common at most the points $s_i'=t_i$ $(i=1, 2, \cdots, n)$ which are invariant.

The same thing is true of $/s_nt_1/$ and $\diagdown s_1't_n'\diagdown$ if t_n' follows t_n, or s_n' follows s_n, or if s_1' precedes s_1, or t_1' precedes t_1.* When t_n' follows t_n the proof is this. As $(s_n't_n')$ is free s_n' does not precede t_n, and if $s_n'=t_n$ then $t_n\neq s_n$ by observation (1). But then if s_1' precedes t_1 a point, namely s_1' preceding t_1, is joined to a point, namely s_n' following s_n. This contradicts (2) and the statement follows as above.

(4) From (3) it appears that in any set of switches of the same type the switch construction may be made simultaneously as any two have in common only invariant points, the two sets of diverted arcs having in no wise interfered while changing position. Also from (3) it follows that two switches of different type when determined by pairs of corresponding intervals may have the switch construction made independently in both, as if $C\ D$ determines $/s_nt_1/$ and $C'\ D'$ following $C\ D$ determines $\diagdown s_1't_n'\diagdown$ then some chain from M_n must have ends on M_1 following C and preceding C' so $C\times C'=0$, and as t_n does not follow D while t_n' follows D' t_n' follows t_n, even though D and D' reduce to coincident points.

(5) Only a or b can be the limit of an infinite set of switches, for among any infinite set must be an infinite subset of the same type and between any two of the same type is a chain; then apply I(c).

On Pockets. (1) In $/s_nt_1/$ if (s_1t_1) is free the switch contains a pocket, otherwise a given switch and pocket have only invariant common points as between the two lies a chain.

(2) Pockets of the first of the four sorts enumerated are simple and cause no difficulties. Given then a set of pockets made up of the other sorts. Such a set if infinite can have only a or b as set limit. If between any two of the set there is a chain then any two have at most invariant common points and in all of the sets simultaneously and without interference the pocket construction can be made, as it does not affect invariant points. What then is the relation between two pockets not having a chain between them? Evidently one must depend on a maximum free segment (l_1f_1) of (M_1), the other on

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[*] Other relations between switches of these types are possible and for these the statement is not necessarily true as may be seen by drawing a figure, but for using switches of these sorts fortunately no need arises.

one $(l_n'f_n')$ of (M_n). But the γ chain of the first pocket precedes the second pocket and the γ' chain of the second precedes the first, so these two chains must coincide. Similarly the δ and δ' chains must coincide. Hence $l_j = l_j'$ $(j = 1, 2, \cdots, n)$ and $f_j = f_j'$. The common points of the b-boundary arc $l_1l_2, \cdots, l_{n-1}l_n$ of the γ chain and the a-boundary arc $f_1f_2, \cdots, f_{n-1}f_n$ of the δ chain are not invariant, as the two pockets have no chain between. If there is more than one such point the two pockets have no common point, if there is one such point they have it in common, if there is no such point actually there is but one pocket of sort (4) (see pocket classification), but it will be convenient still to look upon this set as two pockets. Now if $(l_1l_n) \times (f_1f_n)$, these symbols representing the chain boundary arcs in question, is vacuous or but a single point, evidently pocket constructions in each of the pockets may interfere, fig. 2(d). Such a pair of pockets as well as the free segments determining them will be said to be *opposite.* Evidently their sum is a switch of either type, $/l_nf_1/$ or $\diagdown l_1f_n \diagdown$.

(3) In section 4 pocket constructions will be used when available to divert M_1 or M_n from one or other but not both of the intervals in each pair of a given set of corresponding intervals. Among these intervals certain ones by being contained in maximum free segments, for example (l_1f_1), will identify an associated set of such segments and pockets. Suppose in fact that $(l_1f_1) \supset C_{i_s} + \cdots + C_u + \cdots + C_{i_t}$ $(i_s \leq u \leq i_t)$ but no others of the set, while (l_nf_n) the opposite of (l_1f_1), if it exists, contains $D_{j_s} + \cdots + D_v + \cdots + D_{j_t}$ $(j_s \leq v \leq j_t)$. The circumstances under which pocket constructions may and will be subsequently used to divert M_1 or M_n and the other arcs affected from one of each pair of corresponding intervals in the set is here summarised as condition (P).*

(P): *Among the given set of corresponding intervals at least one of each pair is free, and one at least of the free ones of each pair is in a maximum free segment, as* (l_1f_1), *which either* (1) *has no opposite among the pockets determined by the intervals of the set* (that is (l_nf_n) fails to exist or fails to contain intervals of the set), *or* (2) *has an opposite in which a pocket construction can be made independent of that in itself, or* (3) *has an opposite containing intervals corresponding only to those in itself or an opposite containing every interval to which an interval in itself corresponds,* (that is $i_s \leq j_s$ and $j_t \leq i_t$, or $j_s \leq i_s$ and $i_t \leq j_t$.).

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* This condition can be made some what less restrictive but not without difficulty out of proportion to the benefit accruing.

These conditions not being satisfied, if $j_s < i_s$ and therefore $j_t < i_t$, the sum of the pockets will be taken as the switch $/l_nf_1/$ or $/s_nt_1/$; whereas if $i_s < j_s$ and hence $i_t < j_t$ the sum will be taken as $\diagdown l_1f_n\diagdown$ or $\diagdown s_1t_n\diagdown$.

The next section will be devoted to the proof of the theorem mentioned at the end of section 2.

4. THEOREM. *If none of the pairs of corresponding intervals of M_{1n} is of type* 17 *then F contains $n+1$ independent arcs from a to b.*

Assuming that none of the pairs is of this type construction of the $n+1$ arcs, to be designated as $L_1, L_2, \cdots, L_n, L_{n+1}$, will be attempted. Convenience divides the argument into the following three cases; in case 1 it will be assumed that the arc of II(c) does not occur upon C_0 or D_0, and that the entire set of corresponding intervals satisfies (P); in case 2 that the same arc is not on C_0 or D_0 but that (P) is not satisfied; in case 3 it will be assumed that the arc of II(c) is on C_0.

Proof: Case 1. Let $G = I + K + M_{1n}$. The topological properties of G are precisely those of any set F_{jj+1}, in fact by a transformation of the type described in II(e) G is converted into such a set. Let $\sum_{i=-\infty}^{+\infty} C_i$ be the set of intervals of M_1, and $\sum_{i=-\infty}^{+\infty} D_i$ those of M_n. Subsets $[C]$ and $[D]$ will now be chosen as follows from these. For each j let $[C]$ contain C_j if (1) C_j is free but D_j is not, or if (2) both C_j and D_j are free and the pocket determined by C_j either has no opposite in the set of pockets determined by $\sum_{i=-\infty}^{+\infty} D_i$ or has one in which the pocket construction may be made independently of one in itself or has one containing only intervals which correspond to intervals contained in itself; and let $[D]$ contain D_j if D_j is free and C_j is not or if both C_j and D_j are free and either the pocket determined be D_j has no opposite among those determined by $\sum_{i=-\infty}^{+\infty} C_i$ or has an opposite in which a pocket construction can be made independently of one in itself or an opposite containing only intervals corresponding to those in itself but not all of them. As the entire set of intervals satisfies (P) this analysis for each pair of intervals puts C_j into $[C]$, or D_j into $[D]$, or both. Moreover, pocket constructions may be made simultaneously in all of the pockets determined by all the intervals of $[C]$ and $[D]$, and independently.

Let the cross domains of G be $\cdots \phi_{-p}, \cdots, \phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \phi_2, \cdots, \phi_q, \cdots$, ($\phi_{-1} = \phi_0$ if $K_0 = 0$). From the definition of K and from II(e)

$\sum_{i=-\infty}^{+\infty} \Phi_i \times (M_1) = \sum_{i=-\infty}^{+\infty} C_i$ and $\sum_{i=-\infty}^{+\infty} \Phi_i \times (M_n) = \sum_{i=-\infty}^{+\infty} D_i$. Let $[\Phi]$ be the subset of $\sum_{i=-\infty}^{+\infty} \Phi_i$ for which $\sum_{i=-\infty}^{+\infty} \Phi_i \times (M_1) = [C]$, then by the classification determining $[C]$ and $[D]$ $(\sum_{i=-\infty}^{+\infty} \Phi_i - [\Phi]) \times (M_n) \subset [D]$. Following III(b) divert the arcs $M_1, \cdots, M_n$ from the original positions of M_1 and M_n through the pockets associated with the intervals of $[C]$ and $[D]$. The resulting point sets are still independent by reason of an assumption made throughout this case, and are arcs owing to that property for pockets corresponding to (5) for switches. Call them $L_1, \cdots, L_n$; evidently $F_{j-1j+1} \supset L_j$ $(j = 2, 3, \cdots, n-1)$, $F_{12} \supset L_1$, $F_{n-1n} \supset L_n$,; and $L_1 \times [C] = 0$, $L_n \times [D] = 0$, $L_i \times L_j = a + b$ $(i \neq j)$.

In G L_{n+1} will now be constructed. To do this use I(d) in G, $j = 1$ $i = 1$, and $Q = \sum_{i=-\infty}^{+\infty} C_i - [C]$. The set Q_2, which is clearly on (M_n) not on (M_2), will then, since $\sum_{i=-\infty}^{+\infty} C_i - [C] = (\sum_{i=-\infty}^{+\infty} \Phi_i - [\Phi]) \times (M_1)$, be equal to the set $(\sum_{i=-\infty}^{+\infty} \Phi_i - [\Phi]) \times (M_n)$, that is $Q_2 \subset [D]$. As N_2 constructed by I(d) is for any Q such that $N_2 \times (M_1) \subset \sum_{i=-\infty}^{+\infty} \Phi_i \times (M_1)$ and $N_2 \times (M_n) \subset \sum_{i=-\infty}^{+\infty} \Phi_i \times (M_n)$; for this particular Q, $N_2 \times (M_1) = [C]$ and so $N_2 \times (L_1) = 0$; also $N_2 \times (M_n) \subset Q_2$, hence $N_2 \times (M_n) \subset [D]$ and so $N_2 \times (L_n) = 0$. Likewise $N_2 \times (\sum_{j=2}^{n-1} (L_j)) = 0$ as $N_2 \times I_{1n} = 0$ whereas $I_{1n} \supset \sum_{j=2}^{n-1} (L_j)$. The set N_2 is clearly an arc; let it be L_{n+1}. But then $L_j (j = 1, 2, \cdots, n+1)$ is a set of arcs in F from a to b contradicting the theorem in this case since independent and $n + 1$ in number.

Case 2: The suppositions on this case imply at least one pair of corresponding intervals belonging to type 8, or to type 9, or to types 1 to 7 with one or both free and subsets of maximal free segments determining pockets having opposites among the set of pockets determined by $\sum_{i=-\infty}^{+\infty} C_i$ and $\sum_{i=-\infty}^{+\infty} D_i$, the pocket constructions in an opposite pair not being independent and the pair itself being such that, (a) $j_s < i_s$ and $j_t < i_t$, or (b) $i_s < j_s$ and $i_t < j_t$. Both 8 and the free case with (a) imply a switch of the sort $/s_n t_1/$ with s_n preceding D of the determining pair and t_1 following C; the construction for these two cases is identical and hereafter the two will not be kept separate. Similarly 9 and the free case with (*b*) yield to the same treatment; in fact these two cases need not be considered at the outset at all since upon reflecting

F with respect to any straight line parallel to the straight line ab and then renumbering the arcs M_j in the reverse order, these two are the same as the first two. If a contradiction is obtained when a start is made with the first two one may similarly be derived for the other two. The procedure being the same whatever the corresponding interval pair upon which 8 or (a) occurs, suppose it to be $C_0 D_0$.

Let $/s_n t_1/$ be any switch determined by 8 or the free case with (a) in connection with $C_0 D_0$. As t_1 follows C_0 it follows some last interval C_{l_1-1} of M_1, $l_1 - 1 \geqq 0$; and as s_n precedes D_0 it precedes some first interval D_m of M_n, $m \leqq 0$. Deflect L_1 from M_1 into K_{l_1} through a_{1l_1}; in the switch $L_2, \cdots, L_n$, whose construction is thus begun, move from $M_2, \cdots, M_n$ to $M_1, \cdots M_{n-1}$, L_2 first reaching M_1 at t_1 while L_n leaves M_n at s_n. Finally from any convenient point of K_m introduce L_{n+1} upon M_n at b_{nm}. The construction of the $n+1$ independent arcs is now begun. Their continuation depends entirely upon the nature of the remaining pairs of intervals and for convenience various sub-cases will now be distinguished. As the situations met while constructing the $n+1$ arcs to a are no more general than those met toward b. only the construction to b will be completed as that to a follows the same lines.

The following possibilities call for separate treatment: (1) there is no pair of corresponding intervals subsequent to $C_{l_1-1} D_{l_1-1}$; (2) the next or some subsequent pair implies a switch $\diagdown s_1' t_n' \diagdown$ with s_1' preceding C_{l_1}, t_1' following D_{l_1}; (3) the next or some subsequent pair implies a switch $/s_n' t_1'/$ with s_n' preceding D_{l_1} and t_1' following C_{l_1}; (4) neither the next nor any subsequent pair implies a switch; (5) there exist subsequent pairs which imply switches but none of these has the property explicitely added in (2) or (3).

Case 2 (1). Observe in this connection that t_1 is different from b; this is obvious if the switch is determined by 8 while if determined by (a) $t_1 = b$ would mean $(s_1 b)$ on M_1 and thus $(s_n b)$ on M_n were both free and so $i_t = j_t$, a contradiction. But if $t_1 \neq b$, since no pair of corresponding intervals follows $C_{l_1-1} D_{l_1-1}$, $K_{l_1} = B \neq 0$. But L_1 being thus in B may be drawn through it to b, $L_2, \cdots, L_{n+1}$ meanwhile following $M_1, \cdots, M_n$ independently of each other and of L_1 to b, thus in this case completing the $n+1$ arcs to b.

Case 2 (2). Special circumstances attend the occurrence of this case also. Owing to note (2) on switches s_1' must not precede t_1 and so, since C_{l_1-1} precedes t_1 and C_{l_1} follows s_1', $a_{1l_1} \neq b_{1l_1}$. This being the case proceed as follows. Let $\diagdown s_1' t_n' \diagdown$ be the last such switch with s_1' preceding C_{1l_1} and let the last interval of a corresponding pair preceding t_n' be D_{l_2-1}, $l_2 > l_1$. In the switch $\backslash s_1't_n'\backslash$ let the arcs $L_2, \cdots, L_n$ drop back from $M_1, \cdots, M_{n-1}$ to $M_2, \cdots, M_n$; let L_1 return upon M_1 from K_{l_1} through b_{1l_1} the first point of C_{l_1} and thus independent of L_2 which leaves M_1 at a preceding point namely s_1'; finally let L_{n+1} follow M_n to a_{nl_2} and there pass into K_{l_2} quite independent of L_n which first reaches M_n at t_n' subsequent to a_{nl_2} which is the last point

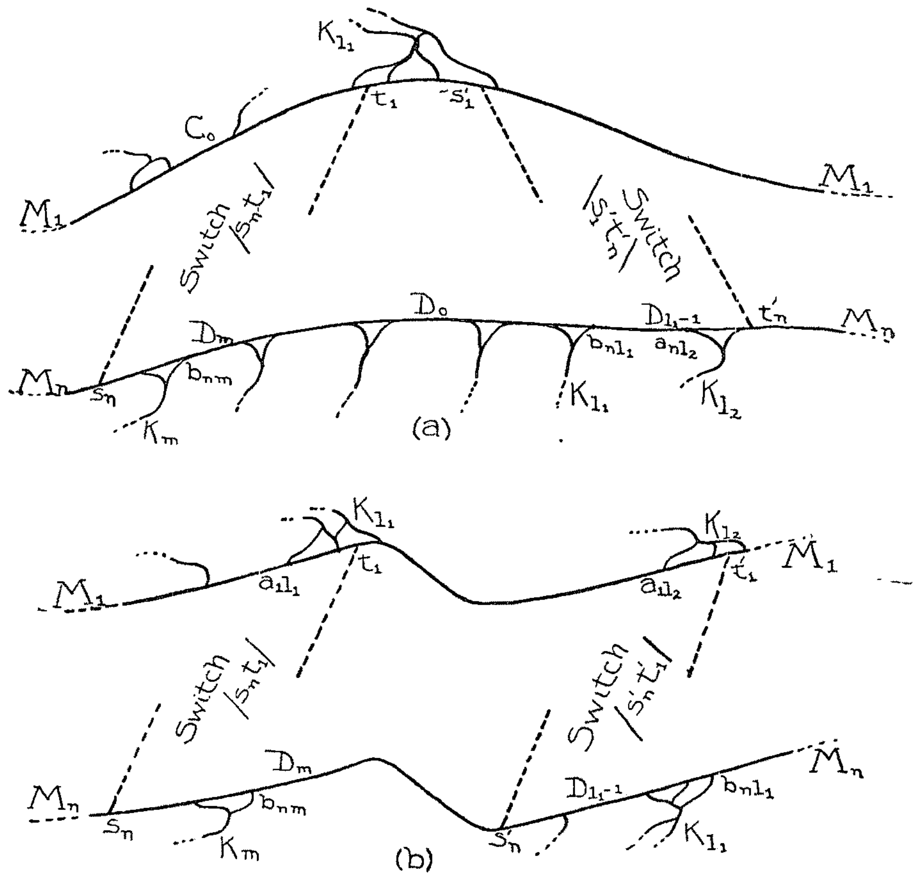

FIG. 4.

of D_{l_2-1}, fig. 4(a). The $n+1$ independent arcs have now been carried on into the next pair of corresponding intervals, or a pair subsequent to these.

Case 2 (3). This case is slightly more general than the two preceding as it imposes no conditions upon I or K. Two suppositions requiring somewhat different handling will be treated separately. As $/s_n't_1'/$ will at the outset be selected the last switch of this sort having s_n' before D_{l_1}. Suppose

s_n' does not precede D_m. Let C_{l_2-1} be the last interval of M_1 preceding t_1', $l_2 > l_1$. The switch construction can be employed immediately. Let $L_3, \cdots, L_{n+1}$ follow $M_2, \cdots, M_n$ from their last points in the switch $/s_nt_1/$ to their first in $/s_n't_1'/$, in so doing clearly they remain stationary or approach b owing to note (3) on switches, in particular are these statements true of L_{n+1} as s_n' does not precede D_m. Let them in $/s_n't_1'/$ then be raised from $M_2, \cdots, M_n$ to $M_1, \cdots, M_{n-1}$, L_{n+1} leaving M_n at s_n' previous to point b_{nl_1}, L_3 reaching M_1 first at t_1' after a_{1l_2}; take L_2 out into K_{l_2} through a_{1l_2}; and finally bring L_1 in from K_{l_1} upon M_n through b_{nl_1}. $L_1, \cdots, L_{n+1}$ have meanwhile remained independent and have been introduced upon the next or a subsequent pair of intervals, fig. 4 (b).

Before undertaking the next case the previous one must be completed for the assumption s_n' precedes D_m; this may certainly occur as $/s_nt_1/$ was chosen as any switch having s_n before D_0, t_1 after C_0; it can not occur if $/s_n't_1'/$ is the last switch having s_n before D_0, and consequently can not occur for $/s_n''t_1''/$ after $/s_n't_1'/$ nor $\backslash s_1''t_n''\backslash$ after $\backslash s_1't_n'\backslash$ in constructing after cases 2 (2) and the first part of 2 (3) as $/s_n't_1'/$ and $\backslash s_1't_n'\backslash$ were selected as the last switches satisfying other requirements as imposed there. It is undesirable to pick $/s_nt_1/$ with the thought of avoiding this case as doing that makes the construction toward b less general than that toward a. However, the situation s_n' precedes D_m affects the fate of only one of the arcs involved and that arc is L_{n+1}. When the switch construction is made in $/s_n't_1'/$ $L_1, \cdots, L_n$ behave as above, but L_{n+1} instead of remaining where it is or approaching b must turn from b_{nm} toward a along M_n until it reaches s_n' and there be acted upon by the switch. Two difficulties seem to arise. It looks as if L_{n+1} might have to go as far as s_n before encountering s_n' for possibly s_n coincides with t_n; the independence of the arcs would thus be lost as L_n already occupies s_n. This difficulty vanishes upon referring to note (1) on switches where it is seen that if $s_n = t_n$ then Δ_{n-1} has a point on M_n following t_n which s_n' can not precede. The other difficulty is this. Perhaps in the next step a switch $/s_n''t_1''/$ will need to be used which has s_n'' also preceding D_m so that L_1 to reach it will have to approach a from b_{1l_1} thus encountering L_{n+1} at b_{nm}; this is impossible because $/s_n't_1'/$ was the last switch for which s_n' preceded D_{l_1} so $/s_n''t_1''/$ must have s_n'' on or following D_{l_1}. Or perhaps a pocket required on D_{l_2} depends on a maximum free segment having an end not following b_{nm} so that L_1 might have to come to this point to enter the pocket construction thus colliding with L_{n+1}; this is impossible owing to the situation on $C_{l_1}D_{l_1}$ for D_{l_1} either has a chain actually ending upon it (e. g. when C_{l_1}

D_{l_1} are of type 8) or is contained in a pocket which with its opposite forms a switch $/s_n' t_1'/$ which is followed by a chain from t_n to t_1, both which circumstances prevent any pockets on D_{l_2} having an end preceding D_{l_1} much less one not following b_{nm}. The $n+1$ arcs are thus successfully extended in such a way that subsequent construction along similar lines will not be interfered with.

Case 2 (4). The statement of this case indicates that among the pairs of corresponding intervals $C_{l_1}D_{l_1}$, $C_{l_1+1}D_{l_1+1}, \cdots$ no pair is of types 8 or 9 and no pair contains only free intervals exhibiting (a) or (b) as described at the beginning of this section. Thus this set of intervals satisfies condition (P). Represent those of the set on M_1 by $\sum_{i=l_1}^{+\infty} C_i$ and those on M_n by $\sum_{i=l_1}^{+\infty} D_i$ and using the same criterion as in case 1 select from these two summations the subsets $[C]$ and $[D]$. Now in G with Q on M_1 equal to $\sum_{i=l_1}^{+\infty} C_i - [C]$ construct N_2 by I(d). Since for any Q on M_1 and N_2 in $G, N_2 \times (M_1) = \sum_{i=-\infty}^{+\infty} C_i$ and $N_2 \times (M_n) \subset \sum_{i=-\infty}^{+\infty} D_i$ and since $b_{1l_1}b \times \sum_{i=-\infty}^{+\infty} C_i = \sum_{i=l_1}^{+\infty} C_i$ and $b_{nl_1}b \times \sum_{i=-\infty}^{+\infty} D_i = \sum_{i=l_1}^{+\infty} D_i$ for this particular Q $N_2 \times b_{1l_1}b) = [C]$ and $N_2 \times b_{nl_1}b) = [D]$, $b_{1l_1}b)$ and $b_{nl_1}b)$ representing arcs of M_1 and M_n respectively which lack the end point b. But the intervals of $[C]$ and $[D]$ are subsets of maximal free segments of M_{1n} all following t_1 and t_n in the pockets determined by which the pocket construction can be made simultaneously and independently. Once the arcs $L_2, \cdots, L_{n+1}$ can be brought to position at $t_1, t_2, \cdots, t_{n-1}, t_n$ without points upon $[C]$ and $[D]$ they may be subjected to this construction and carried on independently to b in such a way that $L_2 \times [C] = 0$ and $L_{n+1} \times [D] = 0$. But the switch construction has already left $L_2, \cdots,$ L_n at $t_1, \cdots, t_{n-1}$ respectively, so the only question is as to the whereabouts of L_{n+1}. The arc L_{n+1} had been brought to M_n at b_{nm}, the first point of D_m and clearly preceding any interval of $[D]$, from K_m. If the first pocket required by $[D]$ is determined by the maximal free segment (pq) and $b_{nm} = p$ then construction may proceed without any trouble; if p follows b_{nm} then draw L_{n+1} along M_n until p is reached; if p precedes b_{nm}[*] draw L_{n+1} along M_n toward a from b_{nm} until it meets p. As q follows b_{nl_1}, it follows b_{nm}, so no difficulties arise. The pocket construction can then be used and $L_2, \cdots, L_{n+1}$ taken on independently to b, with $L_2 \times [C] = 0$ and

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[*] Of course p does not precede t_n and if $s_n = t_n$ it can not coincide with t_n. Compare the situation here with that in the last paragraph of case 2 (3).

$L_{n+1} \times [D] = 0$. But N_2 in passing from the boundary of ϕ_{l_1}, whose limits on M_1 and M_n are C_{l_1} and D_{l_1} respectively, to the boundary of ϕ_{l_1-1} must pass through a point or limit point of K_{l_1} which is clearly not on L_2 or L_{n+1}. Now L_1 when last heard from was in K_{l_1}; join it then in K_{l_1} to any convenient point of N_2 and let it follow N_2 to b. The construction of $n+1$ independent arcs has thus been in this case as in case 1 completed to b.

Case 2 (5). Let $C_r D_r$ be the first pair of corresponding intervals ($r > l_1$) which belong to types 8 or 9 or form with the pairs $C_{l_1} D_{l_1}$, $C_{l_1+1} D_{l_1+1}$, $\cdots$, $C_{r-1} D_{r-1}$ a set not satisfying condition (P). The pair implies a switch of type $/s_n' t_1'/$ or $\diagdown s_1' t_n' \diagdown$, suppose in order to make the discussion slightly less abstract, and for this reason only as the argument for either case follows similar lines, that the switch is $/s_n' t_i'/$. Let the symbol represent the last switch for which s_n' precedes D_r, and suppose D_{l_2} is the first that s_n' precedes $l_1 < l_2 \leq r$. The pairs of corresponding intervals $C_{l_1} D_{l_1}$, $\cdots$, $C_{l_2-1} D_{l_2-1}$ satisfy the condition (P). Form the subset $[C]$ of $\sum_{i=l_1}^{l_2-1} C_i$ and the subset $[D]$ of $\sum_{i=l_1}^{l_2-1} D_i$. Let $Q = \sum_{i=l_1}^{l_2-1} C_i - [C]$ and proceed precisely as in (4) to construct N_2. When N_2 is constructed join L_1 to it from K_{l_1} and let L_1 follow N_2 toward b until it reaches the first limit point of K_{l_2}, that is necessarily a_{1l_2} or a_{nl_2}, and then carry L_1 into K_{l_2}, out of the way. Meanwhile as in 2 (4) bring L_2, $\cdots$, L_{n+1}, with precisely the same circumspection as regards L_{n+1}, on until they arrive at the switch $/s_n' t_1'/$ having on the way avoided all the intervals of $[C]$ and $[D]$ and thus independent of L_1. In the switch then execute exactly the same maneuver as in case 2 (3). The $n+1$ independent arcs are thus introduced upon an interval pair subsequent to the pair $C_{l_1-1} D_{l_1-1}$.

In summary upon case 2 notice that in subcases (1) and (4) the problem is complete, while in the other cases the work paused at the point where the next remark would be, "there are now five possibilities," seen at a glance to be identical with the five just handled for the interval pair $C_{l_1} D_{l_1}$ and disposable in essentially the same way for the pair, say, $C_{l_2} D_{l_2}$, even although here they may follow a switch construction $\diagdown s_1' t_n' \diagdown$ rather than merely $/s_n t_1/$. So continue the process. Either eventually a pair of intervals is reached exhibiting case 2 (1) or (4) at which the construction ends, or it runs on through an infinite number of stages bringing $n+1$ independent arcs indefinitely near to b. To see that in the limit these arrive as independent arcs at b consult II(d), note (5) on switches, and the first few sentences of (2) on pockets.

Accordingly if the limit points of J are not on C_0 (or D_0) one pair of corresponding intervals is of type 17, as otherwise from a to b in F there are $n+1$ independent arcs.

Case 3. Suppose then K_0 is vacuous[*] and the limits of J occur on C_0. Omit J from F. Then $F-J$ contains on M_{1n} a pair of intervals of type 17. Restore J to the set. It will now be shown that if in $F-J$ the intervals $C_0\ D_0$ are the only corresponding ones of type 17 and that all the limits of the joining chains on C_0 occur between the extreme limits of J so that D_0 is no longer joined to either of the intervals into which as mentioned in II(e) J divides C_0, then a contradiction arises—there are $n+1$ arcs from a to b in F. For assume these conditions. For J the notation attached to K_0 will be used as far as it applies. There is at least one chain from D_0 to $(a_{10}\, b_{10})$ and from any convenient point of that constituent of J which connects a_{10} and b_{10} and any convenient points invariant and n in number one from each of the arcs $M_1, \cdots, M_n$ and common to the successive links of such a chain $n+1$ arcs will now be constructed. When the extra arc arrives in K_m, $m > 0$, troubles cease, as no subsequent corresponding intervals are of type 17 and so previous methods apply.

Now D_0 is not free. Suppose C_0 is free. If the pocket containing it has no opposite among the pockets determined by the intervals $\sum\limits_{i=0}^{+\infty} D_i$ or has an opposite for which $i_t > j_t$, pocket construction will be used depending upon the maximum free segment (pq) p preceding b_{10}, q following C_0, in fact following some last interval C_{m-1}. From the points indicated in the last paragraph extend $L_1, \cdots, L_n$ and let them be diverted from $M_1, \cdots, M_n$ only when in the pocket (pq) in which case let them follow its pocket construction; they are thus independent and $L_1 \times (pq) = 0$. From the chosen point of J bring L_{n+1} in through b_{10} upon M_1 which let it follow to a_{1m} last point of C_{m-1} from which it passes into $K_m (m > 0)$. Farther construction now follows previous lines.

If C_0 is not in a pocket, or if when in one is in one which has an opposite with $i_t < j_t$ then a switch $\diagdown s_1\, t_n \diagdown$ is determined with s_1 on $(a_{10}\, b_{10})$ and t_n following $D_{m-1} (m > 0)$. This switch drops the arcs $L_1, \cdots, L_{n-1}$ from the points at which they are begun to $M_2, \cdots, M_n$ L_1 leaving M_1 at s_1, L_{n-1} reaching M_n at t_n; let meanwhile L_n leave M_n at a_{nm} for $K_m (m > 0)$ and let

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[*] If K_0 is not vacuous the limits of J must occupy a subinterval of $a_{10}b_{10}$ or of $a_{n0}b_{n0}$ owing to the nature of the set K_0, see section 2.

L_{n+1} come in from J upon M_1 through b_{10}. As the $n+1$st arc is now in K_m instead of J nothing farther need be said.

The procedure toward a is similar, it being noted that although J redivides C_0 into C_{-1} and C_0 it does not reseparate D_{-1} and D_0 which still coincide. Thus there are $n+1$ arcs in F from a to b in this case also unless a pair of corresponding intervals is joined.

5. THEOREM I. *F posseses n complementary domains, of which one may be its unbounded complementary domain, λ_{12}, λ_{23}, $\cdots$, λ_{n-1n}, λ_{n1}, where $\lambda_{jj\text{-}1}$ is a subset of I_{jj+1} and λ_{n1} is exterior to M_{1n}, such that:*

(1) *Λ_{jj+1} has at least one point in common with $\Lambda_{j-1\,j}$ and at least one in common with Λ_{j+1j+2} but none whatever in common with any of the others unless possibly a or b (subscripts reduced mod n), $(j=1, 2 \cdots n-1)$.*

(2) *all the points common to $\Lambda_{j-1\,j}$ and Λ_{jj+1} are on the arc M_j,*

(3) *at least one of the common points of $\Lambda_{j-1\,j}$ and Λ_{jj+1} is on (M_j), that is, is different from a and from b; this point will be designated as p_j.*

Proof: Any chain joining two corresponding intervals plus the domain of whose boundary these intervals is a subset form a set of domains satisfying these conditions.

Let any one of the points of $\Lambda_{j-1\,j} \times \Lambda_{jj+1} \times (M_j)$ be called p_j.

THEOREM II. *$F - \sum_{i=1}^{n} p_i$ consists of two mutually separated sets F_a and F_b where $F_a \supset a$ and $F_b \supset b$.*

Proof: Let c_{jj+1} be a point of λ_{jj+1}. As any boundary point of a complementary domain of a continuous curve is accessible from any point of the domain there is an arc $(p_j\,c_{jj+1}\,p_{j+1})$ in λ_{jj+1}. Draw all n of these arcs. As they are finite in number, join only at their ends, and join in an order determined by part (1) of theorem I above, the sum $\sum_{j=1}^{n} p_j\,c_{jj+1}\,p_{j+1}$ is a simple closed curve C, having only the points $\sum_{i=1}^{n} p_i$ in common with F. Moreover a and b are separated by this curve for suppose both are interior. Then b must be interior to one of the following simple closed curves with mutually exclusive interiors $a\,m_1\,p_1\,c_{12}\,p_2\,m_2\,a$, $a\,m_2\,p_2\,c_{23}\,p_3\,m_3\,a, \cdots$, $a\,m_{n-1}\,p_{n-1}\,c_{n-1n}\,p_n\,m_n\,a$, $a\,m_n\,p_n\,c_{n1}\,p_1\,m_1\,a$ where m_j is some point of M_j between a and p_j. Suppose for instance that it lies in $a\,m_1\,p_1\,c_{12}\,p_2\,m_2\,a$. Then M_j $(j=3, 4, \cdots, n)$ must to reach b from a cross $a\,m_1\,p_1$, or $p_1\,c_{12}\,p_2$, or $p_2\,m_2\,a$, none of which it may do; or else lie already within M_{12} entirely, which the preliminary ordering forbids. Similarly it is proved that not both a and b lie outside the simple closed curve. Thus F consists of two parts F_a and F_b, $F_a \supset a$ and $F_b \supset b$, one exterior to C and one interior, and of course the points $\sum\limits_{i=1}^{n} p_i$. The theorem is thus seen to be proved.

THEOREM III. *$E - \sum\limits_{i=1}^{n} p_i$ is the sum of two mutually separated sets E_a and E_b where $E_a \supset a$ and $E_b \supset b$.*

Proof: Suppose not. Then a and b are points of some maximal connected subset of $E - \sum\limits_{i=1}^{n} p_i$ in which there can be drawn an arc W from one to the other. This arc cuts C in some point p, as a is separated from b by C. But p is not a point of F as it is not a point of $\sum\limits_{i=1}^{n} p_i$. Proceed along W from p toward a and toward b until reaching the first points p_a and p_b in these directions which W has in common with F, these must be distinct as W is an arc, and must exist as a and b are in F and F is closed. But then $(p_a\,p\,p_b)$ is a part of some maximal connected subset of $E - F$ which has at least two limit points, for example p_a and p_b, in F. This is a contradiction.*

COROLLARY I. *Under the hypotheses of the general theorem E possesses a set of complementary domains having the property of Theorem I above.*†

THEOREM IV. *If among the subsets of the Continuous Curve E whose omission from the continuous curve separates it into two mutually separated sets E_a and E_b, $E_a \supset a$ and $E_b \supset b$ where a and b are given points, is one consisting of n distinct points but none consisting of fewer than n, then there is a set of n independent arcs in E from a to b and no set of independent arcs of E from a to b consists of more than n.*

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* W. L. Ayres, "Arc Curves and Basic Sets of a Continuous Curve."

† For F has such a set and then see W. L. Ayres, "Concerning Arc-curves and Basic sets of a Continuous Curve."

Proof: Obviously if the set of independent arcs from a to b in E were $n+1$ in number, $F - \sum_{i=1}^{n} p_i$ would not be separated between a and b, as at least one of the $n+1$ arcs would fail to contain any point of $\sum_{i=1}^{n} p_i$, so in this case $\sum_{i=1}^{n} p_i$ could not separate E in this way either. On the other hand, if the arc curve were to contain at most $n-j$ $(0 < j \leq n)$ independent arcs from a to b, then by the general theorem there would be $(n-j) < n$ points whose omission would separate E between a and b into two mutually separated sets one containing a and the other b. As this contradicts hypothesis, the arc curve of $a+b$ with respect to E contains for some construction n independent arcs from a to b but for no construction more than n.

THEOREM V. *Under the hypotheses of theorem IV there is a set of n complementary domains of E related to any chosen set of n separating points in the manner of the domains of theorem I and the points* $\sum_{i=1}^{n} p_i$.

Proof: For by theorem IV the arc curve of $a+b$ in E contains a set of n independent arcs but no set of more than n. Let a set of n be $M_1, \cdots, M_n$ ordered as heretofore and for each i let p_i be on (M_i), $\sum_{i=1}^{n} p_i$ being the chosen separating set; evidently one point of this set must be on each of the arcs M_i and no more than one can be. Now the set F_{jj+1} as previously described must be cut between a and b by the omission from it of $\sum_{i=1}^{n} p_i$ as it is a subset of F containing a and b and hence must also be cut by the omission of the points p_j and p_{j+1} which are the only ones of $\sum_{i=1}^{n} p_i$ which it contains. But then as before p_j and p_{j+1} are on the boundary of some bounded complementary domain of F_{jj+1}, hence also one of F like λ_{jj+1} and finally one of E. The set of domains thus determined is easily seen to possess the properties of those in theorem I.

THEOREM VI. *Either under the hypotheses of the general theorem or under the hypotheses of theorem IV, there is a simple closed curve C whose interior contains either a or b and such that* $C \times E = \sum_{i=1}^{n} p_i$ *where* $\sum_{i=1}^{n} p_i$ *is any set of n points whose omission divides E into two mutually separated sets one containing a and the other b.*

Proof: As the sets of n points dividing the curve E are identical with those dividing the curve F, the theorem follows upon the application of theorem V and the process of theorem II. It is to be observed that C can not enclose a or b chosen arbitrarily as the set of domains of theorem V may possibly not include the unbounded complementary domain of E.

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A Direct Treatment of Systems of Linear Differential Equations whose Coefficients have Uniform Singularities.

By J. A. Nyswander.

A complete theory of solution of linear differential equations of the form

$$\frac{d^n x}{dt^n} + \frac{p_1(t)}{t - t_0} \frac{d^{n-1}x}{dt^{n-1}} + \cdots + \frac{p_n(t)}{(t - t_0)^n} x = 0,$$

in which the $p_i(t)$ are analytic about $t = t_0$, was given by Fuchs[*] about 1868. He treated exhaustively the various cases that can arise, classified with respect to the nature of the roots of the characteristic equation

$$f(\lambda) = \lambda(\lambda - 1) \cdots (\lambda - n + 1) + \lambda(\lambda - 1) \cdots (\lambda - n + 2) p_1(0) + \cdots + \lambda p_{n-1}(0) + p_n(0) = 0.$$

The more comprehensive problem of obtaining, under completely general hypotheses with respect to the roots of $f(\lambda) = 0$, a fundamental set of solutions for the system of differential equations

$$t(dx_i/dt) = \theta_{i1}(t)x_1 + \cdots + \theta_{in}(t)x_n \qquad (i = 1, \cdots, n),$$

was first solved by Horn about 1891. In his theory Horn employed a principle originally due to Frobenius[†] and later developed by Grünfeld.[‡] The treatment given by Horn is based on the use of sequential transformations; the various cases that arise being classified in terms of the elementary divisors of the characteristic determinant $D(\lambda)$.

The present paper gives a direct and relatively simple treatment of the general problem, and shows how to obtain the solutions in all cases without recourse to either transformations or the theory of elementary divisors.

1. Consider the system of differential equations

$$(1) \qquad t(dx_i/dt) = \theta_{i1}(t)x_1 + \cdots + \theta_{in}(t)x_n \qquad (i = 1, \cdots, n)$$

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[*] Horn, "Gewöhnliche Differentialgleichungen beliebiger Ordnung"; Sauvage, "Théorie Générale des Systèmes d'Équations Differentielles."

[†] *Crelle's Journal*, Vol. 74.

[‡] Grünfeld, "Über die Integration eines Systems," *Denkschriften der Wiener* [illegible]*emie*, Mathematisch-Naturwissenschaftliche Klasse, 1888.

in which the $\theta_{ij}(t)$ are analytic for $t=0$ and hence may be written in the form

$$\theta_{ij}(t)=\sum_{k=0}^{\infty}\theta_{ij}{}^{(k)}t^{k}\qquad(i,j=1,\cdots,n).$$

Suppose, also, that at least one $\theta_{ij}{}^{(0)}\neq 0$. It will be shown, first, that under certain conditions system (1) will have a solution of the form

$$x_i=t^{\lambda}\sum_{k=0}^{\infty}a_i{}^{(k)}t^{k}\equiv t^{\lambda}P_i(t:\lambda)\qquad(i=1,\cdots,n)\tag{2}$$

in which the λ and $a_i{}^{(k)}$ are constants to be determined.

On substituting (2) in (1) there results the following relations, which must be identities in t if (2) is to be a solution of (1):

$$\lambda t^{\lambda}P_i(t:\lambda)+t^{\lambda+1}\frac{dP_i(t:\lambda)}{dt}=\sum_{j=1}^{n}\sum_{k=0}^{\infty}\theta_{ij}{}^{(k)}t^{\lambda+k}P_j(t:\lambda)\qquad(i=1,\cdots,n).\tag{3}$$

On equating the coefficients of corresponding powers of t in (3), one obtains as sufficient conditions that (2) shall be a solution of (1).

$$\tag{4}\left\{\begin{array}{l}(\theta_{11}{}^{(0)}-\lambda)a_1{}^{(0)}+\theta_{12}{}^{(0)}a_2{}^{(0)}+\cdots+\theta_{1n}{}^{(0)}a_n{}^{(0)}=0\\ \theta_{21}{}^{(0)}a_1{}^{(0)}+(\theta_{22}{}^{(0)}-\lambda)a_2{}^{(0)}+\cdots+\theta_{2n}{}^{(0)}a_n{}^{(0)}=0\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \theta_{n1}{}^{(0)}a_1{}^{(0)}\qquad\qquad+\cdots+(\theta_{nn}{}^{(0)}-\lambda)a_n{}^{(0)}=0,\end{array}\right.$$

$$\tag{5}\left\{\begin{array}{l}(\theta_{11}{}^{(0)}-\lambda-p)a_1{}^{(p)}+\cdots+\theta_{1n}{}^{(0)}a_n{}^{(p)}=F_1{}^{(p)}(a_j{}^{(0)},\cdots,a_j{}^{(p-1)})\\ \qquad\theta_{21}{}^{(0)}a_1{}^{(p)}+\cdots+\theta_{2n}{}^{(0)}a_n{}^{(p)}=F_2{}^{(p)}(a_j{}^{(0)},\cdots,a_j{}^{(p-1)})\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \qquad\theta_{n1}{}^{(0)}a_1{}^{(p)}+\cdots+(\theta_{nn}{}^{(0)}-\lambda-p)a_n{}^{(p)}=F_n{}^{(p)}(a_j{}^{(0)},\cdots,a\end{array}\right.$$

$$(p=1,2,$$

where

$$F_i{}^{(p)}=-\sum_{j=1}^{n}\sum_{k=1}^{p}\theta_{ij}{}^{(k)}a_j{}^{(p-k)}\qquad(i=1,\cdots,n).\tag{6}$$

Equations (4) which are linear and homogeneous in the $a_j{}^{(0)}$ will have a non-trivial solution if and only if λ is a root of the characteristic equation

$$D(\lambda)=\begin{vmatrix}\theta_{11}{}^{(0)}-\lambda, & \theta_{12}{}^{(0)}, & \cdots, & \theta_{1n}{}^{(0)}\\ \theta_{21}{}^{(0)}, & \theta_{22}{}^{(0)}-\lambda, & \cdots, & \theta_{2n}{}^{(0)}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \theta_{n1}{}^{(0)}, & \theta_{n2}{}^{(0)}, & \cdots, & \theta_{nn}{}^{(0)}-\lambda\end{vmatrix}=0\tag{7}$$

We shall first dispose of the simplest case, in which the n roots of $D(\lambda) = 0$ are distinct and no two of the λ_j differ by an integer. For each λ_j at least one first minor in $D(\lambda_j)$ will be different from zero, and hence the solutions of equations (4) may be written in the form

$$a_i^{(0)} = c_j \eta_{ij}^{(0)} \qquad (i = 1, \cdots, n;\ j = 1, \cdots, n),$$

where the $\eta_{ij}^{(0)}$ are the first minor-cofactors in $D(\lambda_j)$ which arise in the solution of equations (4) by determinants.

In view of the hypothesis regarding the roots of $D(\lambda) = 0$, it is evident that

$$D(\lambda_j + p) \neq 0 \qquad (j = 1, \cdots, n;\ p = 1, 2, \cdots).$$

Thus equations (5) determine the $a_i^{(p)}$ $(i = 1, \cdots, n;\ p = 1, 2, \cdots)$ for each λ_j and consequently one obtains n solutions of system (1) of the form (2). These solutions can be written in the form

$$x_i = \sum_{j=1}^{n} c_j t^{\lambda_j} P_{ij}(t) \qquad (i = 1, \cdots, n;\ j = 1, \cdots, n),$$

where

$$P_{ij}(t) = a_{ij}^{(0)} + a_{ij}^{(1)} t + a_{ij}^{(2)} t^2 + \cdots.$$

The convergence of the solutions obtained in the present and later sections will [illegible]d in Section 7. It will be shown in Section 6, under entirely [illegible]ses in regard to the nature of the roots of the characteristic [illegible] the n solutions obtained by the method of the present paper [illegible] fundamental set.

2. *Solutions when two Roots of* $D(\lambda) = 0$ *differ by an Integer.* It may happen that two or more of the λ_j differ by an integer.

Suppose that $\lambda_1 - \lambda_2 = d$, where d is a positive integer, and that there are no other congruences (mod 1) among the λ_j. Assume also that λ_1 and λ_2 are simple roots. Starting with a possible solution of form (2), one arrives, as in section 1, at equations (4) and (5). Clearly $D(\lambda_1 + p) \neq 0$ $(p = 1, 2, \cdots)$ and consequently a solution of form (2) for $\lambda = \lambda_1$, is obtained exactly as in section 1.

In attempting to arrive at a solution of form (2) for $\lambda = \lambda_2$ one observes that $D(\lambda_2 + d) = D(\lambda_1) = 0$, and hence equations (5) are inconsistent for $\lambda = \lambda_2$; $p = d$ unless the corresponding $F_i^{(d)}$ satisfy certain linear relations. Designate by $D_i(\lambda + p)$ the result of replacing the elements of the ith column of $D(\lambda + p)$ by $F_1^{(p)}, \cdots, F_n^{(p)}$. Obviously equations (5) for $\lambda = \lambda_2$ and $p = d$ will be consistent if and only if

$$D_i(\lambda_2 + d) = 0 \qquad (i = 1, \cdots, n). \tag{8}$$

CASE I. Assume that relations (8) are satisfied. The values of the $a_i^{(d)}$ as obtained from (5) may be thought of as a sum of two terms, one carrying an arbitrary constant as a factor, the other an expression involving the $F_i^{(d)}$. It is readily seen that the terms involving the arbitrary constant are exactly the values previously found for the $a_i^{(0)}$ of (4) with $\lambda = \lambda_1$. Since these terms if retained in determining the $a_i^{(p)}$, $p = d + 1, d + 2, \cdots$ would give rise to the solution of system (1) already obtained for $\lambda = \lambda_1$, it is desirable to omit them in computing the $a_i^{(p)}$. For $p > d$, it is seen that $D(\lambda_2 + p) \neq 0$, and consequently a solution of (1) of the form (2) exists in Case I.

CASE II. Suppose that the $F_i^{(d)}$ do not satisfy relations (8). We shall show that the desired solution of system (1) is of the type

$$(9) \qquad x_i = t^\lambda (Q_i(t) + P_i(t) \log t) \qquad (i = 1, \cdots, n),$$

where P_i and Q_i are of the form

$$P_i(t) = \sum_{k=0}^{\infty} a_i^{(k)} t^k \qquad (i = 1, \cdots, n),$$

$$Q_i(t) = \sum_{k=0}^{\infty} b_i^{(k)} t^k \qquad (i = 1, \cdots, n).$$

On substituting (9) in (1), it is seen that the resulting relations will be identities in t, $t = 0$ being excluded, provided that the $a_i^{(k)}$ and $b_i^{(k)}$ satisfy the following relations:

$$(10) \quad (\theta_{ii}^{(0)} - \lambda) a_i^{(0)} + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(0)} = 0 \qquad (i = 1, \cdots, n;\ k \neq i),$$

$$(11) \quad (\theta_{ii}^{(0)} - \lambda - p) a_i^{(p)} + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(p)} = F_i^{(p)}(a_j^{(0)}, \cdots, a_j^{(p-1)}) \qquad (i = 1, \cdots, n;\ k \neq i;\ p = 1, 2, \cdots),$$

$$(12) \quad (\theta_{ii}^{(0)} - \lambda - p) b_i^{(p)} + \sum_{k=1}^{n} \theta_{ik}^{(0)} b_k^{(p)} = a_i^{(p)} + F_i^{(p)}(b_j^{(0)}, \cdots, b_j^{(p-1)}) \qquad (i = 1, \cdots, n;\ k \neq i;\ p = 0, 1, 2, \cdots),$$

where

$$F_i^{(0)} \equiv 0 \qquad (i = 1, \cdots, n).$$

In order to determine the $a_i^{(p)}$ and $b_i^{(p)}$ satisfying (10), (11), and (12), start with the n cofactors $\eta_1(\lambda), \cdots, \eta_n(\lambda)$ of the elements $\theta_{n1}^{(0)}, \cdots, \theta_{nn}^{(0)} - \lambda$ in the determinant $D(\lambda)$. Assume, for simplicity, that the first $n - 1$ rows of $D(\lambda_2)$ contain a non-vanishing first minor; then $a_i^{(0)} = \eta_i^{(0)}(\lambda_2)$ $(i = 1, \cdots, n)$ is a solution of (10). Equations (10) are homogeneous in the $a_j^{(0)}$, and hence will be satisfied for

$$(13)\qquad a_i^{(0)}(\lambda) = (\lambda - \lambda_2)\,\eta_i^{(0)}(\lambda) \qquad (i = 1, \cdots, n;\ \lambda = \lambda_2),$$

where the $a_i^{(0)}(\lambda)$ are to be regarded as functions of λ. Substitute (13) into the right members of (11) and solve for the $a_i^{(p)}$ $(p = 1, 2, \cdots)$, where again the $a_i^{(p)}(\lambda)$ are to be thought of as functions of λ. In view of (13) one can write

$$a_i^{(p)}(\lambda) = (\lambda - \lambda_2)\,\eta_i^{(p)}(\lambda) \qquad (i = 1, \cdots, n;\ p = 1, 2, \cdots, d - 1).$$

Since $\lambda - \lambda_2$ is a factor of $D(\lambda + d)$ and likewise of each $F_i^{(d)}(a_j^{(0)}, \cdots, a_j^{(d-1)})$ $(i = 1, \cdots, n)$, it follows that equations (11) give n functions of λ, $a_i^{(d)}(\lambda)$ $(i = 1, \cdots, n)$, which, for $\lambda = \lambda_2$, yield a non-trivial solution of the equations

$$(\theta_{ii}^{(0)} - \lambda_2 - d)a_i^{(d)} + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(d)} = F_i^{(d)} = 0 \qquad (i = 1, \cdots, n;\ k \neq i).$$

For all values of $p > d$ equations (11) give functions of λ, $a_i^{(p)}(\lambda)$ $(i = 1, \cdots, n)$ not all of which are zero for $\lambda = \lambda_2$. Thus $a_i^{(p)}(\lambda_2)$ $(i = 1, \cdots, n;$ $p = d, d + 1, \cdots)$ satisfy equations (4) and (5) with $\lambda = \lambda_1$, it being noted that the $a_i^{(d)}(\lambda_2)$ satisfy (4); the $a_i^{(d+1)}(\lambda_2)$ satisfy (5) for $p = 1$, and so on. It is thus apparent that

$$x_i = [t^\lambda P_i(t:\lambda)]_{\lambda=\lambda_2} \qquad (i = 1, \cdots, n),$$

is a solution of system (1) and differs by a constant factor only from the solution already obtained, for $\lambda = \lambda_1$.

If we choose

$$(14)\qquad b_i^{(p)} = [\partial a_i^{(p)}(\lambda)/\partial\lambda]_{\lambda=\lambda_2} \qquad (i = 1, \cdots, n;\ p = 1, 2, \cdots),$$

it is easily shown that equations (12) are satisfied. For $p < d$ one has

$$b_i^{(p)} = [(\partial/\partial\lambda)(\lambda - \lambda_2) - \eta_i^{(p)}(\lambda)]_{\lambda=\lambda_2} = \eta_i^{(p)}(\lambda_2) \qquad (i = 1, \cdots, n).$$

Also for $p \leqq d$ equations (11) enable one to write the identities,

$$(15)\quad (\theta_{ii}^{(0)} - \lambda - p)a_i^{(p)}(\lambda) + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(p)}(\lambda) \overset{\lambda}{\equiv} (\lambda - \lambda_2) F_i^{(p)}(\eta_j^{(0)}(\lambda), \cdots, \eta_j^{(p-1)}(\lambda)) \qquad (i = 1, \cdots, n;\ k \neq i).$$

After differentiating (15) with respect to λ and setting $\lambda = \lambda_2$ there results

$$(16)\quad (\theta_{ii}^{(0)} - \lambda_2 - p)(\partial a_i^{(p)}(\lambda)/\partial\lambda)_{\lambda=\lambda_2} + \sum_{k=1}^{n} \theta_{ik}^{(0)}(\partial a_k^{(p)}(\lambda)/\partial\lambda)_{\lambda=\lambda_2} = a_i^{(p)} + F_i^{(p)}[\eta_j^{(0)}(\lambda_2), \cdots, \eta_j^{(p-1)}(\lambda_2)] \qquad (i = 1, \cdots, n;\ k \neq i),$$

from which it is observed that the values of the $b_i^{(p)}$ in (14) satisfy (12) for $p \leqq d$. Note that the $a_i^{(p)}(\lambda_2) = 0$ for $p = 1, 2, \cdots, d-1$. If $p > d$ equations (11) give the identities

$$(17)\begin{cases} (\theta_{ii}^{(0)} - \lambda - p)a_i^{(p)}(\lambda) + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(p)}(\lambda) \overset{\lambda}{\equiv} F_i^{(p)}(a_j^{(0)}(\lambda), \cdots, a_j^{(p-1)}(\\ \quad \equiv -\sum_{j=1}^{n} \sum_{k=1}^{p-d} \theta_{ij}^{(k)} a_j^{(p-k)}(\lambda) - (\lambda - \lambda_2) \sum_{j=1}^{n} \sum_{k=p-d+1}^{p} \theta_{ij}^{(k)} \eta_j^{(p-k)}(\lambda) \\ \qquad\qquad (i = 1, \cdots, n;\ k \neq \end{cases}$$

On differentiating (17) with respect to λ and setting $\lambda = \lambda_2$, one obtains

$$\begin{aligned} &(\theta_{ii}^{(0)} - \lambda_2 - p)(\partial a_i^{(p)}(\lambda)/\partial\lambda)_{\lambda=\lambda_2} + \sum_{k=1}^{n} \theta_{ik}^{(0)} (\partial a_k^{(p)}(\lambda)/\partial\lambda)_{\lambda=\lambda_2} \\ &\quad = a_i^{(p)}(\lambda_2) - \sum_{j=1}^{n} \sum_{k=1}^{p-d} \theta_{ij}^{(k)} b_j^{(p-k)}(\lambda_2) - \sum_{j=1}^{n} \sum_{k=p-d+1}^{p} \theta_{ij}^{(k)} b_j^{(p-k)}(\lambda_2) \\ &\quad = a_i^{(p)}(\lambda_2) + F_i^{(p)}(b_j^{(0)}(\lambda_2), \cdots, b_j^{(p-1)}(\lambda_2)) \\ &\qquad\qquad\qquad (i = 1, \cdots, n;\ k \neq i). \end{aligned}$$

Thus it has been shown that (14) yield the values of the $b_i^{(p)}$ satisfying (12) for $p = 0, 1, 2, \cdots$, and consequently (9) gives as the solution of system (1) associated with $\lambda = \lambda_2$,

$$x_i = t^{\lambda_2}[(\partial P_i(t:\lambda)/\partial\lambda) + P_i(t:\lambda) \log t]_{\lambda=\lambda_2} \qquad (i = 1, \cdots, n).$$

The methods of this section (Case II) have been presented in considerable detail because of their basic importance in the more general cases treated in later sections.

3. *Solutions corresponding to a Group of Roots of* $D(\lambda) = 0$ *mutually Congruent* (mod 1). Let $\lambda_1, \cdots, \lambda_e$ be e simple roots of $D(\lambda) = 0$ such that

$$(18) \qquad \lambda_i - \lambda_j = d_{ij} \qquad (i, j = 1, \cdots, e;\ i < j),$$

where the d_{ij} are positive integers. Suppose further that no other root of $D(\lambda) = 0$ differs from any one of the foregoing roots by an integer. A solution of system (1) of the form

$$(19) \qquad x_i = t^{\lambda_1} P_{i1}(t) \qquad (i = 1, \cdots, n),$$

is obviously obtainable by the method of Section 1. Solution (19) may, in exact analogy with the scheme of Section 2, Case II, be represented by each of the e forms

$$x_i = t^{\lambda_\sigma}[P_{i\sigma}(t:\lambda)]_{\lambda=\lambda_\sigma} \qquad (i=1,\cdots,n;\ \sigma=1,\cdots,e), \tag{20}$$

where the $P_{i\sigma}(t:\lambda)$ are power series in t which will now be defined. Let the constant term of $P_{i\sigma}(t:\lambda)$ be $(\lambda-\lambda_\sigma)^{g_\sigma}\eta_i^{(0)}(\lambda)$, the $\eta_i^{(0)}(\lambda)$ having been defined in section 2. The coefficients of t^k are derived from equations (11) after choosing g_σ so that the coefficients of t^k vanish for $k < d_{1\sigma}$, while for $k = d_{1\sigma}$ at least one coefficient of t^k in $P_{i\sigma}(t:\lambda)$ $(i=1,\cdots,n)$ is different from zero. The value of g_σ depends upon the vanishing or non-vanishing of the $D_i(\lambda_\sigma + d_{j\sigma})$ $(i=1,\cdots,n;\ j=1,\cdots,\sigma-1)$, and will be given explicitly in the subsequent discussion.

In considering the solution associated with $\lambda=\lambda_2$ it is seen that this solution will be of the form

$$x_i = t^{\lambda_2}P_{i2}(t) \qquad (i=1,\cdots,n),$$

if and only if

$$D_i(\lambda_2 + d_{12}) = 0 \qquad (i=1,\cdots,n). \tag{21}$$

Conditions (21) will in general not be satisfied, in which case the solution in question, just as in Case II of section 2, will be of the form

$$x_i = t^{\lambda_2}[(\partial P_{i2}(t:\lambda)/\partial\lambda) + P_{i2}(t:\lambda)\log t]_{\lambda=\lambda_2} \quad (i=1,\cdots,n).$$

If we assume for any one of the group of e roots of $D(\lambda)=0$, say for $\lambda=\lambda_\sigma$, that

$$D_i(\lambda_\sigma + d_{j\sigma}) \neq 0 \tag{22}$$

for at least one of the values $i=1,\cdots,n$ when $j=1,2,\cdots,\sigma-1$ respectively, then corresponding to $\lambda=\lambda_\sigma$, one readily verifies the following solution of system (1):

$$x_i = t^{\lambda_\sigma}\left\{\sum_{h=0}^{\sigma-1}\binom{\sigma-1}{h}\frac{\partial^{\sigma-h-1}P_{i\sigma}(t:\lambda)}{\partial\lambda^{\sigma-h-1}}\log^h t\right\}_{\lambda=\lambda_\sigma} \qquad (i=1,\cdots,n), \tag{23}$$

Clearly $g_\sigma = \sigma-1$ in the $P_{i\sigma}(t:\lambda)$ of solution (23). On substituting (23) in (1) it is observed that the coefficient of $\log^h t$ vanishes for $h=1,2,\cdots,\sigma-1$ respectively. The vanishing of the coefficients of the respective powers of $\log t$ is a consequence of the consistency of pairs of equation-sets of precisely the forms given by (11) and (12).

If it should happen for any root, $\lambda=\lambda_\epsilon$, of the group that

$$D_i(\lambda_\epsilon + d_{j\epsilon}) = 0 \qquad (i=1,\cdots,n)$$

for $j=1,2,\cdots,\epsilon-1$ respectively, then equations (11) will be consistent when $\lambda = \lambda_\epsilon$ for all values of p. The corresponding solution of system (1) is therefore of the form

$$x_i = t^{\lambda_\epsilon} P_{i\epsilon}(t) \qquad (i = 1, \cdots, n),$$

in which the $P_{i\epsilon}(t)$ $(i = 1, \cdots, n)$ are power series in t, at least one of which has a non-vanishing constant term. It is observed that the factor $(\lambda - \lambda_\sigma)^{g_\sigma}$ is introduced in the constant terms of the $P_{i\sigma}(t:\lambda)$ for the purpose of forming consistent sets of equations (12) when p takes on values $d_{j\sigma}$ for which the determinant $D(\lambda_\sigma + d_{j\sigma}) = 0$. Thus g_σ is reduced by one for each value of j for which

$$(24) \qquad D_i(\lambda_\sigma + d_{j\sigma}) = 0 \qquad (i = 1, \cdots, n);\ (j = 1, 2, \cdots, \sigma - 1).$$

One can now generalize solution (23) and write for any root $\lambda = \lambda_\sigma$ of $D(\lambda) = 0$

$$(25) \qquad x_i = t^{\lambda_\sigma} \left\{ \sum_{h=0}^{g} \binom{g}{h} \left(\frac{\partial^{g-h} P_{i\sigma}(t:\lambda)}{\partial \lambda^{g-h}} \right) \log^h t \right\}_{\lambda=\lambda_\sigma} \qquad (i = 1, \cdots, n),$$

where, in view of the foregoing discussion,

$$g = \sigma - 1 - \rho,$$

ρ being the number of times (24) holds for the values $j = 1, 2, \cdots, \sigma - 1$.

Before treating the most general situation that can arise, namely, the case for which one or more of the $\lambda_1, \cdots, \lambda_e$ are multiple roots, we shall take up in some detail the method of obtaining the solutions corresponding to a multiple root which is not congruent (mod 1) to any other root of $D(\lambda) = 0$.

4. *Solutions corresponding to a Multiple Root of* $D(\lambda) = 0$. Let $\lambda = \lambda_\sigma$ be a multiple root of $D(\lambda) = 0$ such that λ_σ does not differ from any other root of $D(\lambda) = 0$ by an integer. The nature of the solutions of system (1) for the present case depend upon the properties of the determinant $D(\lambda_\sigma)$ and will be obtained from hypotheses regarding $D(\lambda_\sigma)$ and its minors, that include every special case that can arise.

Suppose (A) that $\lambda = \lambda_\sigma$ is a root of $D(\lambda) = 0$ of multiplicity s_0 and (B) that all 1st, 2nd, $\cdots$, $(q-1)$st minors of $D(\lambda_\sigma)$ are zero, $q \leqq s_0$, while at least one qth minor of $D(\lambda_\sigma)$ is different from zero.

In consequence of hypotheses (A) and (B) equations (10) will have q linearly independent solutions each of which gives rise to q solutions of (1) of the form

$$x_i = t^{\lambda_\sigma} P_{ikq}(t) \qquad (i = 1, \cdots, n;\ k = 1, \cdots, q),$$

where the $P_{ikq}(t)$ are power series in t of the form

$$P_{ikq}(t) = \sum_{h=0}^{\infty} a^{(h)}{}_{ikq} \quad t^h \qquad (i = 1, \cdots, n).$$

The method of obtaining the $a^{(h)}{}_{ikq}$ has already been given in section 1.

The remaining $s_0 - q$ solutions, as will be shown, occur in groups of the form

$$(26) \qquad x_i = t^{\lambda_\sigma} \left\{ \sum_{h=0}^{k} \frac{\partial^{k-h} P_i(t:\lambda)}{\partial \lambda^{k-h}} \; \frac{\log^h t}{(r+k-h)!\,h!} \right\}_{\lambda=\lambda_\sigma}$$
$$(i = 1, \cdots, n;\ k = 1, \cdots, \omega),$$

where

$$P_i(t:\lambda) = \sum_{h=0}^{\infty} a_i^{(h)}(\lambda) t^h.$$

The integers r and ω, and the coefficients $a_i^{(h)}(\lambda)$ will now be determined.

On substituting (26) in (1) it is seen that (26) will be a solution of system (1) for $k = 1, 2, \cdots, \omega$ provided the $a_i^{(0)}(\lambda)$ can be determined so as to satisfy the following conditions:

$$(27) \qquad (\theta_{ii}^{(0)} - \lambda) a_i^{(0)}(\lambda) + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(0)}(\lambda) = 0$$
$$(i = 1, \cdots, n;\ k \neq i;\ \lambda = \lambda_\sigma),$$

$$(28) \qquad (\theta_{ii}^{(0)} - \lambda) \frac{\partial^m a_i^{(0)}(\lambda)}{\partial \lambda^m} + \sum_{k=1}^{n} \theta_{ik}^{(0)} \frac{\partial^m a_k^{(0)}(\lambda)}{\partial \lambda^m} = (r+m) \frac{\partial^{m-1} a_i^{(0)}(\lambda)}{\partial \lambda^{m-1}}$$
$$(i = 1, \cdots, n;\ k \neq i;\ \lambda = \lambda_\sigma;\ m = 1, \cdots, \omega).$$

It will now be shown that under certain conditions equations (27) and (28) will be consistent and that the sufficient conditions for their consistency are implied in the hypotheses of the present case.

From the determinant $D(\lambda)$ one verifies the identities,

$$(29) \quad \begin{cases} (\theta_{ii}^{(0)} - \lambda)\delta_{ij}(\lambda) + \sum_{k=1}^{n} \theta_{ik}^{(0)} \delta_{kj}(\lambda) \overset{\lambda}{\equiv} 0 \\ \qquad\qquad (i = 1, \cdots, n-j;\ k \neq i), \\ (\theta_{ii}^{(0)} - \lambda)\delta_{ij}(\lambda) + \sum_{k=1}^{n} \theta_{ik}^{(0)} \delta_{kj}(\lambda) \overset{\lambda}{\equiv} \delta_{i,j-1}(\lambda) \\ \qquad\qquad (i = n-j+1, \cdots, n;\ k \neq i), \end{cases}$$

where the $\delta_{ij}(\lambda)$ $(i = 1, \cdots, n-j+1)$ are the jth minor-cofactors respectively (of order $n-j$), of the elements $\theta^{(0)}{}_{n-j+1,i}$ $(i = 1, \cdots, n-j+1)$, in the upper left $(j-1)$st minor of $D(\lambda)$; and the $\delta_{i,j-1}(\lambda)$, $(i = n-j+1, \cdots, n)$ are the upper left $(j-1)$st minors of $D(\lambda)$ obtained by replacing the $(n-j+1)$st row by the $(n-j+1)$st, $(n-j+2)$nd, $\cdots$, nth rows respectively. For the foregoing particular choice in the position of the $(j-1)$st minor-cofactors occurring in (29) the $\delta_{kj}(\lambda)$, $(k=n-j+2, \cdots, n$ are identically zero, by definition, and have been written for the purpose of securing generality in notation and logic, and also for the purpose of making the coefficients in (29) agree exactly, in form, with the coefficients in the left numbers of equations (27) and (28). This specialization in the position, in $D(\lambda)$, of the $(j-1)$st minors in question has been employed solely for the purpose of effecting simplicity in presentation; all subsequent work, however, is entirely general and applies to any minor-cofactors satisfying certain conditions which will now be specified.

Suppose (a) that $(\lambda-\lambda_\sigma)^s$ is a factor of every $(j-1)$st minor (of order $n-j+1$) of $D(\lambda)$, while at least one $(j-1)$st minor does not contain the factor $(\lambda-\lambda_\sigma)^{s+1}$; and (b) that $(\lambda-\lambda_\sigma)^r$ is a factor of every jth minor in $D(\lambda)$, while at least one jth minor does not contain $(\lambda-\lambda_\sigma)^{r+1}$, where obviously $s-r \geqq 1$. Let identities (29) be differentiated $s-1$ times successively with respect to λ, setting $\lambda=\lambda_\sigma$ in each of the s sets of relations thus obtained. By hypothesis (a), the right members of the resulting s sets of relations vanish and, by (b), each term of the left members of the first r sets of equations also vanishes. There remain, however, the following $s-r$ non-trivial sets of relations:

$$(30)\qquad \begin{cases} (\theta_{ii}^{(0)}-\lambda_\sigma)\,\dfrac{\partial^r\delta_{ij}}{\partial\lambda^r}+\displaystyle\sum_{k=1}^{n}\theta_{ik}^{(0)}\,\dfrac{\partial^r\delta_{kj}}{\partial\lambda^r} = r\,\dfrac{\partial^{r-1}\delta_{ij}}{\partial\lambda^{r-1}}=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (i=1,\cdots,n;\ k\neq i) \\ (\theta_{ii}^{(0)}-\lambda_\sigma)\,\dfrac{\partial^{r+h}\delta_{ij}}{\partial\lambda^{r+h}}+\displaystyle\sum_{k=1}^{n}\theta_{ik}^{(0)}\,\dfrac{\partial^{r+h}\delta_{kj}}{\partial\lambda^{r+h}} = (r+h)\,\dfrac{\partial^{r+h-1}\delta_{ij}}{\partial\lambda^{r+h-1}} \\ \qquad\qquad (i=1,\cdots,n;\ k\neq i;\ h=1,\cdots,s-r-1), \end{cases}$$

where

$$\frac{\partial^\nu\delta_{ij}}{\partial\lambda^\nu} \equiv \left[\frac{\partial^\nu\delta_{ij}(\lambda)}{\partial\lambda^\nu}\right]_{\lambda=\lambda_\sigma}:$$

On comparing relations (30) with equations (27) and (28), it is seen that, under hypotheses (a) and (b), the latter equations have as solutions

$$(31)\qquad \begin{cases} a_i^{(0)}(\lambda) = \dfrac{\partial^r\delta_{ij}(\lambda)}{\partial\lambda^r} \qquad\qquad (i=1,\cdots,n;\ \lambda=\lambda_\sigma) \\ \dfrac{\partial^m a_i^{(0)}(\lambda)}{\partial\lambda^m} = \dfrac{\partial^{r+m}\delta_{ij}(\lambda)}{\partial\lambda^{r+m}} \\ \qquad\qquad (i=1,\cdots,n;\ m=1,\cdots,\ s-r-1;\ \lambda=\lambda_\sigma).. \end{cases}$$

in which the value of r is defined by (b) and where $\omega = s - r - 1$. Thus on assumptions (a) and (b) one obtains $\omega = s - r - 1$ solutions of system (1) which are given by (26), for $k = 1, 2, \cdots, s - r - 1$.

In consequence of hypotheses (B) of the present section all derivatives with respect to λ, up to and including the $(q - j - 1)$st, of all jth minors of $D(\lambda)$ vanish when $\lambda = \lambda_\sigma$ for $j = 1, 2, \cdots, (q - 1)$. This follows from the fact that the λ-derivative of every $(j - 1)$st minor of $D(\lambda)$ is a sum of jth minors of $D(\lambda)$. Hence (B) implies that $(\lambda - \lambda_\sigma)^{q-j}$ is a factor of every jth minor of $D(\lambda)$, $j = 1, \cdots, q - 1$. If there exists at least one first minor of $D(\lambda)$ which does not contain $(\lambda - \lambda_\sigma)^q$ as a factor, then in view of the foregoing discussion the remaining $s_0 - q$ solutions of system (1), associated with $\lambda = \lambda_\sigma$, are given by (26) with

$$s = s_0, \quad r = q - 1, \quad j = 1, \quad \omega = s_0 - q.$$

It may happen, however, that $(\lambda - \lambda_\sigma)^\alpha$ where $\alpha \geqq q$ is a common factor of all first minors of $D(\lambda)$ in which case there are fewer than $s_0 - q$ solutions of system (1) which are expressed in terms of first minor-cofactors of $D(\lambda)$ and their λ-derivatives. In fact a similar special situation may exist for the jth minors of $D(\lambda)$ for $j = 2, \cdots, q - 1$. The assumptions with respect to this question will therefore be generalized so as to include every possible special case.

Hence let it be assumed (C) that every jth minor of $D(\lambda)$, has the factor $(\lambda - \lambda_\sigma)^{s_j}$ while at least one jth minor of $D(\lambda)$ does not contain $(\lambda - \lambda_\sigma)^{s_j+1}$, $(j = 1, \cdots, q)$, where obviously $s_j \geqq q - j$ and $s_{j-1} - s_j \geqq 1$. Hypotheses (C) are identical in character with the assumptions (a) and (b) underlying the derivation of (31), and consequently form (26) gives solutions of system (1) in which the coefficients of the respective powers of t are expressed in terms of minor-cofactors in $D(\lambda_\sigma)$ and their λ-derivatives. In obtaining solutions of system (1) for which the $a_i^{(0)}$ are expressed in terms of jth minor-cofactors of $D(\lambda_\sigma)$, $(j = 1, \cdots, q)$ one makes the following substitutions in (26) and (31):

$$s = s_{j-1}; \quad r = s_j; \quad \omega = s_{j-1} - s_j - 1 \equiv \omega_j. \tag{32}$$

The total number of solutions, corresponding to $\lambda = \lambda_\sigma$, which one obtains in this way is clearly given by

$$\begin{aligned} \omega_1 + \omega_2 + \cdots + \omega_q &= \\ &= (s_0 - s_1 - 1) + (s_1 - s_2 - 1) + \cdots + (s_{q-1} - s_q - 1) \\ &= s_0 - s_q - q = s_0 - q, \end{aligned}$$

since, by (B), $s_q = 0$.

Thus we have obtained s_0 solutions of equations (1) associated with $\lambda = \lambda_\sigma$ which is a root of $D(\lambda) = 0$ of order s_0. These solutions may be conveniently represented by the notation

$$(33) \qquad x_i = \sum_{k=1}^{q} c_{\sigma k}\, t^{\lambda_\sigma} P_{ikq}(t) + \sum_{j=1}^{q} \sum_{\omega=1}^{\omega_j} c_{\sigma\omega j}\, t^{\lambda_\sigma} R_{i\omega j}(t) \qquad (i = 1, \cdots, n),$$

where the $t^{\lambda_\sigma} R_{i\omega j}(t), \omega = 1, \cdots, \omega_j$ are the ω_j solutions given by (26) in terms of jth minor-cofactors of $D(\lambda_\sigma)$ and their λ-derivatives.

5. *Solutions Associated with a Group of Multiple Roots. General Case.* Let $\lambda_1, \cdots, \lambda_e$ be a group of multiple roots of $D(\lambda) = 0$ of orders $\beta_1, \cdots, \beta_e$ respectively, such that

$$\lambda_i - \lambda_j = d_{ij} \qquad (i, j = 1, \cdots, n;\ i < j),$$

where the d_{ij} are positive integers. It is also assumed that no other root of $D(\lambda) = 0$ differs from any one of the λ_j $(j = 1, \cdots, e)$ by an integer.

The results of sections 3and 4 enable one to obtain, without difficulty, the $\beta_1 + \cdots + \beta_e$ solutions of system (1) corresponding to the foregoing group of multiple roots. Consider λ_σ, any one of the e multiple roots, and assume that $D(\lambda_\sigma)$ has properties (A), (B), and (C) of section 4. Set $\beta_\sigma = s_0$, in order that the results of section 4 may be applied with as little change in notation as possible.

On referring to equations (11) and taking $p = d_{k\sigma}$ $(k = 1, \cdots, \sigma - 1)$ it is noted that $D(\lambda + d_{k\sigma})$ contains the factor $(\lambda - \lambda_\sigma)^{\beta_k}$ but does not contain the factor $(\lambda - \lambda_\sigma)^{\beta_k + 1}$. It may happen that some power of $\lambda - \lambda_\sigma$ is a common factor of the n determinants $D_i(\lambda + d_{k\sigma})$, $(i = 1, \cdots, n)$. We shall therefore assume that $(\lambda - \lambda_\sigma)^{\gamma_k}$ is a factor of every $D_i(\lambda + d_{k\sigma})$, $(i = 1, \cdots, n)$ while at least one of these n determinants does not contain $(\lambda - \lambda_\sigma)^{\gamma_k + 1}$. The foregoing hypotheses apply for $k = 1, 2, \cdots, \sigma - 1$. Let

$$(34) \qquad (\beta_1 + \beta_2 + \cdots + \beta_{\sigma-1}) - (\gamma_1 + \gamma_2 + \cdots + \gamma_{\sigma-1}) \\ = g(\sigma; \delta_i(\lambda)) \equiv g,$$

and choose

$$(35) \qquad a_i^{(0)}(\lambda) = (\lambda - \lambda_\sigma)^g \delta_i(\lambda) \qquad (i = 1, \cdots, n).$$

In view of the hypotheses (A) and (B), equations (10), for $\lambda = \lambda_\sigma$, have q linearly independent sets of solutions which are expressed in terms of qth minor-cofactors of $D(\lambda_\sigma)$. Replace the λ_σ by λ in these solutions and thus obtain q sets of functions $\delta_{iqk}(\lambda)$.

Set $\delta_i(\lambda) = \delta_{iqk}(\lambda)$, $(k = 1, \cdots, q)$ respectively, in (35), giving q sets of values of the $a_i^{(0)}(\lambda)$. Equations (11) now determine the $a_i^{(p)}(\lambda)$

$(p = 1, 2, \cdots)$ as functions of λ. Substitute the q sets of values for the coefficients $a_i^{(p)}$ in the series

$$P_{ikq}(t:\lambda) = \sum_{h=0}^{\infty} a_i^{(h)} t^h \qquad (i = 1, \cdots, n),$$

and indicate the resulting series by the notation

$$P_{ikq}(t;\lambda) = \sum_{h=0}^{\infty} a_{ikq}^{(h)}(\lambda)\, t^h$$
$$(i = 1, \cdots, n;\ k = 1, \cdots, q).$$

Since, in general, the value of g depends upon the choice of the $\delta_i(\lambda)$ we shall designate by α_k the value of g corresponding to $\delta_i(\lambda) = \delta_{ikq}(\lambda)$. One can now write q of the desired s_0 solutions of system (1) in the form

$$(36) \qquad x_i = t^{\lambda\sigma} \left\{ \sum_{h=0}^{a_k} \frac{\partial^{a_k-h} P_{ikq}(t:\lambda)}{\partial\lambda^{a_k-h}} \frac{\log^h t}{(\alpha_k - h)!\, h!} \right\}_{\lambda=\lambda\sigma}$$
$$(i = 1, \cdots, n;\ k = 1, \cdots, q).$$

In order to obtain the remaining $s_0 - q$ solutions of system (1) choose for the $\delta_i(\lambda)$ in (35) exactly the λ-derivatives of jth minor-cofactors of $D(\lambda)$, $(j = 1, \cdots, q)$ given in (31), which were used in setting up the $R_{i\omega j}(t)$ of (33). Designate by μ_j the value of g resulting from the choice

$$\delta_i(\lambda) = \frac{\partial^{s_j} \delta_{ij}(\lambda)}{\partial\lambda^{s_j}} .$$

Let the corresponding sets of infinite series in t be represented by $R_{ij}(t:\lambda)$ $(i = 1, \cdots, n)$ and set

$$(37) \qquad \sum_{h=0}^{\mu_j} \frac{\partial^{\mu_j-h} R_{ij}(t:\lambda)}{\partial\lambda^{\mu_j-h}} \frac{\log^{h+k} t}{(h+k)!\,(\mu_j - h)!} = H_{ijk}(t:\lambda) \quad (i = 1, \cdots,$$

We shall find that the remaining $s_0 - q$ solutions of system (1) corresponding to $\lambda = \lambda_\sigma$ can be expressed in the form

$$(38) \qquad x_i = t^{\lambda\sigma} \left\{ \frac{H_{ijk}(t:\lambda)}{s_j!} + \sum_{\nu=1}^{k} \frac{\partial^{\mu_j+\nu} R_{ij}(t:\lambda)}{\partial\lambda^{\mu_j+\nu}} \frac{\log^{k-\nu} t}{(k-\nu)!\,(s_j+\nu)!} \right\}_{\lambda=\lambda}$$
$$(i = 1, \cdots, n;\ k = 1, \cdots, \omega_j;\ j = 1, \cdots, q).$$

The hypotheses regarding $D(\lambda_\sigma)$ and its minor-cofactors are identical with those underlying the derivation of solutions (33) and hence the results of section 4 show that (38) gives $s_0 - q$ solutions of system (1).

If the x_i of (38) be substituted in (1) the coefficients of $\log^m t$ $(m = 0, 1, \cdots, \mu_j + \omega_j - 1)$ vanish in each of the resulting n equations.

For each of the foregoing values of m, the coefficient of $\log^m t$ when equated to zero, gives identities of exactly the forms (11) and (12); while for $m = \mu_j + \omega_j$ the coefficient of $\log^m t$ vanishes in consequence of identities of the form (10).

6. *Independence of the n Solutions of System* (1). Let

$$x_i = \phi_{ij}(t) \qquad (i = 1, \cdots, n;\ j = 1, \cdots, n) \tag{39}$$

be a set of n solutions of system (1) obtained by the foregoing method. These solutions will form a fundamental set if and only if their determinant

$$\Delta(t) = | \phi_{ij}(t) | \neq 0.$$

The n solutions given by (39) occur in groups of the form

$$x_i = t^{\lambda_\sigma} \{ \sum_{k=1}^{l_{\sigma j}} V_{ij\sigma}^{(k-1)}(t) \log^{k-1} t \} \tag{40}$$
$$(j = 1, \cdots, q_\sigma;\ \sigma = 1, \cdots, n;\ i = 1, \cdots, n)$$

where $$1 \leqq l_{\sigma j} \leqq n \quad \text{and} \quad \sum_{\sigma=1}^{n} \sum_{j=1}^{q_\sigma} l_{\sigma j} = n.$$

Thus $\Delta(t)$ may be reduced to the form

$$\Delta(t) = t^N | V_{ij\sigma}^{(0)}(t) | = t^N K(t) \tag{41}$$
$$(j = 1, \cdots, q_\sigma;\ \sigma = 1, \cdots, n;\ i = 1, \cdots, n),$$

where the $V_{ij\sigma}^{(0)}(t)$ are the power series in t, that are not multiplied by a power of $\log t$, in each of the n solutions; and N is the sum of the n roots of $D(\lambda) = 0$.

Clearly $\Delta(t) \overset{t}{\equiv} 0$ if and only if $K(t) \overset{t}{\equiv} 0$. Since $K(t)$ is an analytic function of t in the neighborhood of $t = 0$, $K(0) \neq 0$ implies that $\Delta(t_1) \neq 0$ for $| t_1 |$ sufficiently small. Thus the independence of the n solutions of system (1) is established if it can be shown that $K(0) \neq 0$.

If we write

$$V_{ij\sigma}^{(0)}(t) = a_{ij\sigma}^{(0)} + a_{ij\sigma}^{(1)} t + a_{ij\sigma}^{(2)} t^2 + \cdots$$
$$(j = 1, \cdots, q_\sigma;\ \sigma = 1, \cdots, n;\ i = 1, \cdots, n),$$

it is observed that

$$K(0) = | a_{ij\sigma}^{(0)} | \qquad (j = 1, \cdots, q_\sigma;\ \sigma = 1, \cdots, n;\ i = 1, \cdots, n), \tag{42}$$

where $a_{ij\sigma}^{(0)} = 0$, by definition, if there are no $V_{ij\sigma}^{(0)}(t)$ in which the $a_{ij\sigma}^{(0)}$ are expressed in terms of jth minor-cofactors of $D(\lambda_\sigma)$.

Now consider the system of differential equations

$$dx_i/dt = \theta_{i1}^{(0)} x_1 + \cdots + \theta_{in}^{(0)} x_n \qquad (i = 1, \cdots, n), \tag{43}$$

obtained from system (1) by omitting the factor t from the left members and retaining as coefficients of the $x_1, \cdots, x_n$ the constant terms of the $\theta_{i1}(t), \cdots, \theta_{in}(t)$. The n solutions of system (43) as obtained by a direct method due to the author,[*] are shown to form a fundamental set. The proof of this property of the n solutions, in the paper referred to, is effected by showing that the determinant $E(t) = E(0)$

$$\exp \int \sum \theta_{ii}^{(0)} dt$$

of the solutions is different from zero for $t = 0$. The determinant $E(0)$ is identical with the determinant $K(0)$ of the present paper, except for a constant non-zero multiplier.

Hence the independence of the n solutions of system (1) follows as a consequence of the corresponding proof for the solutions of a system of linear differential equations having constant coefficients.

7. *Convergence of the Solutions.* Let it be assumed that the $\theta_{ij}(t)$ of system (1) converge for $|t| \leqq R$. Choose $t = t_0$ any point within the domain of convergence. Let

$$x_i = \phi_{ij}(t) \qquad (i, j = 1, \cdots, n), \tag{44}$$

be a fundamental set of solutions for the regular point $t = t_0$. Make analytic continuations of the ϕ_{ij} about $t = 0$ along paths which neither enclose nor pass through any other singular point.

Let $\phi_{ij}(t)$ become $\bar{\phi}_{ij}(t)$ after making this circuit. Since the $\bar{\phi}_{ij}$ that result from this circuit are also solutions of system (1), one can write

$$\bar{\phi}_{ij}(t) = \sum_{k=1}^{n} c_{kj}\, \phi_{ik}(t) \qquad (i = 1, \cdots, n) \tag{45}$$

for $j = 1, \cdots, n$.

We shall raise the question of the existence of a solution of system (1),

$$x_i = \phi_i(t) \qquad (i = 1, \cdots, n),$$

having the property

$$\bar{\phi}_i(t) = r\, \phi_i(t) \qquad (i = 1, \cdots, n). \tag{46}$$

If such a solution exists it can be written in the form

$$\phi_i(t) = \sum_{k=1}^{n} b_k\, \phi_{ik}(t) \qquad (i = 1, \cdots, n), \tag{47}$$

where, after a circuit about $t = 0$ one has

$$\bar{\phi}_i(t) = \sum_{k=1}^{n} b_k\, \bar{\phi}_{ik}(t). \tag{48}$$

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[*] *American Journal of Mathematics*, Vol. 47 (1925), pp. 257-276.

After expressing both members of (46) as linear functions of the $\phi_{ij}(t)$ by substituting from (45), (47), and (48), there results

$$r\sum_{k=1}^{n} b_k\,\phi_{ik} = \sum_{k=1}^{n} b_k\left(\sum_{j=1}^{n} c_{kj}\,\phi_{ij}(t)\right) \qquad (i=1,\cdots,n). \tag{49}$$

Relations (49) may be regarded as a system of linear homogeneous equations in the b_i $(i=1,\cdots,n)$; and they will have a solution other than $b_i=0$ $(i=1,\cdots,n)$ if and only if the determinant of the coefficients

$$\Gamma(r)\Delta(r)\equiv \tag{50}$$

$$\begin{vmatrix} c_{11}-r, & c_{12}, & \cdots, & c_{1n} \\ c_{21}, & c_{22}-r, & \cdots, & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1}, & c_{n2}, & \cdots, & c_{nn}-r \end{vmatrix} \cdot \begin{vmatrix} \phi_{11}, & \phi_{12}, & \cdots, & \phi_{1n} \\ \phi_{21}, & \phi_{22}, & \cdots, & \phi_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \phi_{n1}, & \phi_{n2}, & \cdots, & \phi_{nn} \end{vmatrix} = 0.$$

Since, by hypothesis, (44) is a fundamental set of solutions of system (1) it is evident that (50) is equivalent to $\Gamma(r)=0$. Thus for each root of $\Gamma(r)=0$ there exists a set of values of the b_i given by (49) which upon substitution in (47) yield the solution $x_i=\phi_i(t)$, satisfying condition (46).

Suppose first that the n roots $r_1,\cdots,r_n$ of $\Gamma(r)=0$ are distinct and that the corresponding n sets of solutions of system (1) having property (46) are given by

$$x_i=\psi_{ij}(t) \qquad (i,j=1,\cdots,n), \tag{51}$$

where

$$\overline{\psi}_{ij}(t)=r_j\,\psi_{ij}(t) \qquad (i=1,\cdots,n). \tag{52}$$

Let r_σ be any root of $\Gamma(r)=0$ and define λ_σ by the relation

$$r_\sigma=e^{2\pi i\lambda_\sigma}, \tag{53}$$

from which one has

$$\lambda_\sigma=(1/2\pi i)\log r_\sigma+h, \tag{54}$$

where h is any integer. Consider the function

$$U_\sigma(t)=t^{\lambda_\sigma}$$

which, after a circuit about the point $t=0$ becomes

$$\overline{U}_\sigma(t)=r_\sigma U_\sigma(t). \tag{55}$$

Thus one observes that the ratio

$$\psi_{ij}(t)/U_j(t)\equiv P_{ij}(t) \qquad (i,j=1,\cdots,n),$$

On Related Difference and Differential Systems.*

By William M. Whyburn.

1. *Introduction.* Sturm's † work on difference equations has been reconstructed for the most part by Porter,‡ Bocher,§ Fort,¶ Merrill,‖ Carmichael,** and others. In addition numerous authors including Hilbert †† and Fredholm ‡‡ have made extensive use of the intimate relationship that exists between algebraic and transcendental problems. Porter ‡ was the first to actually carry difference equation results over to the limiting differential equations. Others have carried this limiting process through, but all previous results have been confined to the case where the coefficients of the differential systems are continuous functions. It is the purpose of the present paper to establish results for difference systems and to carry these results over to the limiting differential systems. This passage to the limit is accomplished for a differential system with Lebesgue summable coefficients by making use of a subdivision of the interval of the independent variable.

In parts 2 and 3, preliminary theorems concerning a system of two linear difference equations are proved. In part 4, we show that the equations, solutions, and many properties of a system of m difference equations of the first order go over in the limit to the equations, solutions, and properties, respectively of a system m differential equations with summable coefficients. Similar results are obtained for general non-linear systems. In part 5, the Green's function and the adjoint system are defined for difference systems and these quantities are carried over to the limiting differential systems.

The notation of vector and tensor calculus is used in parts 2, 4, and 5. A capital letter denotes an m-dimensional tensor (square matrix) whose element in the ith row and jth column is given by the same letter with the sub-

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[*] Presented to the American Mathematical Society, September 9, 1926 and Dec. 31, 1926.

[†] *Journal des Mathématiques*, t. 1 (1836), pp. 106-186.

[‡] *Annals of Mathematics*, Vol. 3 (1901-1902), pp. 55-70.

[§] *Bulletin of the American Mathematical Society*, Vol. 18 (1911-1912), p. 1 ff.

[¶] *American Journal of Mathematics*, Vol. 39 (1917), pp. 1-26.

[‖] *Transactions of the American Mathematical Society*, Vol. 4 (1903), p. 423 ff.

[**] *American Journal of Mathematics*, Vol. 43 (1921), pp. 69-101, 232-270.

[††] Six papers in *Göttinger Nachrichten*, 1904-1910.

[‡‡] *Acta Mathematica*, Bd. 27, S. 384.

script ij and whose vector component that makes up its jth column is given by the same letter with the single subscript j. A lowercase letter or a Greek letter is used to denote an m-dimensional vector whose jth component is given by the same letter with the subscript j. To say that a vector or tensor has a certain property means that each of its components has that property. The dot product, $A \cdot B$, of the tensor A by the tensor B is a tensor C such that $C_{rj} = \sum_{k=1}^{m} A_{rk} B_{kj}$. The product $A \cdot u$ of the tensor A by the vector u is a vector b such that $b_j = \sum_{k=1}^{m} A_{jk} u_k$. The letters j, k, i, m, n, p, r, s, are used to denote variables of iteration. All exceptions to the above notations are signalized where they occur.

Throughout the paper we will suppose that our difference systems are defined with reference to the finite set of distinct points $E_n : x_{0n} = a, x_{1n}, x_{2n}, \cdots, x_{nn} = b$ lying on the finite interval $X : a \leqq x \leqq b$. Bold-faced type denotes a function that is defined on E_n only. By $\boldsymbol{f}(i)$ we mean $\boldsymbol{f}(x_{in})$ and by $\Delta \boldsymbol{f}(i)$ we mean $\boldsymbol{f}(i+1) - \boldsymbol{f}(i)$. We use the notation $j = (k, r)$ to mean that j takes on the values k and r together with all positive integral values that lie between k and r.

2. *The Second Order System.* Consider the second order difference system

$$\frac{\Delta \boldsymbol{y}_j(i)}{\Delta x(i)} - \boldsymbol{A}_{j1}(i)\boldsymbol{y}_1(i) - \boldsymbol{A}_{j2}(i)\boldsymbol{y}_2(i) = 0 \qquad (j = 1, 2) \tag{2.1''}$$

which we can write in the form

$$\frac{\Delta \boldsymbol{y}(i)}{\Delta x(i)} - \boldsymbol{A}(i) \cdot \boldsymbol{y}(i) = 0 \tag{2.1'}$$

where $\boldsymbol{A}(i)$ is a two dimensional tensor and $\boldsymbol{y}(i)$ is a two dimensional vector. Let E denote the unit tensor which is defined so that $E_{ij} = 0$ for $i \neq j$ and $E_{jj} = 1$. Let $\boldsymbol{B}(i) = \boldsymbol{A}(i)\Delta x(i) + E$ and equation (2.1′) takes the form

$$\boldsymbol{y}(i+1) = \boldsymbol{B}(i) \cdot \boldsymbol{y}(i). \tag{2.1}$$

Throughout this discussion we will assume that $| \boldsymbol{B}(i) | \neq 0$ for every point of E_n, where $| \boldsymbol{B}(i) |$ means the determinant (second order) of $\boldsymbol{B}(i)$. We will frequently use continuous functions $f_n(x)$ that are defined in the following manner:

$$f_n(x) = \boldsymbol{f}(i) + \frac{\Delta \boldsymbol{f}(i)}{\Delta x(i)} [x - x_{in}] \quad \text{on } x_{in} \leqq x \leqq x_{i+1,n}.$$

The notation $f_n(x)$ will be used consistently to denote a function of this type that is constructed from $\boldsymbol{f}(i)$.

Definition: If $\boldsymbol{u}(i)$ and $\boldsymbol{v}(i)$ are any two solutions of (2.1), then by the wronskian of these solutions we mean $\boldsymbol{W}(i)$, where

$$\boldsymbol{W}(i) = \begin{vmatrix} \boldsymbol{u}_1(i) & \boldsymbol{v}_1(i) \\ \boldsymbol{u}_2(i) & \boldsymbol{v}_2(i) \end{vmatrix} .$$

THEOREM 2.1: *$\boldsymbol{W}(i)$ either vanishes identically or it does not vanish at all on E_n and if $| \boldsymbol{B}(i) | > 0$ on E_n, $\boldsymbol{W}(i)$ maintains the same sign on E_n.*

Proof: Substitute $\boldsymbol{u}(i)$ and $\boldsymbol{v}(i)$ into (2.1) and write the resulting equations in component form.

$$\text{(a)} \qquad \boldsymbol{u}_1(i+1) = \boldsymbol{B}_{11}(i)\boldsymbol{u}_1(i) + \boldsymbol{B}_{12}(i)\boldsymbol{u}_2(i)$$

$$\text{(b)} \qquad \boldsymbol{u}_2(i+1) = \boldsymbol{B}_{21}(i)\boldsymbol{u}_1(i) + \boldsymbol{B}_{22}(i)\boldsymbol{u}_2(i)$$

$$\text{(c)} \qquad \boldsymbol{v}_1(i+1) = \boldsymbol{B}_{11}(i)\boldsymbol{v}_1(i) + \boldsymbol{B}_{12}(i)\boldsymbol{v}_2(i)$$

$$\text{(d)} \qquad \boldsymbol{v}_2(i+1) = \boldsymbol{B}_{21}(i)\boldsymbol{v}_1(i) + \boldsymbol{B}_{22}(i)\boldsymbol{v}_2(i).$$

Multiply equation (c) by $\boldsymbol{u}_2(i)$ and subtract this from (a) after the latter has been multiplied through by $\boldsymbol{v}_2(i)$. Multiply (d) by $\boldsymbol{u}_1(i+1)$ and subtract this from (b) after multiplying (b) through by $\boldsymbol{v}_1(i+1)$. This gives

$$\begin{aligned} &\boldsymbol{v}_2(i)\boldsymbol{u}_1(i+1) - \boldsymbol{u}_2(i)\boldsymbol{v}_1(i+1) = \boldsymbol{B}_{11}(i)\boldsymbol{W}(i), \\ &-\boldsymbol{W}(i+1) = \boldsymbol{B}_{21}(i)[\boldsymbol{u}_1(i)\boldsymbol{v}_1(i+1) - \boldsymbol{v}_1(i)\boldsymbol{u}_1(i+1)] + \\ &\qquad\qquad \boldsymbol{B}_{22}(i)[\boldsymbol{u}_2(i)\boldsymbol{v}_1(i+1) - \boldsymbol{v}_2(i)\boldsymbol{u}_1(i+1)]. \end{aligned}$$

Combining these equations and replacing $\boldsymbol{v}_1(i+1)$ and $\boldsymbol{u}_1(i+1)$ by their values as given by (a) and (c), we get

$$\boldsymbol{W}(i+1) = | \boldsymbol{B}(i) | \, \boldsymbol{W}(i).$$

From this equation we conclude immediately that $\boldsymbol{W}(i)$ either vanishes identically or does not vanish at all and if $| \boldsymbol{B}(i) | > 0$ on E_n, then $\boldsymbol{W}(i)$ maintains the same sign on E_n.

Porter[*] proved many properties of second order difference systems that carry over without difficulty to our system. He proved that a necessary and sufficient condition that two solutions be linearly independent is that $\boldsymbol{W}(i)$ be different from zero on E_n. An examination of the matrix of coefficients reveals that the system (2.1) has a two parameter family of solutions since this matrix contains one determinant of the $2n$th order whose value is $\prod_0^{n-1} | \boldsymbol{B}(i) | \neq 0$. It follows from Porter's work that if $\boldsymbol{u}(i)$ and $\boldsymbol{v}(i)$ are linearly independent solutions of (2.1), then $\boldsymbol{u}_1(i)$ and $\boldsymbol{v}_1(i)$ cannot both vanish for any point of E_n. It is also true that $\mathbf{v}_1(i)$ and $\mathbf{v}_2(i)$ cannot both vanish at the same point of E_n and, $\mathbf{v}_1(i)$ and $\mathbf{v}_1(i+1)$ cannot both vanish for any value of i.

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[*] *Loc. cit.*, pp. 59-60.

THEOREM 2. 2: *If $\mathbf{u}(i)$ and $\mathbf{v}(i)$ are two linearly independent solutions of (2. 1) and $\varphi(i) = \boldsymbol{\alpha}(i) \cdot \mathbf{u}(i)$, $\theta(i) = \boldsymbol{\alpha}(i) \cdot \mathbf{v}(i)$, where $\alpha_1^2(i) + \alpha_2^2(i) \neq 0$, then the zeros of $\phi_n(x)$ and $\theta_n(x)$ separate each other on X provided $|\mathbf{B}(i)| > 0$ and $\{\alpha_1\alpha_2\}$ is different from zero and maintains the same sign on E_n, where*

$$\begin{aligned}\{\alpha_1\alpha_2\} = {} & \alpha_2(i)\alpha_2(i+1)\mathbf{B}_{21}(i) + \alpha_1(i+1)\alpha_2(i)\mathbf{B}_{11}(i) \\ & - \alpha_1(i)\alpha_2(i+1)\mathbf{B}_{22}(i) - \alpha_1(i)\alpha_1(i+1)\mathbf{B}_{12}(i).\end{aligned}$$

Proof: Neither $\phi_n(x)$ nor $\theta_n(x)$ can have more than a finite number of zeros on X. For, if we suppose that $\phi_n(x)$ has an infinite number of zeros on X, then there must be at least one pair of consecutive values of $\varphi(i)$ that are both zero. Let $\varphi(r) = 0$ and $\varphi(r+1) = 0$, then

$$\begin{aligned}&\boldsymbol{\alpha}(r) \cdot \mathbf{u}(r) = 0 \\ &\boldsymbol{\alpha}(r+1) \cdot \mathbf{B}(r) \cdot \mathbf{u}(r) = 0.\end{aligned}$$

The determinant of coefficients of $\mathbf{u}_1(r)$ and $\mathbf{u}_2(r)$ is $\{\alpha_1\alpha_2\}$ which is different from zero. This leads to a contradiction when we note that $\mathbf{u}_1(r)$ and $\mathbf{u}_2(r)$ cannot both vanish.

$\phi_n(x)$ and $\theta_n(x)$ cannot both vanish at the same point of X. Suppose that at some point t on $x_{rn} \leqq t \leqq x_{r+1,n}$ we have $\phi_n(t) = 0$ and $\theta_n(t) = 0$. It follows from the statement that we have just proved that $\Delta\varphi(r) \neq 0$, $\Delta\theta(r) \neq 0$ and hence we can solve the equations $\phi_n(t) = 0$, $\theta_n(t) = 0$ for t.

$$t = x_{rn} - \frac{\varphi(r)\Delta x(r)}{\Delta\varphi(r)} = x_{rn} - \frac{\theta(r)\Delta x(r)}{\Delta\theta(r)} .$$

The foregoing equation yields

$$\frac{\Delta\varphi(r)}{\varphi(r)} = \frac{\Delta\theta(r)}{\theta(r)}$$

or

$$\mathbf{W}(r)\{\alpha_1\alpha_2\} = 0.$$

Since neither $\mathbf{W}(i)$ nor $\{\alpha_1\alpha_2\}$ can vanish, our assumption that $\phi_n(x)$ and $\theta_n(x)$ vanish simultaneously leads to a contradiction.

If neither $\phi_n(x)$ nor $\theta_n(x)$ vanish twice on X, our theorem is trivial. Suppose that c and d are two zeros of $\phi_n(x)$ and assume that $\theta_n(x)$ does not vanish for any x between c and d. The function $\phi_n(x)/\theta_n(x)$ is continuous

$c \leqq x \leqq d$ and has a derivative everywhere on this interval except at the finite subset of E_n that lies on $c \leqq x \leqq d$.

$$(d/dx)[\phi_n(x)/\theta_n(x)] = [\theta(i)\Delta\varphi(i) - \varphi(i)\Delta\theta(i)]/\theta_n^2(x)\Delta x(i)$$
$$= W(i)\{\alpha_1\alpha_2\}/\theta_n^2(x)\Delta x(i) \quad \text{on } x_{in} < x < x_{i+1,n}.$$

Since $W(i)$ and $\{\alpha_1\alpha_2\}$ both maintain the same sign on X, we see that $\phi_n(x)/\theta_n(x)$ is a monotonic, continuous, non-identically zero, function on $c \leqq x \leqq d$ and therefore cannot vanish at both c and d. This contradicts our hypothesis that $\phi_n(x)$ vanishes at both of these points and hence $\theta_n(x)$ must vanish at least once between every pair of zeros of $\phi_n(x)$. If we apply the above argument to $\theta_n(x)/\phi_n(x)$ we arrive at the conclusion that between every pair of zeros of $\theta_n(x)$ there is at least one zero of $\phi_n(x)$. Hence the zeros of $\theta_n(x)$ and $\phi_n(x)$ separate each other on X.

COROLLARY 1: *If* $|B(i)| > 0$ *and* $B_{12}(i)$ *maintains the same sign on and is different from zero on* E_n, *then the zeros of* $u_{1n}(x)$ *and* $v_{1n}(x)$ *separate each other on* X.[*]

COROLLARY 2: *If* $|B(i)| > 0$ *and* $B_{21}(i)$ *maintains the same sign and is different from zero on* E_n, *then the zeros of* $u_{2n}(x)$ *and* $v_{2n}(x)$ *separate each other on* X.

THEOREM 2.3: *If* $y(i)$ *is a non-identically vanishing solution of* (2.1) *and* $\varphi(i) = \alpha(i)\cdot y(i)$, $\theta(i) = \beta(i)\cdot y(i)$, *where* $\alpha_1^2(i) + \alpha_2^2(i) \neq 0$, $\beta_1^2(i) + \beta_2^2(i) \neq 0$, $\{\alpha_1\alpha_2\} \neq 0$, $\{\beta_1\beta_2\} \neq 0$, *then the zeros of* $\phi_n(x)$ *and* $\theta_n(x)$ *separate each other on* X *provided that* $R(i) < 0$ *and* $S(i)$ *maintains the same sign on* E_n, *where*

$$\begin{aligned} R(i) = {} & \{B_{22}(i)[\alpha_2(i+1)\beta_1(i) - \alpha_1(i)\beta_2(i+1)] + B_{12}(i)[\alpha_1(i+1)\beta_1(i) \\ & - \alpha_1(i)\beta_1(i+1)] + B_{21}(i)[\alpha_2(i)\beta_2(i+1) \\ & - \alpha_2(i+1)\beta_2(i)] \\ & + B_{11}(i)[\alpha_2(i)\beta_1(i+1) - \alpha_1(i+1)\beta_2(i)]\}^2 \\ & - 4d(i)d(i+1)|B(i)| \\ S(i) = {} & B_{11}(i)[\beta_1(i)\alpha_1(i+1) - \beta_1(i+1)\alpha_1(i)] + [\alpha_2(i+1)\beta_1(i) \\ & - \beta_2(i+1)\alpha_1(i)]B_{21}(i) \\ d(i) = {} & \alpha_2(i)\beta_1(i) - \alpha_1(i)\beta_2(i). \end{aligned}$$

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[*] If (2.1) is specialized to the forms used by these writers, this corollary becomes the theorem on the separation of the nodes that was first proved by Porter, *loc. cit.*, p. 63, and later by Moulton, *Annals of Mathematics*, Vol. 13 (1911), p. 137; Carmichael, *American Journal of Mathematics*, Vol. 43 (1921), p. 80, and Dunkel, *Bulletin of the American Mathematical Society*, Vol. 32, p. 333.

Proof: The same proof that was given under theorem 2.2 shows that $\phi_n(x)$ and $\theta_n(x)$ have only a finite number of zeros on X. To show that $\phi_n(x)$ and $\theta_n(x)$ cannot both vanish at the same point of X, we assume that they both vanish at $x = t$, $x_{rn} < t \leqq x_{r+1,n}$. Solving the equations $\phi_n(t) = 0$, $\theta_n(t) = 0$, we get

$$t = x_{rn} - \frac{\varphi(r)\Delta x(r)}{\Delta\varphi(r)} = x_{rn} - \frac{\theta(r)\Delta x(r)}{\Delta\theta(r)}$$

or $Q(r) = 0$, where $Q(i)$ is a quadratic form in $y_1(i)$ and $y_2(i)$ that has its discriminant equal to $R(i)$ and the coefficient of $y_1^2(i)$ equal to $S(i)$. The conditions on $R(i)$ and $S(i)$ are sufficient to make this quadratic form definite on X and hence the equation $Q(r) = 0$ cannot be satisfied for any real values of $y_1(r)$ and $y_2(r)$. This contradiction proves that $\phi_n(x)$ and $\theta_n(x)$ cannot both vanish at the same point of X.

If neither of the two functions vanish twice then our theorem is meaningless. Let c and d be two zeros of $\phi_n(x)$ and assume that $\theta_n(x)$ does not vanish between $x = c$ and $x = d$. The function $\phi_n(x)/\theta_n(x)$ will be continuous on $c \leqq x \leqq d$ and will have a derivative everywhere on this interval except at a finite number of points.

$$\frac{d}{dx}\left[\frac{\phi_n(x)}{\theta_n(x)}\right] = \frac{Q(i)}{\theta_n^2(x)\Delta x(i)} \text{ on } x_{in} < x < x_{i+1,n}.$$

Since $Q(i)$ has the same sign on X, we see that $\phi_n(x)/\theta_n(x)$ is a continuous, monotonic, non-identically zero, function on $c \leqq x \leqq d$ and hence cannot vanish at both c and d. Since this contradicts our hypothesis that c and d are zeros of $\phi_n(x)$ it follows that $\theta_n(x)$ must vanish at least once between c and d. A similar consideration of $\theta_n(x)/\phi_n(x)$ proves that between every pair of zeros of $\theta_n(x)$ there is at least one zero of $\phi_n(x)$ and hence it follows that the zeros of $\theta_n(x)$ and $\phi_n(x)$ separate each other on X.

COROLLARY: *The following special cases of theorem 2.3 are important:*[*]

(a). *If* $\beta_2(i) \equiv 0$ *and* $\beta_1(i) \equiv 1$, *we get separation of the zeros of* $y_{1n}(x)$ *and* $\phi_n(x)$.

(b). $\beta_2(i) \equiv 1$ *and* $\beta_1(i) \equiv 0$ *yields separation of the zeros of* $y_{2n}(x)$ *and* $\phi_n(x)$.

(c). $\alpha_1(i) \equiv 1$, $\alpha_2(i) \equiv 0$, $\beta_1(i) \equiv 0$, $\beta_2(i) \equiv 1$ *yields separation of the zeros of* $y_{1n}(x)$ *and* $y_{2n}(x)$.

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[*] The conditions simplify very much in these special cases. Simpler conditions are derived if we assume that $\alpha_1 \neq 0$, $\beta_1 \neq 0$ but in doing this we lose some of the above special cases.

THEOREM 2.4: *If $\mathbf{y}(i)$ is a non-identically vanishing solution of (2.1) and $\varphi(i) = \alpha(i) \cdot \mathbf{y}(i)$, where $\alpha_2(i) > 0$ and $\{\alpha_1\alpha_2\} \neq 0$, then the product $y_{1n}(x)\phi_n(x)$ cannot vanish more than once on X provided $\mathbf{y}(i)$ and $\varphi(i)$ do not both vanish for any i, $T(x,i) < 0$ on XE_n* and $P(x,i)$ maintain the same sign on XE_n, where*

$$
\begin{aligned}
P(x,i) &= [1 + 2A_{11}(i)\,(x - x_{in})]\,[(\Delta\beta(i)/\Delta x(i)) \\
&\quad + \beta(i+1)A_{11}(i) + A_{21}(i)] + \beta(i)A_{11}(i) \\
T(x,i) &= \{A_{12}(i)[\beta(i+1) + \beta(i) + 2(x - x_{in})[2A_{11}(i)\beta(i+1) \\
&\quad + (\Delta\beta(i)/\Delta x(i)) + A_{21}(i)]\} + A_{22}(i) + A_{11}(i) \\
&\quad + 2(x - x_{in})A_{11}(i)A_{22}(i)\}^2 - 4P(x,i)A_{12}(i)[1 + 2(x - x_{in}) \\
&\quad \{\beta(i+1)A_{12}(i) + A_{22}(i)\}] \quad \text{and} \quad \beta(i) = \alpha_1(i)/\alpha_2(i).
\end{aligned}
$$

Proof: Let $\theta(i) = \varphi(i)/\alpha_2(i)$. Clearly the product $y_{1n}(x)\theta_n(x)$ will have the same number of zeros on X as the product $y_{1n}(x)\phi_n(x)$. The derivatives of these products exist everywhere on X except for a finite number of points. The derivative of $y_{1n}\theta_n$ can be written as a quadratic form in $\mathbf{y}_1(i)$ and $\mathbf{y}_2(i)$ and this quadratic form has $T(x,i)$ for its discriminant and $P(x,i)$ as a coefficient of one of its squared terms. The conditions on $T(x,i)$ and $P(x,i)$ are sufficient to insure that this quadratic form have the same sign on X. It follows that the product $y_{1n}(x)\theta_n(x)$ is continuous and monotonic on X and since it cannot vanish for two consecutive values of i it cannot vanish more than once on X.†

COROLLARY 1: *If we let $\alpha_1(i) \equiv 0$, we get conditions for the non-oscillation of $y_{1n}(x)$ and $y_{2n}(x)$.*

COROLLARY 2: *If $A_{11}(i) \equiv A_{22}(i) \equiv 0$, $A_{12}(i) \equiv 1/K(i)$, $A_{21}(i) \equiv G(i)$, the conditions for the non-oscillation of the solution become $G(i)$ of constant sign on E_n and $0 < G(i)/K(i) < 4/[\Delta x(i)]^2$. These conditions serve to insure non-oscillation for the self-adjoint system when the points of E are sufficiently close together.*

3. *Existence Theorems.* In this section we will consider the second order system

$$
\begin{aligned}
&\Delta y(i)/\Delta x(i) = z(i)/K(i,\lambda) \\
&\Delta z(i)/\Delta x(i) = G(i,\lambda)y(i) + G(i,\lambda)\,\Delta x(i)z(i)/K(i,\lambda)
\end{aligned}
\tag{3.1}
$$

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* This means for every x on X and for every x_{in} on E_n.

† If the requirement that $y(i)$ and $\varphi(i \pm 1)$ not both vanish is removed, the above argument shows that the product $y_{1n}(x)\phi_n(x)$ cannot vanish for two distinct values of x without vanishing for every x that lies between these values.

where $\boldsymbol{K}$ and $\boldsymbol{G}$ are continuous functions of λ on L: $L_1 < \lambda < L_2$ for each i. System (3. 1) can be written in the form

$$\frac{\Delta}{\Delta x(i)}\left[\boldsymbol{K}(i,\lambda)\,\frac{\Delta \boldsymbol{y}(i)}{\Delta x(i)}\right] - \boldsymbol{G}(i,\lambda)\boldsymbol{y}(i+1) = 0. \tag{3. 2}$$

Equations (3. 1) show immediately that $\boldsymbol{y}(i,\lambda)$ and $\boldsymbol{z}(i,\lambda)$ are continuous functions of λ on L. The continuity of the zeros of the functions $y_n(x,\lambda)$, $z_n(x,\lambda)$, and $\phi_n(x,\lambda)$ follows from the continuity of $\boldsymbol{y}(i,\lambda)$, $\boldsymbol{z}(i,\lambda)$ and $\boldsymbol{\varphi}(i,\lambda)$ in λ. We note that if $f_n(t,\lambda) = 0$, then

$$t(\lambda) = x_{in} - \frac{f(i,\lambda)\Delta x(i)}{\Delta f(i,\lambda)} \qquad \text{when } x_{in} < t < x_{i+1,n}.$$

THEOREM 3. 1: *If* $\boldsymbol{y}(i,\lambda)$ *is a solution of* (3. 2) *and* $\boldsymbol{K}(i,\lambda) > 0$, *then the zeros of* $y_n(x,\lambda)$ *move to the left on* X *as* λ *increases provided that* $\boldsymbol{G}(i,\lambda)/\boldsymbol{K}(i,\lambda)$ *and* $\boldsymbol{y}(1,\lambda)/\boldsymbol{y}(0,\lambda)$ *are non-increasing functions of* λ *and these ratios are not both constant over any part of the* λ*-interval.*

Proof: This theorem was proved by Fort (*loc. cit.*, pp. 8-9). He stated his hypotheses in a slightly different form but they are equivalent to the above.

THEOREM 3. 2: *Any equation of the type*

$$\boldsymbol{N}(i,\lambda)(\Delta^2\boldsymbol{y}(i)/\Delta x(i)^2) + \boldsymbol{L}(i,\lambda)(\Delta\boldsymbol{y}(i)/\Delta x(i)) + \boldsymbol{M}(i,\lambda)\boldsymbol{y}(i) = 0$$

where $\boldsymbol{N}$ *and* $[1 + \boldsymbol{M}/\boldsymbol{N}[\Delta x(i)]^2 - \boldsymbol{L}/\boldsymbol{N}\Delta x(i)]$ *are different from zero, can be written in the form of* (3. 2).[*]

Proof: Let

$$\boldsymbol{K}(i,\lambda) = \prod_{0}^{i-1}\,[1 + \frac{\boldsymbol{M}(j,\lambda)}{\boldsymbol{N}(j,\lambda)}\,\Delta x(j)^2 - \frac{\boldsymbol{L}(j,\lambda)}{\boldsymbol{N}(j,\lambda)}\Delta x(j)]^{-1}$$

$$\boldsymbol{G}(i,\lambda) = -\boldsymbol{M}(i,\lambda)/\boldsymbol{N}(i,\lambda)\boldsymbol{K}(i+1,\lambda)$$

and our equation takes the desired form.

THEOREM 3. 3: *Under the hypotheses of theorem* 3. 1, $\boldsymbol{y}(i,\lambda)$ *changes signs when it vanishes.*

Proof: Let $\boldsymbol{y}(i,\lambda) = 0$. Neither $\boldsymbol{y}(i-1,\lambda)$ nor $\boldsymbol{y}(i+1,\lambda)$ can vanish and since these functions are continuous functions of λ, we can determine an interval, $\lambda' < \lambda < \lambda''$, such that neither $\boldsymbol{y}(i-1,\lambda)$ nor $\boldsymbol{y}(i+1,\lambda)$ can vanish on this interval. Equation (3. 2) gives

$$\boldsymbol{K}(i-1,\lambda)\boldsymbol{y}(i-1,\lambda) + \boldsymbol{K}(i,\lambda)\boldsymbol{y}(i+1,\lambda) = 0$$

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[*] Porter (*loc. cit.*, p. 60) proved this theorem by means of a change of variable.

for since $K(i,\boldsymbol{\lambda}) > 0$ on E_nL, $y(i-1,\boldsymbol{\lambda})$ and $y(i+1,\boldsymbol{\lambda})$ differ in sign. Since these two functions do not vanish on the λ-interval that we have cho $y(i+1,\lambda)$ and $y(i-1,\lambda)$ differ in signs for every λ on $\lambda' < \lambda < \lambda''$. There is only one zero of $y_n(x,\lambda)$ on $x_{i-1,n} \leqq x \leqq x_{i+1,n}$ and this zero mo to the left as λ increases from λ′ to λ″. For $\lambda' < \lambda < \boldsymbol{\lambda}$, $y(i,\lambda)$ has the sa sign as $y(i+1,\lambda)$ and for $\boldsymbol{\lambda} < \lambda < \lambda''$, $y(i,\lambda)$ has the sign of $y(i-1,\lambda)$ and hence $y(i,\lambda)$ changes sign when it vanishes.

COROLLARY: *$y_n(x,\lambda)$ can neither gain nor lose zeros except at the the x-interval.*

Let $\varphi(i,\lambda) = \alpha(i,\lambda)z(i,\lambda) - \beta(i,\lambda)y(i,\lambda)$, where $\alpha(i,\lambda) \neq 0$ and α and β are continuous functions of λ on L for each fixed i. Compute $\Delta\varphi(i,\lambda)$, eliminate y and z between this equation and the above equation for z; application of the transformation given in theorem 3.2 yields

$$\frac{\Delta}{\Delta x(i)}\left[H(i,\lambda)\frac{\varphi(i,\lambda)}{\Delta x(i)}\right] - J(i,\lambda)\varphi(i+1,\lambda) = 0 \tag{3.3}$$

where

$$H(i,\lambda) = \prod_{0}^{i-1}\frac{\{\theta(j)\}}{\{\theta(j+1)\}}, \quad \theta(j) = \beta(j)/\alpha(j)$$

$$\{\theta(j)\} = \theta(j)[G(j)\Delta x(j) - \theta(j+1)]/K(j) + G(j) - \frac{\Delta\theta(}{\Delta x(}$$

$$J(i,\lambda) = -H(i+1,\lambda)\left[\frac{\theta(i+2) - G(i+2)\Delta x(i+2)}{K(i+1)} - \frac{\theta(i)\{\theta(i-1)}{K(i)\{\theta(i}\right.$$

The above transformation demands that $\{\theta(i)\}$ be different from zero. We can make the zeros of $\phi_n(x,\lambda)$ move to the left as λ increases by imposing the same conditions on H and J that were imposed on K and G by theorem 3.1. We will suppose that such conditions are satisfied and that the hypotheses of theorem 2.3 are met by φ and θ for each fixed λ on L, where $\theta(i,\lambda) = \nu(i,\lambda)z(i,\lambda) - \delta(i,\lambda)y(i,\lambda)$ and ν and δ are continuous functions of λ on L.

Definition: By a characteristic number of a difference or differential system we mean a value of the parameter for which the system has a non identically vanishing solution.

THEOREM 3.4: *The equation* (3.1) *together with the boundary conditions*

$$\varphi(0,\lambda) = \varphi(n,\lambda) = 0 \tag{3.4}$$

has $n-1$ *characteristic numbers,* $p_1, p_2, \cdots p_{n-1}$, such that if $\mathbf{y}(i, p_j)$, $\mathbf{z}(i, p_j)$ *is the solution that goes with* p_j *then* $y_n(x, p_j)$ *has exactly* j *zeros on* X, *provided condition* (3.5) *is satisfied.*

$$(3.5) \qquad \begin{aligned} &\frac{\min \boldsymbol{G}(i)}{\min \boldsymbol{K}(i)} \geqq 0 \quad \text{for each fixed } \lambda \text{ on } L_1 < \lambda < \mu_1, \text{ and} \\ &\frac{\max \boldsymbol{G}(i)}{\max \boldsymbol{K}(i)} \leqq \frac{-4}{[\max \Delta x(i)]^2} \text{ for every fixed } \lambda \text{ on } \mu_2 < \lambda < L_2, \end{aligned}$$

where μ_1 and μ_2 are fixed points such that $L_1 < \mu_1 < \mu_2 < L_2$.

Proof: We can choose a solution $\mathbf{y}(i, \lambda)$, $\mathbf{z}(i, \lambda)$ of (3.1) such that $\mathbf{y}(0, \lambda) = \boldsymbol{\alpha}(0, \lambda)$ and $\mathbf{z}(0, \lambda) = \boldsymbol{\beta}(0, \lambda)$ for all values of λ on L. This solution satisfies the condition $\boldsymbol{\varphi}(0, \lambda) = 0$ for all values of λ on L. Condition (3.5) insures that $\varphi_n(x, \lambda)$ does not vanish on $a < x \leqq b$ for any λ on $L_1 < \lambda < \mu_1$ while this same condition, together with the fact that the zeros of $\phi_n(x, \lambda)$ and $y_n(x, \lambda)$ separate each other on X, assures us that $\phi_n(x, \lambda)$ has $n-1$ zeros on X for every fixed λ on $\mu_2 < \lambda < L_2$.[*] If we extend the interval X indefinitely and let the points of this extended interval be at a distance of one unit from each other, and if we let $\boldsymbol{K}(i, \lambda) \equiv 1$ and $\boldsymbol{G}(i, \lambda) \equiv -4$ outside of X, we see that $\mathbf{y}_n(x)$ and $\phi_n(x)$ will have as many zeros as you please on this extended interval. As λ increases the zeros of $\phi_n(x, \lambda)$, Which are continuous functions of λ, move into the X interval at $x = b$ and when $\phi_n(b, \lambda) = 0$ we get the characteristic numbers of the system (3.1), (3.4). Since no zeros of $y_n(x, \lambda)$ and $\phi_n(x, \lambda)$ are lost and since the only point where zeros enter the interval X is at $x = b$, our oscillation theorem follows.

COROLLARY: $\phi_n(x, p_j)$ *has exactly* j *zeros on* X.

THEOREM 3.5: *If conditions* (3.5) *are met, there exists a set of characteristic numbers,* $q_1, q_2, \cdots, q_n - 1$, *for the general self-adjoint system* (3.1), (3.6), *where*

$$(3.6) \qquad \begin{aligned} \boldsymbol{\varphi}(0, \lambda) &= \boldsymbol{\varphi}(n, \lambda) \\ \boldsymbol{\vartheta}(0, \lambda) &= \boldsymbol{\vartheta}(n, \lambda) \end{aligned}$$

and

$$\boldsymbol{\alpha}(0, \lambda)\boldsymbol{\delta}(0, \lambda) - \boldsymbol{\beta}(0, \lambda)\boldsymbol{\gamma}(0, \lambda) = \boldsymbol{\alpha}(n, \lambda)\boldsymbol{\delta}(n, \lambda) - \boldsymbol{\beta}(n, \lambda)\boldsymbol{\gamma}(n, \lambda).$$

Case I. $\boldsymbol{\alpha}(0)\boldsymbol{\delta}(0) - \boldsymbol{\beta}(0)\boldsymbol{\gamma}(0) = 0$.

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[*] Compare (3.1) with the equations: $\mathbf{y}(i+2) + 2\mathbf{y}(i+1) + \mathbf{y}(i) = 0$ and $\mathbf{y}(i+2) - 2\mathbf{y}(i+1) + \mathbf{y}(i) = 0$.

This case can be reduced back to the case considered under theorem 3.4 by making linear combinations of the boundary conditions.

Case II. $\alpha(0)\delta(0)-\beta(0)\gamma(0)\neq 0$. *In this case we make* $\alpha(0)\delta(0)-\beta(0)\gamma(0)=1$ *by division.*

It is easy to see that there is always a solution y_1, z_1 for the system (3.1), (3.7) and there is always a solution y_2, z_2 for (3.1), (3.8), where

$$(3.7)\qquad \begin{aligned}\varphi_1(0,\lambda)&=0\\ \theta_1(0,\lambda)&=1\end{aligned}\qquad\qquad (3.8)\qquad \begin{aligned}\varphi_2(0,\lambda)&=1\\ \theta_2(0,\lambda)&=0\end{aligned}$$

and

$$\varphi_1=\alpha z_1-\beta y_1,\ \varphi_2=\alpha z_2-\beta y_2,\ \theta_1=\gamma z_1-\delta y_1,\ \theta_2=\gamma z_2-\delta y_2.$$

If we solve (3.7) and (3.8) we get

$$z_1(0)=\beta(0),\ y_1(0)=\alpha(0),\ z_2(0)=-\delta(0),\ y_2(0)=-\gamma(0).$$

y_1, z_1 and y_2, z_2 form a fundamental system of solutions of (3.1) since $W(i,\lambda)=1$. We also have

$$(3.9)\quad \varphi_1(n)\theta_2(n)-\varphi_2(n)\theta_1(n)=-W(n)[\alpha(n)\delta(n)-\beta(n)\gamma(n)]=-1.$$

We write the solution of (3.1) as

$$\begin{aligned}y(i,\lambda)&=c_1y_1(i,\lambda)+c_2y_2(i,\lambda)\\ z(i,\lambda)&=c_1z_1(i,\lambda)+c_2z_2(i,\lambda)\end{aligned}$$

and substitute this general solution into the boundary conditions.

$$\begin{aligned}c_1[\varphi_1(0)-\varphi_1(n)]+c_2[\varphi_2(0)-\varphi_2(n)]&=0\\ c_1[\theta_1(0)-\theta_1(n)]+c_2[\theta_2(0)-\theta_2(n)]&=0.\end{aligned}$$

In order that there exist a solution for the system (3.1), (3.6) it is necessary that $D(\lambda)$ be zero, where $D(\lambda)$ is the determinant of coefficients of c_1 and c_2 in the above equations. If we make use of (3.7), (3.8), and (3.9), $D(\lambda)$ takes the form

$$D(\lambda)=-2+\varphi_2(n,\lambda)+\theta_1(n,\lambda).$$

If all of the elements of $D(\lambda)$ vanish for a value of λ, then that value of λ is a double characteristic number of system (3.1), (3.6), since in that case both of the solutions y_1, z_1 and y_2, z_2 satisfy the system. Let us consider the case where $D(\lambda)$ is of rank $\geqq 1$. Let $p_1, p_2, \cdots p_{n-1}$, be the characteristic numbers of the system (3.1), (3.4) and substitute p_j into (3.9). This gives

$$\varphi_2(n, p_j) = 1/\theta_1(n, p_j)$$

and

$$D(p_j) = [1 - \theta_1(n, p_j)]^2/\theta_1(n, p_j)$$

so that $D(p_j)$ either has the same sign as $\theta_1(n, p_j)$ or it is zero in which case p_j is a characteristic number of our system.

By theorem 4.3 we have that the zeros of $\varphi_{1n}(x, \lambda)$ and $\theta_{1n}(x, \lambda)$ separate each other on X for a fixed λ. Also, $\varphi_1(n, \lambda)$ and $\theta_1(n, \lambda)$ change signs when they vanish. Since $\varphi_1(n, p_j) = 0$, it follows that $\theta_1(n, p_j) \neq 0$ and that it is either positive for $j = 1, 3, \cdots$, and negative for $j = 2, 4, \cdots$, or negative for $j = 1, 3, 5, \cdots$, and positive for $j = 2, 4, \cdots$. Since $D(p_j)$ has the same sign as $\theta_1(n, p_j)$ or else it is zero, it follows that $D(\lambda)$ has at least one root between (or coincident with one of the two) each pair of characteristic numbers of system (3.1), (3.4). This establishes our existence theorem.

4. *The Passage to the Limit.* In this section we will let the number of points in E_n increase indefinitely in such a way that these points form a *fine* subdivision of the interval X, where by a *fine* subdivision we mean one such that the length of the maximum subdivision approaches zero as the number of these subdivisions becomes infinite. All integrals are to be taken in the sense of Lebesgue.

1°. *The Method of Successive Approximations.* Consider the m^{th} order differential system

$$(dy_j/dx) - \sum_{r=1}^{r=m} A_{jr}y_r = \beta_j \qquad j = (1, m)$$

$$y_j(a) = a_j$$

which we will write in the form*

$$(4.1) \qquad (dy/dx) - A \,.\, y = \beta$$

$$y(a) = a$$

following the notation described in the introduction of this paper. A_{jk} and β_j are assumed to be Lebesgue summable functions of x on X and a_j is a constant. We set up the difference system

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* An alternate form of (4.1) is $y(x) = \int_a^x [A(t) \cdot y(t) + \beta(t)]dt + a$.

$$(4.2) \qquad \frac{\Delta y(i)}{\Delta x(i)} - A(i) \cdot y(i) = \beta(i) \qquad y(0) = \alpha$$

which takes the form

$$(4.3) \qquad y(i+1) = \sum_{r=0}^{r=i} [B(r) \cdot y(r) + \varphi(r)] + \alpha \qquad y(0) = \alpha$$

when we let $B(i) = A(i)\Delta x(i)$ and $\varphi(i) = \beta(i)\Delta x(i)$. Let

$$y_0(i+1) = \alpha + \sum_{r=0}^{r=i} \varphi(r), \qquad y_0(0) = \alpha$$

and let $y_1(i)$ be determined by

$$y_1(i+1) = \alpha + \sum_{r=0}^{r=i} [B(r) \cdot y_0(r) + \varphi(r)], \qquad y_1(0) = \alpha.$$

In general let $y_k(i)$ be determined by

$$y_k(i+1) = \alpha + \sum_{r=0}^{r=i} [B(r) \cdot y_{k-1}(r) + \varphi(r)], \qquad y_k(0) = \alpha.$$

We wish to show that $\lim_{k\to\infty} y_k(i)$ exists and is the unique solution of system (4.2). Let

$$z_0(i) = y_0(i)$$

and

$$z_k(i) = y_k(i) - y_{k-1}(i) \text{ for } k \geqq 1.$$

We see immediately that if $k > 0$

$$z_k(i+1) = \sum_{r=0}^{r=i} B(r) \cdot z_{k-1}(r)$$

while $y_k(i+1)$ is given by

$$y_k(i+1) = \sum_{j=0}^{j=k} z_j(i+1).$$

Our problem is to show that the series

$$z_0(i+1) + z_1(i+1) + \cdots$$

converges to a sum $y(i+1)$ which is the solution of system (4.2). Let $u(r)$ be the sum of the quantities $| B_{pj}(r) |$, α and $\varphi^*(r)$ be the respective sums of the quantities $| \alpha_j |$ and $| \varphi_j(r) |$, and let the constant M be defined by

$$M = m \left[\alpha + \sum_{r=0}^{r=n} \varphi^*(r)\right].$$

The foregoing series is made up of m component series and when we say that it converges we mean that each of its components converges. We have

$$| z_{0j}(i+1) | \leqq M$$

$$| z_{1j}(i+1) | \leqq (mM/1!) \sum_{r=0}^{r=i} u(r)$$

and in general

$$(4.4) \qquad | z_{kj}(i+1) | \leqq (m^k M/k!) [\sum_{r=0}^{r=i} \boldsymbol{u}(r)]^k$$

where z_{kj} is any one of the m components of z_k. In order to prove that (4.4) is valid for all positive integral values of k, we note that it is true for $k=0$ and $k=1$. If there are indices for which it is not true, then there is a first such index and we will call that one k. Now

$$| z_{kj}(i+1) | = | \sum_{r=0}^{r=i} \sum_{p=1}^{p=m} B_{jp}(r) z_{k-1,\,p}(r) | \leqq \sum_{r=0}^{r=i} \boldsymbol{u}(r) \sum_{p=1}^{p=m} | z_{k-1,\,p}(r) |$$

and making use of the fact that (4.4) is valid for $k-1$ and $z_k(0)=0$ for $k \geqq 1$, we get

$$| z_{kj}(i+1) | \leqq [m^k M/(k-1)!] \sum_{r=1}^{r=i} \boldsymbol{u}(r) \; [\sum_{s=0}^{s=r-1} \boldsymbol{u}(s)]^{k-1}$$

Let

$$\boldsymbol{v}(r-1) = \sum_{s=0}^{s=r-1} \boldsymbol{u}(s),$$

then $\Delta \boldsymbol{v}(r-1) = \boldsymbol{u}(r)$ and

$$\sum_{r=1}^{r=i} \boldsymbol{u}(r) \; [\sum_{s=0}^{s=r-1} \boldsymbol{u}(s)]^{k-1} = \sum_{r=1}^{r=i} [\boldsymbol{v}(r-1)]^{k-1} \Delta \boldsymbol{v}(r-1).$$

Since

$$\begin{aligned} \Delta[\boldsymbol{v}(r-1)]^k &= [\boldsymbol{v}(r-1) + \Delta \boldsymbol{v}(r-1)]^k - [\boldsymbol{v}(r-1)]^k \\ &= k[\boldsymbol{v}(r-1)]^{k-1} \boldsymbol{v}(r-1) + k(k-1)/2!\, [\boldsymbol{v}(r-1)]^{k-2} [\Delta \boldsymbol{v}(r-1)]^2 \\ &\qquad + \cdots + [\Delta \boldsymbol{v}(r-1)]^k \end{aligned}$$

and both $\boldsymbol{v}(r-1)$ and $\Delta \boldsymbol{v}(r-1)$ are positive or zero, we have

$$[\boldsymbol{v}(r-1)]^{k-1} \Delta \boldsymbol{v}(r-1) \leqq \Delta \, [\boldsymbol{v}(r-1)]^k / k$$

and hence

$$\begin{aligned} | z_{kj}(i+1) | &\leqq (m^k M/k!) \sum_{r=1}^{r=i} \Delta[\boldsymbol{v}(r-1)]^k \\ &\leqq (m^k M/k!) [\boldsymbol{v}(i)]^k - [\boldsymbol{v}(0)]^k \\ &\leqq (m^k M/k!) [\sum_{r=0}^{r=i} \boldsymbol{u}(r)]^k. \end{aligned}$$

Therefore (4.4) is valid for k which contradicts our hypothesis that k is an index for which it is not valid and hence (4.4) is valid for all positive integral values of k.

Let C_k be defined by

$$C_k = (m^k M/k!) [\sum_{r=0}^{r=n-1} \boldsymbol{u}(r)]^k.$$

Since $u(r) \geqq 0$, it follows from (4.4) that $| z_{kj}(i+1) | \leqq C_k$ independent of the value of i. Since the series of constants C_k is absolutely convergent, it follows that each of the component series of the series

$$\sum_{k=0}^{k=\infty} z_k(i+1)$$

is uniformly convergent. Let $y(i+1)$ denote the vector whose components are the respective sums of these series.

THEOREM 4.1: *$y(i)$ is the unique solution of system* (4.2).

Proof: The condition $y(0) = \alpha$ is satisfied since $z_0(0) = \alpha$, $z_k(0) = 0$ for $k \geqq 1$. Also

$$\begin{aligned} y(i+1) - y(i) &= \lim_{k\to\infty} [y_k(i+1) - y_k(i)] \\ &= \lim_{k\to\infty} [B(i)\cdot y_{k-1}(i) + \varphi(i)] \\ &= B(i)\cdot y(i) + \varphi(i) \end{aligned}$$

and dividing by $\Delta x(i)$ we get the equation of (4.2). The uniqueness of this solution follows from the fact that the difference between $y(i)$ and any other solution of (4.2) would satisfy the system that is obtained from (4.2) by replacing α and $\beta(i)$ with zeros. However, this system has no solution other than the identically vanishing one as is easily seen when we calculate $y(1)$, $y(2), \cdots, y(n)$ successively by means of the equations (4.3).

Let us now define the coefficients of system (4.2) as follows:

$$A_{rj}(i) = \frac{\int_{x_{in}}^{x_{i+1,n}} A_{rj}(t)\,dt}{\Delta x(i)}, \qquad \beta_j(i) = \frac{\int_{x_{in}}^{x_{i+1,n}} \beta_j(t)\,dt}{\Delta x(i)}$$

where $A_{rj}(t)$ and $\beta_j(t)$ are coefficients of the differential system (4.1). Let n become infinite in such a way that the subdivision of the interval X is *fine.*

Definition: By a solution of the differential equation (4.1) we mean a set of absolutely continuous functions that satisfy this equation almost everywhere on X.

THEOREM 4.2: *If x is a fixed point of X and if the index p varies in such a way that at every stage of the limiting process $x_{pn} \leqq x \leqq x_{p+1,n}$, then $\lim_{n\to\infty} y(p) = y(x)$, where $y(x)$ is the unique solution of system* (4.1).

Lemma: If $z_{0j}(x) = \alpha_j + \int_a^x \beta_j(t)\,dt$

$$z_{kj}(x) = \int_a^x \sum_{s=1}^{s=m} A_{js}(t) z_{k-1,\,s}(t)\,dt, \text{ for } k = 1, 2, \cdots,$$

then $\lim_{n\to\infty} z_{kj}(p+1) = z_{kj}(x)$ *for every* x *on* X *and for* $k = 0, 1, 2, \cdots$.

Proof of Lemma: Since $z_{kj}(p+1)$ consists of only m terms, where m is finite and independent of n, it is sufficient to show that the typical term of $z_{kj}(p+1)$ has the corresponding term of $z_{kj}(x)$ for its limit. We wish to show that

$$\begin{aligned} &\lim_{n\to\infty} [\alpha_j + \sum_{r=0}^{r=p} \varphi_j(r)] = \alpha_j + \int_a^x \beta_j(t)\,dt \\ (4.\,5)\qquad &\\ &\lim_{n\to\infty} [\sum_{r=0}^{r=p} B_{js}(r)\, z_{k-1,s}\,(r) = \int_a^x A_{js}(t) z_{k-1,s}(t)\,dt, \text{ for } k \geqq 1 \end{aligned}$$

(4. 5) is valid for all values of k and for every x on X. It is valid for $k = 0$, since

$$\sum_{r=0}^{r=p} \varphi_j(r) = \int_a^{x_{p+1,\,n}} \beta_j(t)\,dt = \int_a^x \beta_j(t)\,dt + \int_x^{x_{p+1,\,n}} \beta_j(t)\,dt$$

and the limit of the last integral is zero. If there are values of k for which (4. 5) is not valid for every x on X, then let r be the first such value and let p be associated with a value of x for which (4. 5) is not valid. Consider the difference

$$(4.\,6)\qquad \sum_{i=0}^{i=p} B_{js}(i) z_{r-1,s}(i) - \int_a^x A_{js}(t)\, z_{r-1,s}(t)\,dt$$

which we will write in the form $I_1 + I_2 + I_3$, where

$$I_1 = \int_a^x A_{js}(t)\,[z^{(p)}{}_{r-1,s}(t) - z_{r-1,s}(t)]\,dt$$

$$I_2 = \sum_{i=0}^{i=p} B_{js}(i) z_{r-1,s}(i) - \int_a^{x_{p+1,\,n}} A_{js}(t) z^{(p)}{}_{r-1,s}(t)\,dt$$

$$I_3 = \int_x^{x_{p+1,\,n}} A_{js}(t)\, z^{(p)}{}_{r-1,s}(t)\,dt$$

and

$$z^{(p)}{}_{r-1,s}(t) = z_{r-1,s}(i) \text{ on } x_{in} < t < x_{i+1,n}.$$

Let $f_n(t) = A_{js}(t)\,[z^{(p)}{}_{r-1,s}(t) - z_{r-1,s}(t)]$. It follows from the fact that (4. 5) is valid for $k = r - 1$ that $\lim_{n\to\infty} f_n(t) = 0$ for every t on X. Furthermore, of $g(x)$ and $\beta^*(x)$ denote the respective sums of $|A_{jk}(x)|$ and $|\beta_j(x)|$, then these quantities are summable and it is easy to see that

$$u(i) = \int_{x_{in}}^{x_{i+1,n}} g(t)\,dt \text{ and } M = m\,[\alpha + \int_a^b \beta^*(t)\,dt.$$

We have then

$$|z_{kj}(i+1)| \leqq (m^k M/k!)[\int_a^b g(t)dt]^k = C_k$$

independent of n and i. We also note that $z_{kj}(x)$ is continuous on X and hence it is bounded. Let G be the sum of this bound and C_k. The sequence of functions $f_n(t)$ is less than the summable function $g(t)G$ for all n and for every t on X. We may now apply Lebesgue's theorem[*] concerning the limit of the integral of a sequence of functions to get $\lim_{n\to\infty} I_1 = 0$.

Now consider I_2. We see immediately that

$$I_2 = \sum_{i=0}^{i=p} \int_{x_{in}}^{x_{i+1,n}} A_{js}(t)[z_{r-1,s}(i) - z^{(p)}{}_{r-1,s}(t)]dt \equiv 0$$

for all values of n since $z^{(p)}{}_{r-1,s}(t) = z_{r-1,s}(i)$ on $x_{in} < t < x_{i+1,n}$. Also

$$I_3 = z_{r-1,s}(p) \int_x^{x_{p+1,n}} A_{js}(t)dt \leqq G \int_x^{x_{p+1,n}} g(t)dt$$

and since $\int_x^{x_{p+1,n}} g(t)dt$ is a continuous function of $x_{p+1,n}$ and $\lim_{n\to\infty} x_{p+1,n} = x$, we have $\lim_{n\to\infty} I_3 = 0$.

We have shown that the limit of the expression (4.6) is zero and hence that (4.5) is valid for $k = r$ and for every x on X. This contradicts our hypothesis that (4.5) was not valid for $k = r$ and some x on X and hence it follows that (4.5) is valid for all values of x on X and for $k = 0, 1, 2, \cdots$.

Proof of Theorem 4.2: Theorem 4.2 follows from an application of the above lemma and a theorem due to Tannery.[†] We have

$$z_{0j}(p) + z_{1j}(p) + \cdots + z_{kj}(p)$$

such that $|z_{kj}(p)| \leqq C_k$ independent of n and the series of constants C_k is convergent. Also, from the above lemma we have that $\lim_{n\to\infty} z_{kj}(p) = z_{kj}(x)$. Hence by Tannery's theorem

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[*] *Leçons sur l'intégration etc.*, Paris, 1904, p. 114.

[†] *Introduction à l'étude des fonctions*, Paris, 1866, Sec. 183.

$$\lim_{n\to\infty} y_j(p) = lim_{n\to\infty} \sum_{k=0}^{k=\infty} z_{kj}(p) = \lim_{n\to\infty} \sum_{k=0}^{k=\infty} z_{kj}(x) = y_j(x).$$

By taking the limit of system (4. 3) we get

$$y(x) = \int_a^x [A(t) \cdot y(t) + \beta(t)]\, dt + \alpha$$

for every x on X, or

$$y'(x) = A(x) \cdot y(x) + \beta(x)$$
$$y(a) = \alpha$$

almost everywhere on X. $y(x)$ is a solution of (4. 1). The uniqueness of this solution can be established directly by a method that is the same as the one used to establish this property when the coefficients are continuous.

2°. *The Coefficients Contain a Parameter.* Consider system (4. 1) where the coefficients are continuous functions of λ on L for each fixed x on X, summable functions of x on X for each fixed λ on L, and bounded numerically for all x and λ on XL by a function $M(x)$ that is summable on X. Let α_j be a continuous function of λ on L. Let the coefficients of the difference system (4. 2) be continuous in λ on L for each x_{1n} of E_n, and bounded numerically for all λ and x_{in} on LE_n by a function $u(i)$ such that $\lim_{n\to\infty} u(i)$ exists almost everywhere and is summable on X.

We obtain the solution of the difference system by exactly the same method that was used in the preceding case. Each of the functions $z_{kj}(i,\lambda)$ is continuous in λ on L and since all of the series converge uniformly with respect to λ and i, it follows that the limit functions $y_j(i,\lambda)$ are continuous in λ on L.

If the coefficients of the difference system are defined as they were in the preceding case, we see that all of the above hypotheses are met. These coefficients are continuous* in λ on L and bounded numerically by the function

$$\frac{\int_{x_{in}}^{x_{i+1,n}} M(t)\, dt}{\Delta x(i)} .$$

This function has the summable function $M(x)$ as its limit function almost everywhere on X. All of the proofs of the preceding case apply here and we get

$$\lim_{n\to\infty} y(p,\lambda) = y(x,\lambda)$$

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* See Carathéodory, *Vorlesungen über Reele Funktionen*, Teubner, Berlin, 1918, Chapter XI.

where $y(x, \lambda)$ is continuous in the two variables x and λ and is the unique solution of the differential system.

3°. *A Non-Linear System.* Consider the system

$$\begin{aligned} dy/dx &= f(x, y) \\ y(a) &= \alpha \end{aligned} \tag{4.7}$$

where $f(x, y)$ is the m-dimensional vector whose jth component is

$$f_j[x, y_1(x), y_2(x), \cdots, y_m(x)].$$

$f(x, y)$ satisfies the conditions given by Carathéodory* in his general existence theorem. The corresponding difference system is

$$\begin{aligned} \frac{\Delta y(i)}{\Delta x(i)} &= f[i, y(i)] \\ y(0) &= \alpha. \end{aligned} \tag{4.8}$$

If we define

$$f_j[i, y(i)] = \frac{\int_{x_{in}}^{x_{i+1,n}} f_j[t, y(i)]\, dt}{\Delta x(i)}$$

and apply the same method of proof that was used in treating the linear system, we get that the limit of the system (4. 8) and its solution is the system (4. 7) and its solution respectively.

4°. *Application of Theorem* 4. 2. Theorem 4. 2 enables us to carry over the results of parts 2 and 3 to the limiting differential systems. It follows from the absolute continuity of indefinite integrals that the coefficients of the difference systems as defined in this section, have the summable coefficients of the differential systems as their limit functions almost everywhere on X. Our conditions on $\boldsymbol{B}(i)$ go over into similar conditions on $B(x)$. Theorem 2. 1 goes over into a corresponding theorem for the differential system when we note that theorem 4. 1 carries $\boldsymbol{W}(i)$ over into the wronskian of the differential system. If we define $\phi(x)$ to be equal to $\alpha(x) \cdot y(x)$, where α_1 and α_2 are absolutely continuous functions on X, and if we let $\boldsymbol{\alpha}_1(i)$ and $\boldsymbol{\alpha}_2(i)$ respectively be the values of $\alpha_1(x)$ and $\alpha_2(x)$ at x_{in}, then

$$\begin{aligned} \lim_{n\to\infty} \frac{\{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\}}{\Delta x(i)} = \{\alpha_1(x)\alpha_2(x)\} &= \alpha_2^2(x)A_{21}(x) + \alpha_1(x)\alpha_2(x)[A_{11}(x) - A_{22}(x)] \\ &\quad + \alpha_2(x)\alpha_1'(x) - \alpha_1(x)\alpha_2'(x) - \alpha_1^2(x)A_{12}(x) \end{aligned}$$

---

[*] *Loc. cit.*, p. 672 ff.

almost everywhere on X. If we specialize our differential system to the self-adjoint form by letting $A_{11}(x) = A_{22}(x) = 0$, $A_{21}(x) = G(x)$, $A_{12}(x) = 1/K(x)$, we see immediately that $\{\alpha_1(x)\alpha_2(x)\}$ is Bôcher's* $\{\phi_1\phi_2\}$ relation. Theorem 2.2 goes over into a corresponding theorem for the limiting system when we require that $\{\alpha_1(x)\alpha_2(x)\}$ be different from zero. We can replace the proof for a finite number of zeros by one that does not depend on n. Theorems 2.3 and 2.4 go over for the differential system without introducing any difficulties. $\mathfrak{R}(i)/[\Delta x(i)]^2$ and $\mathfrak{S}(i)/[\Delta x(i)]$ go over into similar expressions. We note that for the self-adjoint system, the conditions that the zeros of $y_1(x)$ and $y_2(x)$ separate each other become that $K(x)$ be positive and $G(x)$ be negative. The condition that the solution of the self-adjoint system be non-oscillatory is that K and G both be positive.

5. *The Green's Function and the Adjoint System.* In this section we deal with mth order systems. We write these systems in vector and tensor form for the sake of compactness.

Consider the differential system

$$
\begin{aligned}
(5.1) \qquad & (dy/dx) - A(x)\cdot y = \beta(x) \\
& \phi[y(a)] - \phi[y(b)] = 0
\end{aligned}
$$

where $\phi[y(x)] = R(x)\cdot y(x)$, $R(x)$ is a tensor of absolutely continuous functions, A and β are summable functions of x on X. Let (5.1′) denote the homogeneous system obtained from (5.1) by making $\beta(x) \equiv 0$. Consider the corresponding difference system

$$
\begin{aligned}
(5.2) \qquad & \frac{\Delta \mathfrak{y}(i)}{\Delta x(i)} - \mathfrak{A}(i)\cdot \mathfrak{y}(i) = \mathfrak{B}(i) \\
& \varphi[\mathfrak{y}(0)] - \varphi[\mathfrak{y}(n)] = 0
\end{aligned}
$$

where $\varphi[\mathfrak{y}(i)] = \mathfrak{R}(i)\cdot\mathfrak{y}(i)$. Let (5.2′) be the system obtained from (5.2) by making $\mathfrak{B}(i) \equiv 0$.

If the system (5.2′) has no solution other than the identically vanishing one, it is said to be incompatible. We suppose that (5.2′) is incompatible and proceed to define its Green's tensor † in the following manner: Let $\mathfrak{Y}(i)$, be a tensor of solutions of (5.2′) such that the (mth ordered) determinant of $\mathfrak{Y}(i)$, $|\mathfrak{Y}(i)|$, is different from zero for all values of i. Let x_{sn} be a point of E_n. We seek to determine a tensor $\mathfrak{G}(i,s)$ in such a way that its vector components $\mathfrak{G}_j(i,s)$ will have the following properties:

---

* *Leçons sur les Méthodes de Sturm*, Paris, 1917, p. 45.

† For a discussion of the Green's tensor for differential systems, see my paper "The Green's Functions for Systems of Differential Equations," *Annals of Mathematics*, Vol. 28 (1927), No. 3, pp. 291-300.

1. $G_j(i,s)$ as a function of i satisfies the boundary conditions of (5.2′).

2. $G_j(i,s)$ satisfies the equation of (5.2′) for every i except for $i = s - 1$ and it satisfies this equation when $G_{jj}(s,s)$ is decreased by unity.[*]

Let

$$G(i,s) = Y(i) \cdot C(s) \qquad \text{for } i < s$$
$$G(i,s) = Y(i) \cdot D(s) \qquad \text{for } i \geqq s$$

and try to determine tensors C and D in such a way that G has the above properties. In order to have property 2, it must satisfy

$$Y(s) \cdot [D(s) - C(s)] = E,$$

where E is the unit tensor and is defined so that $E_{ij} = 0$ for $i \neq j$, $E_{jj} = 1$. Since $| Y(i) | \neq 0$, we can solve this equation for the unique value of $D - C$. This gives

$$D(s) - C(s) = Y^{-1}(s) \cdot E = Y^{-1}(s)$$

where Y^{-1} is the inverse tensor of Y and is defined by $Y \cdot Y^{-1} = E$. Substitute G_j into the boundary conditions and let Q be the tensor such that $Q_{kj} = \varphi_k[Y_j(0)] - \varphi_k[Y_j(n)]$. Let S be the tensor whose general element is $S_{kj} = \varphi_k[Y_j(n)]$.

$$Q \cdot C = S \cdot Y^{-1}(s)$$

and since system (5.2′) is incompatible, we may solve for C and substitute its value into the above equation for $D - C$ to get

$$C(s) = Q^{-1} \cdot S \cdot Y^{-1}(s) \tag{5.3}$$
$$D(s) = Q^{-1} \cdot S \cdot Y^{-1}(s) + Y^{-1}(s).$$

Since D and C are uniquely determined by (5.3) it follows that the Green's tensor is uniquely determined by the properties 1 and 2.

THEOREM 5.1: *The unique solution of system* (5.2) *is* $y(i)$, *where*

$$y_j(i) = \sum_{s=1}^{s=n} G_j(i,s) \cdot \beta(s-1)\Delta x(s-1).$$

Proof: It is evident that $y(i)$ satisfies the boundary conditions since $G_j(i,s)$ satisfies these conditions. Substitute $y(i)$ into the jth equation of (5.2)

$$\sum_{s=1}^{s=n} \frac{\Delta G_j(i,s)}{\Delta x(i)} - \sum_{r=1}^{r=m} A_{jr}(i) G_r(i,s)] \cdot \beta(s-1)\Delta x(s-1) = \beta_j(i).$$

Using property 2 of G_j we see that the left hand side reduces to $\beta_j(i)$ and hence the equation is satisfied. The uniqueness of this solution follows when we note that the difference between any two solutions of (5.2) would satisfy (5.2′) which is assumed incompatible.

---

[*] This is equivalent to requiring that $G(s,s) - G(s,s) = E$.

THEOREM 5.2: *If the coefficients of system* (5.2) *are defined as they were in part* 4 *and if the number of points in* E_n *is allowed to increase indefinitely in such a way that the subdivision of* X *is fine, then the limit of* $\mathbf{G}(i, s)$ *is* $G(x, t)$, *where* $G(x, t)$ *is the Green's tensor for the system* (5.1′), *provided the systems* (5.2′) *are incompatible for all values of* n.

Proof: It follows from theorem 4.2 that $\mathbf{Y}(i)$ goes over in the limit to $Y(x)$, where $Y(x)$ is a tensor of solution of the equation (5.1′) and $| Y(x) \neq 0$ on X. Similarly $\mathbf{Y}^{-1}(i)$ goes over into $Y^{-1}(x)$. Since S and Q are made up of fixed finite linear combinations of the values of the elements of $\mathbf{Y}(i)$ at a and b, it follows that Q, Q^{-1} and S go over into corresponding tensors for the differential system. An examination of (5.3) shows that $C(s)$ and $D(s)$ go over into $C(t)$ and $D(t)$ respectively, and hence

$$\lim_{n\to\infty} \mathbf{G}(i, s) = G(x, t).$$

Let

$$L[\mathbf{y}(i)] = \frac{\Delta \mathbf{y}(i)}{\Delta x(i)} \mathbf{A}(i) \cdot \mathbf{y}(i)$$

and determine a function $\boldsymbol{v}(i)$ in such a way that

$$\boldsymbol{v}(i) \cdot L[\mathbf{y}(i)] = \frac{\Delta[\mathbf{y}(i) \cdot \boldsymbol{v}(i-1)]}{\Delta x(i)} .$$

We see immediately that $\boldsymbol{v}(i)$ must satisfy the equation*

$$M[\boldsymbol{v}(i)] = \frac{\Delta \boldsymbol{v}(i+1)}{\Delta x(i)} + \boldsymbol{v}(i) \cdot \mathbf{A}(i) = 0. \tag{5.4}$$

Equation (5.4) is defined to be the adjoint of the equation $L[\mathbf{y}(i)] = 0$.

Lagrange's identity is

$$\boldsymbol{v}(i) \cdot L[\mathbf{y}(i)] + \mathbf{y}(i) \cdot M[\boldsymbol{v}(i)] = \frac{\Delta \mathbf{y}(i) \cdot \boldsymbol{v}(i-1)]}{\Delta x(i)}$$

while Green's formula is

$$\sum_{i=1}^{i=n-1} \{\boldsymbol{v}(i) \cdot L[\mathbf{y}(i)] + \mathbf{y}(i) \cdot M[\boldsymbol{v}(i)]\} \Delta x(i) = \mathbf{y}(i) \cdot \boldsymbol{v}(i-1) \Big|_{i=1}^{i=n-1} .$$

A repetition of the argument used in proving theorem 4.2 carries the adjoint system (5.4) into the corresponding limiting differential system and with it the Lagrange identity and Green's formula are carried over.

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* L and M are m-dimensional vectors.

On the Motion of a Rigid Body about a Fixed Point in Space of Constant Curvature.

BY JAMES PIERPONT.

1. This problem, in a more general form, has been treated by Clifford and Heath[*] for elliptic space, and in two papers by de Francesco.[†] The first two papers are so general that it is difficult to have much of an idea what actually takes place. Clifford uses his biquaternions, which must first be learned to understand his paper. Heath's treatment rests on six homogeneous line co-ordinates tied by two relations. De Francesco employs sometimes four and sometimes six co-ordinates. The problem is, however, carried out in much greater detail.

In the present paper we restrict the problem to motion about a fixed point; moreover to get formulae we can handle we have supposed the body moves in a constant field of force. This latter assumption is the one made in classical mechanics. The lines of force due to gravity are of course neither parallel nor constant, yet the dimensions of the gyroscope or top are so small relative to the earth that no sensible error is made. Under these assumptions the problem admits a solution of remarkable simplicity, if we make use of a few ideas I have developed in earlier papers.[‡]

2. We assume first a Euclidean space, whose points are defined by rectangular co-ordinates x, y, z for which the metric is given by

$$d\sigma^2 = dx^2 + dy^2 + dz^2.$$

Let us set

$$r^2 = x^2 + y^2 + z^2, \quad \lambda = 4R^2 + r^2, \quad \mu = 4R^2 - r^2.$$

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[*] W. K. Clifford, "Motion of a Solid in Elliptic Space," *London Mathematical Society*, 1874. First published from posthumous notes in his *Mathematical Papers*, p. 374 (1882); R. S. Heath, "On the Dynamics of a Rigid Body in Elliptic Space," *Philosophical Transactions of the Royal Society of London*, Vol. 175 (1885), p. 281.

[†] D. de Francesco, "Alcuni Problemi di Meccanica in uno Spazio a tre Dimensioni di Curvatura constante," *Atti della Reale Academia di Scienze di Napoli* (2), Vol. 10 (1900), No. 4 and 9.

[‡] "The Geometry of Riemann and Einstein," *American Mathematical Monthly*, Vol. 30 (1923), p. 26; "Non-Euclidean Geometry from Non-Projective Standpoint," *Transactions of the Sixth International Congress* (August, 1924), Vol. 1, p. 117.

Using the same co-ordinates, we may define an elliptic space by

$$(1) \qquad ds = \frac{4R^2 d\sigma}{\lambda}$$

and a hyperbolic one by

$$ds = \frac{4R^2 d\sigma}{\mu},$$

where $R > 0$ is called the space constant.

We shall restrict ourselves to elliptic space for the sake of clearness; it is not difficult to make the necessary modifications for hyperbolic space. For brevity let us read *e*- and *E*- as Euclidean and elliptic, respectively.

The points of *E*-space lie within the *e*-sphere $\mu = 0$. Two diametral *e*-points on $\mu = 0$ are regarded as one *E*-point. The length of a segment of an *E*-curve is given by $\int ds$ and the curve is an *E*-straight, if the first variation of this integral is zero. *E*-straights appear to an *e*-observer as circles cutting $\mu = 0$ in diametral points. *E*-straights are closed curves each having the length πR. Two points A, B, on an *E*-straight determine two segments. The length of the shorter one is the length of the segment $A.B$. We write $\delta = \text{dist}\,(A, B)$. *E*-straights thru the origin 0 are also *e*-straights. An *E*-plane is an *e*-sphere cutting $\mu = 0$ along a great circle. The angle between two *E*-straights, planes, etc., is the same in *E*-measure as in *e*-measure, *i. e.*, an *e*-observer and an *E*-observer measuring an angle would get the same result.

If we introduce four new variables

$$(2) \qquad z_1 = 4R^2x/\lambda, \quad z_2 = 4R^2y/\lambda, \quad z_3 = 4R^2z/\lambda, \quad z_4 = R\mu/\lambda,$$

satisfying the identical relation $z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2$, we find that any linear relation $u_1z_1 + u_2z_2 + u_3z_3 + u_4z_4 = 0$ defines an *E*-plane. In these variables $ds^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2$. The four planes $z_1 = 0, \cdots, z_4 = 0$ define a tetrahedron τ. We find $z_i = R \sin\,(\delta_i/R)$, where δ_i is the length of the perpendicular from z on the face $z_1 = 0$, $i = 1, 2, 3, 4$. The distance δ between two points a, b or the *E*-length of the segment $a \cdot b$ is given by

$$R^2 \cos\,(\delta/R) = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4.$$

If $\delta = \pi(R/2)$ we say the points a and b are associated, $a \sim b$.

In the z co-ordinates the equations of an *E*-straight are

$$z_i = a_i \cos\,(s/R) + b_i \sin\,(s/R), \qquad (i = 1, \cdots, 4).$$

Here a, b are two points on the straight and $a \sim b$ and s is dist (a, z), measured in a given sense.

Let us subject the points z of E-space to a linear transformation.

$$z'_i = a_{i1}z_1 + \cdots + a_{i4}z_4, \qquad (i = 1, \cdots, 4). \tag{3}$$

whose determinant $a = |a_{ij}| = +1$. If the scalar product of two rows are orthogonal i. e. if

$$\begin{aligned} a_{i1}a_{j1} + \cdots + a_{i4}a_{j4} &= 0, \quad \text{if } i \neq j \\ &= 1, \quad\ \ i = j, \end{aligned} \tag{4}$$

we find the distance between two points is the same before and after the transformation. Such a transformation is called an E-displacement. Let

$$\begin{aligned} x' &= \alpha_1 x + \beta_1 y + \gamma_1 z \\ y' &= \alpha_2 x + \beta_2 y + \gamma_2 z \\ z' &= \alpha_3 x + \beta_3 y + \gamma_3 z \end{aligned} \tag{5}$$

define an e-rotation about 0. If we multiply these equations on the left by $4R^2/\lambda'$

$$\lambda' = 4R^2 + x'^2 + y'^2 + z'^2$$

and on the right by $4R^2/\lambda$ observing that $\lambda = \lambda'$ we have

$$\begin{aligned} z'_1 &= \alpha_1 z_1 + \beta_1 z_2 + \gamma_1 z_3 + 0z_4 \\ z'_2 &= \alpha_2 z_1 + \beta_2 z_2 + \gamma_2 z_3 + 0z_4 \\ z' &= \alpha_3 z_1 + \beta_3 z_2 + \gamma_3 z_3 + 0z_4 \\ z'_4 &= 0z_1 + 0z_2 + 0z_3 + z_4. \end{aligned} \tag{6}$$

Thus to the e-rotation corresponds a linear transformation in E-space. Its determinant is $+1$, its coefficients satisfy the orthogonal relation (4).

It defines an E-displacement. All this is more easily seen by noting that (6) leaves $d\sigma$ and λ unaltered and hence ds. Since e-straights and planes thru 0 are E-straights and planes, and since angles have the same value in both measures we have a complete picture of an E-rotation about 0. This does not hold for any other point.

Let us apply this to small e-rotations about 0. We see these as compounded precisely as in e-geometry.

Thus three small rotations ω_x, ω_y, ω_z about the x, y, z axes produce the displacement.

$$\begin{aligned} \delta z_1 &= -z_2\omega_z + z_3\omega_y \\ \delta z_2 &= z_1\omega_z - z_3\omega_x \\ \delta z_3 &= -z_1\omega_y + z_2\omega_x \\ \delta z_4 &= 0. \end{aligned} \tag{7}$$

Consider the point a whose co-ordinates are

$$a_1 = \frac{4R^2\omega_x}{\lambda}, \quad a_2 = \frac{4R^2\omega_y}{\lambda}, \quad a_3 = \frac{4R^2\omega_z}{\lambda}, \quad a_4 = (R\mu/\lambda),$$

where

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2, \quad \lambda = 4R^2 + \omega^2, \quad \mu = 4R^2 - \omega^2.$$

If l is the E-straight thru 0 and a, the e-displacement (7) is an E-rotation about l thru the angle ω.

Let $\xi\,\eta\,\zeta$ be an orthogonal e-set of axes thru 0 fixed in the body and defined by the customary scheme (8)

$$(8) \qquad \begin{array}{c|ccc} & x & y & z \\ \hline \xi & \alpha_1 & \beta_1 & \gamma_1 \\ \eta & \alpha_2 & \beta_2 & \gamma_2 \\ \zeta & \alpha_3 & \beta_3 & \gamma_3 \end{array}$$

having the determinant $+1$. The $\xi\,\eta\,\zeta$ axes define a new tetrahedron τ'. If we denote the E-co-ordinates of a point referred to τ and τ' by $(z_1, \cdots z_4)$ and $(\zeta_1, \cdots \zeta_4)$ respectively, we have

$$\begin{aligned} z_1 &= \alpha_1\zeta_1 + \alpha_2\zeta_2 + \alpha_3\zeta_3 \\ z_2 &= \beta_1\zeta_1 + \beta_2\zeta_2 + \beta_3\zeta_3 \\ z_3 &= \gamma_1\zeta_1 + \gamma_2\zeta_2 + \gamma_3\zeta_3 \\ z_4 &= \zeta_4. \end{aligned}$$

Let us give the body the displacement (7). Then

$$(9) \qquad \delta z_1 = \zeta_1\delta\alpha_1 + \zeta_2\delta\alpha_2 + \zeta_3\delta\alpha_3, \quad \text{etc.}$$

By *purely analytical transformations* we find as in e-space

$$(10) \qquad \delta\alpha_1 = \gamma_1\omega_y - \beta_1\omega_z, \quad \delta\alpha_2 = \gamma_2\omega_y - \beta_2\omega_z, \quad \delta\alpha_3 = \gamma_3\omega_y - \beta_3\omega_z, \quad \text{etc.}$$

If we set

$$(11) \qquad \begin{aligned} p = \omega_\xi &= \alpha_1\omega_x + \beta_1\omega_y + \gamma_1\omega_z \\ q = \omega_\eta &= \alpha_2\omega_x + \beta_2\omega_y + \gamma_2\omega_z \\ r = \omega_\zeta &= \alpha_3\omega_x + \beta_3\omega_y + \gamma_3\omega_z \end{aligned}$$

we find that ω_ξ, ω_η, ω_ζ are small E-rotations about the ξ, η, ζ axes which together produce the displacement (7).

3. The E-velocity v of a point is defined by

$$v^2 = (ds^2/dt^2) = \dot{z}_1^2 + \dot{z}_2^2 + \dot{z}_3^2 + \dot{z}_4^2.$$

The kinetic energy T of a body we define by

$$2T = \sum v^2 dm, \qquad dm = \text{mass of an element.}$$

Now by (9)

$$\dot{z}_1 = \dot{\alpha}_1 \zeta_1 + \dot{\alpha}_2 \zeta_2 + \dot{\alpha}_3 \zeta_3, \quad \text{etc.} \qquad \dot{z}_4 = 0.$$

Hence squaring and using (10) (11) we get,

$$2T = \sum \{ p^2(\zeta_2^2 + \zeta_3^2) + q^2(\zeta_1^2 + \zeta_3^2) + r^2(\zeta_1^2 + \zeta_2^2) \\ - 2pq\zeta_1\zeta_2 - 2pr\zeta_1\zeta_3 - 2qr\zeta_2\zeta_3 \} \, dm.$$

Set $A = \sum (\zeta_2^2 + \zeta_3^2) dm$, $B = \sum (\zeta_1^2 + \zeta_3^2) dm$, $C = \sum (\zeta_1^2 + \zeta_2^2) dm$,
$D_{12} = \sum \zeta_1 \zeta_2 dm$, $D_{13} = \sum \zeta_1 \zeta_3 dm$, $D_{23} = \sum \zeta_2 \zeta_3 dm$.

Then

$$(12) \qquad 2T = Ap^2 + Bq^2 + Cr^2 - 2D_{12}pq - 2D_{13}pr - 2D_{23}qr.$$

By a proper choice of the $\xi\eta\zeta$ axes we may make $D_{ij} = 0$ and for such axes

$$(13) \qquad 2T = Ap^2 + Bq^2 + Cr^2.$$

4. With Killing[*] we will suppose that the components of a force acting at a point $(z_1, \cdot \cdot \cdot, z_4)$ are

$$F_i = F \cos \alpha_i, \qquad (i = , \cdot \cdot \cdot , 4),$$

where α_i is the angle made by a plane thru z and perpendicular to F, with the co-ordinate planes $z_i = 0$. We can give F_i a form which is at times more convenient. For if y is the point on the line F associated with x, we find $F_i = y_i(F/R)$,
and hence

$$F^2 = F_1^2 + F_2^2 + F_3^2 + F_4^2.$$

The work done by a small displacement of x to $x' = x + \delta x$ Killing defines by

$$dW = \sum F_i \delta x_i, \qquad (i = 1, \cdot \cdot \cdot, 4).$$

Hence

$$dW = (F/R) \sum y_i dx_i = (F/R) \sum y_i (x_i + dx_i), \text{ as } \sum x_i y_i = 0, \\ = (F/R) \sum x_i y_i' = R^2 \cos (r/R), \qquad r = \text{dist } (y, x').$$

Now $\cos (r/R) = \cos (\pi R/2) \cos (ds/R) + \sin (\pi R/2) \sin (ds/R) \cdot \cos \theta$
where

$$ds = \text{dist } (x, x'), \qquad \theta = \text{Ang } (F, ds).$$

Thus

$$(14) \qquad dW = F \cos \theta \cdot ds,$$

as in e-mechanics.

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[*] W. Killing, "Die Mechanik in den Nicht-Euklidischen Raumformen," *Crelle's Journal*, Vol. 98 (1885), p. 1.

We now find the work done by the force F for a small rotation about 0. Let the line of the force F acting on an element of mass dm at $x = (x_1 \cdots x_4)$ cut $z_4 = 0$ at the point $a = (a_1, \cdots a_4)$.

Any other point x' on this straight has co-ordinates $x'_i = a_i l + x_i m$, $i=1 \cdots 4$, where

$$l = \frac{\sin [(x'x)/R]}{\sin [(ax)/R]}, \qquad m = \frac{\sin [(ax')/R]}{\sin [(ax)/R]}.$$

Suppose $(ax) = \text{dist}\ (a, x) = s$, $(xx') = -ds$, then $(ax') = s - ds$. Then

$$l = -\frac{ds/R}{\sin (s/R)}, \qquad m = 1 - \frac{\cos (s/R)}{\sin (s/R)} \cdot (ds/R)$$

neglecting small quantities of second order. Thus

$$dx_k = -[a_k + x_k \cos (s/R)] \frac{ds}{R \sin (s/R)}, \qquad k = 1, 2, 3$$

$$dx_4 = -x_4 \frac{\cos (s/R)}{\sin (s/R)} \cdot (ds/R).$$

These are the changes in the co-ordinates along the line of force F. On the other hand, a small rotation ω_x about the x axis causes x to describe an arc δs; thereby x passes to $x + \delta x$ where

$$\delta x_1 = 0, \qquad \delta x_2 = -x_3\omega_x, \qquad \delta x_3 = x_2\omega_x, \qquad \delta x_4 = 0.$$

The angle θ between the arcs ds and δs is given by

$$\cos \theta = \sum (dx_i/ds) \cdot (\delta x_i/\delta s)$$

$$= \frac{a_2x_3 - a_3x_2}{R \sin (s/R)} \cdot (\omega_x/\delta s).$$

This in (14) gives the work done by the displacement ω_x.

Let us now suppose (analogous to e-mechanics) that the lines of force F, all issue from a and that F is sensibly proportional to the mass so that $F = mf$. Then for the whole body

$$dW_x = (f\omega_x/R) \{a_2 \sum \frac{x_3 dm}{\sin (s/R)} - a_3 \sum \frac{x_2 dm}{\sin (s/R)}\}$$

with similar expressions for dW_y, dW_z. Now for any rigid body which a physicist in E-space would have to deal with, s is sensibly constant and in the present case equal $\pi R/2$. On this assumption,

$$dW_x = (f\omega_x/R) \{a_2 \sum x_3 dm - a_3 \sum x_2 dm\}$$

and similar expressions of the two other components.

We note next that $| x_i | < R$ hence if $M = \sum dm$

$$\frac{\sum x_i dm}{RM} < 1, \text{ numerically.}$$

We may therefore set this fraction $= \sin (p_i/R)$. If further we set $g_i = R \sin p_i/R$ we have

$$\sum x_i dm = M g_i, \qquad i = 1, 2, 3; \qquad g_4^2 = 1 - (g_1^2 + g_2^2 + g_3^2).$$

This gives us

$$\begin{aligned} dW_x &= f(M/R)\,(a_2 g_3 - a_3 g_2)\,\omega_x, \\ dW_y &= f(M/R)\,(a_3 g_1 - a_1 g_3)\,\omega_y, \\ dW_z &= f(M/R)\,(a_1 g_2 - a_2 g_1)\,\omega_z. \end{aligned} \tag{15}$$

This shows that the work done by a small rotation about 0 is the same as if the mass M were concentrated at the point g.

As a special case let us take $a = (0, 0, R, 0)$. Then the lines of force F are all perpendicular to the x, y plane which we regard as horizontal and

$$dW = dW_x + dW_y + dW_z = fM(g_2\omega_x - g_1\omega_y). \tag{16}$$

On the other hand ω_x gives g_3 the increment $g_2\omega_x$, ω_y gives g_3 the increment $g_1\omega_y$, while ω_z leaves g unaltered. Thus the rotation $\omega = (\omega_x, \omega_y, \omega_z)$ gives g the increment

$$\delta g_3 = g_1\omega_y + g_2\omega_x.$$

Comparing with (16) we have

$$dW = fM \cdot \delta g_3. \tag{17}$$

5. To get the equations of motion we will use Hamilton's principle[*]

$$\int (\delta T + \delta W)\,dt = 0. \tag{18}$$

Instead, however, of using supernumerary variables as Killing and Heath do, tied by various relations, we will use independent variables which fix the position of the body. If q is one of these variables (18) gives

$$(d/dt)\,(\partial T/\partial \dot{q}) - (\partial T/\partial q) = (\partial W/\partial q).$$

If we take Euler's angles as variables we find as in *e*-mechanics

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[*] W. Killing, *loc. cit.*, p. 13.

$$\begin{aligned} p &= \omega_\xi = \dot\psi \sin\theta \sin\phi + \dot\theta \sin\phi, \\ q &= \omega_\eta = \dot\psi \sin\theta \cos\phi - \dot\theta \sin\phi, \\ r &= \omega_\zeta = \dot\psi \cos\theta + \dot\phi. \end{aligned}$$

These in (12) or (13) give T in terms of θ, ϕ, ψ. To get dW we use (17) and observe that

$$g_3 = \zeta_1 \sin\phi \sin\theta + \zeta_2 \cos\phi \sin\theta + \zeta_3 \cos\theta$$

where $\zeta_1 \cdot \cdot \cdot \zeta_4$ are the co-ordinates of g referred to τ'. We see therefore that to an e-observer the body appears to rotate as if its e-center of mass had the e-co-ordinates

$$4R^2 \sum (\xi dm/\lambda), \quad 4R^2 \sum (\eta dm/\lambda), \quad 4R^2 \sum (\zeta dm/\lambda)$$
$$\lambda = 4R^2 + (\xi^2 + \eta^2 + \zeta^2),$$

and the moments of inertia,

$$J = 16R^4 \sum \frac{(\eta^2 + \zeta^2)\,dm}{\lambda^2} \text{ etc,} \qquad \Delta_{\xi\eta} = 16R^4 \sum \frac{\xi\eta dm}{\lambda^2}, \text{ etc.}$$

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On the Prime Ideals of the General Cubic Galois Field.

By Claiborne G. Latimer.

1. In a recent paper, Nowlan[*] treated the problems of determining the rational integers representable by two cubic forms. Each of these forms is the norm of the general integral algebraic number in a cubic Galois field of class unity. The problem is equivalent to that of determining the rational primes which are factorable in the field. We shall consider this problem for the general cubic Galois field. The discriminants of the fields considered by Nowlan are 7^2 and 9^2 respectively. If we let $D = 7$ or $D = 9$ and let the corresponding field be F, his results, except as to discriminantal divisors, may be stated as follows. *A rational prime p, not a divisor of D, is the product of three conjugate ideals of F if and only if p is a cubic residue of D.*

The discriminant of a cubic Galois field is the square of a rational integer D. If p is a rational prime, not a divisor of $6D$, we shall find a necessary and sufficient condition, in terms of D and p, that p be factorable in the field. When D is 9 or a prime and $p > 3$ our results may be stated exactly as in italics above. It will also be shown that a necessary condition that the field be of class unity is that $D = 9$ or a prime.

It may be shown, in a comparatively simple manner, that certain of Eisenstein's results on the canonical cubic and associated forms may be deduced from the results of this paper.

2. Consider the cubic field F, defined by a root θ_1 of

$$x^3 + ax + b = 0, \tag{1}$$

where a and b are integers.[†] After an obvious transformation, we may assume that there is no integer $K > 1$ such that a and b are divisible by K^2 and K^3 respectively. If $a = 9a'$, $b = 9b'$, where a' and b' are integers, it may be verified that $\theta' = (6a' + \theta_1^2)/3$ is a root of $x^3 - 3a'^2x - (2a'^3 + 3b'^2) = 0$. The second and third coefficients of this equation are not both divisible by 9 since b' is prime to 3. We may therefore assume that

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[*] *Bulletin of the American Mathematical Society*, 32 (1926), 374-80.

[†] We shall use "integer" and "prime" to mean "rational integer" and "rational prime" unless some other reference field is mentioned.

(A) *there is no integer $K > 1$ such that a and b are divisible by K^2 and K^3 respectively; and a and b are not both divisible by* 9.

By Woronoj's method of determining a basis* of a cubic field, we find that

$$1, \theta_2 = (\theta_1 - \xi)/\delta, \qquad W_1 = (\xi^2 + a + \xi\theta_1 + \theta_1^2)/\sigma\delta^2,$$

constitute a basis, where δ, σ, ξ are as follows. If each of the following conditions is satisfied, $\delta = 3$; otherwise $\delta = 1$.

$$(2) \qquad a \equiv -3 \pmod 9, \qquad \pm b \equiv a + 1 \pmod{27}.$$

σ is the largest integer such that the congruences

$$(3) \qquad x^3 + ax + b \equiv 0 \pmod{\delta^3\sigma^2}, \qquad 3x^2 + a \equiv 0 \pmod{\delta^2\sigma},$$

have a common solution ξ. If d is the discriminant of (1), we find that the discriminant of F is $d/\delta^6\sigma^2$.

If $\delta = 3$, $\theta_1 = 3\theta_2 + \xi$. It will be found that $W_1 = m + (\xi\theta_2 + \theta^2_2)/\sigma$ where by (3_2) m is an integer. If ξ were divisible by 3, $\theta_1/3$ would be integral, contrary to (A). Hence ξ is prime to 3. θ_2 is a root of $x^3 + \xi x^2 + \alpha x + \beta = 0$, where α and β are integers. We have then

THEOREM 1. *The field F has a basis in the form*

$$1, \theta, W = (s + t\theta + \theta^2)/\sigma$$

where s and t are rational integers and θ is a root of

$$(4) \qquad f(x) = x^3 + Px^2 + Qx + R = 0,$$

where P, Q, R are rational integers. If $P = 0$, then $\delta = 1$, $Q = a$, $R = b$, and if $P \neq 0$, then $P = \xi$, $Q = (3\xi^2 + a)/9$, $R = (\xi^3 + a\xi + b)/27$, and P is prime to 3.

We shall assume hereafter that the discriminant of $f(x)$ is the square of a rational integer $\Delta > 0$. Then $\Delta = \sigma D$, where D^2 is the discriminant of F. It will be seen later that this assumption is equivalent to assuming that F is a Galois field. We have

$$(5) \qquad -\delta^6\sigma^2 D^2 = 4a^3 + 27b^2.$$

It will also be seen later that $3Q - P^2 = fD$, where f is an integer. By Theorem 1, $P = 0$ if and only if fD is divisible by 3, and $3fD = a$ or $9a$ according as $P \neq 0$ or $P = 0$.

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* *Über die ganzen algebraischen Zahlen, die von einer Wurzel der Gleichung dritten Grades abhängen*, St. Petersburg, 1894.

Suppose $\delta = 1$ and that σ is divisible by 3. By (5), $a = 3a'$, a' an integer, and σD is divisible by 9. Then $4a'^3 + b^2 \equiv 0 \pmod{27}$ or $4a'^3 + b^2 \equiv -3 \pmod 9$ according as σD is or is not divisible by 27. The first congruence holds only if the conditions (2) hold, whence $\delta = 3$, contrary to hypothesis. If the second congruence holds, $a = 3a' \equiv -3$, $b \equiv \pm 1 \pmod 9$, and then by (3_1), $\xi^3 - 3\xi \pm 1 \equiv 0 \pmod 9$, which is obviously impossible. We have then (a) of the following lemma. Let k be the g. c. d. of fD and $\sigma = k\sigma'$.

Lemma 1. (a) *If* $\delta = 1$ *then* σ *is prime to* 3;

(b) $D = 3^j D_1$, *where* $j = 0$ *or* 2 *and* D_1 *is prime to* 3;

(c) k *contains no square factor and is prime to* $3\sigma'$;

(d) k^2 *divides* a *and* b;

(e) $f = f'k$, $D = D'k$, *where* f' *is prime to* σ', *and* D' *is prime to* σ.

If D is divisible by 3, $fD = 3Q = 3a$, $P = 0$, $\delta = 1$ and by (*a*), σ is prime to 3. Then by (5), D is divisible by 9. If D were divisible by 27, a would be divisible by 9 and hence by (5) the same would be true of b, contrary to (A). This proves (*b*).

If k were divisible by 3, as before $\delta = 1$ and we would have a contradiction of (a). Hence k is prime to 3. Suppose p^2 divided $\sigma = k\sigma'$, where p is a prime divisor of k. Since $3\,fD = a$ or $9a$, p divides a. By (3_2) a and ξ are divisible by p^2 and p respectively. Then by (3_1) p^3 would divide b, contrary to (A). We have therefore (c). If p is a prime factor of k we find, as in the proof of (c), that a and b are divisible by p and p^2 respectively. Then by (5) p^2 divides a. Since k contains no square factor, (d) follows.

By (a), (d) and (5), k divides $\sigma' D$ and hence by (c) it divides D. Suppose p^2 divided $D = kD'$, where p is a prime factor of k. Then by (5) and $3fD = a$ or $9a$, a and b would be divisible by p^2 and p^3 respectively, contrary to (A). Hence D' is prime to k, and k divides f, since k^2 divides a. f' and D' are prime to σ' by definition of k. This completes the proof of the lemma.

It may be verified that if $\lambda = \alpha_0 + \alpha\,\theta_1 + \alpha_2\theta_1^2$ then the norm of λ is

$$\psi(\alpha_i) = \alpha_0^3 + a\alpha_0\alpha_1^2 - b\alpha_1^3 + 3b\alpha_0\alpha_1\alpha_2 - 2a\alpha_0^2\alpha_2 + a^2\alpha_0\alpha_2^2 - ab\alpha_1\alpha_2^2 + b^2\alpha_2^3. \tag{6}$$

Every integer of $F(\theta)$ may be expressed in the forms

$$\lambda = x + y(\theta_1 - \xi)/\delta + z(\xi^2 + a + \xi\theta_1 + \theta_1^2)/\sigma\delta^2 = (\alpha_0 + \alpha_1\theta_1 + \alpha_2\theta_1^2)/\sigma\delta^2$$

where the x, y, z, α_i are integers. Then $\delta^6\sigma^3 N(\lambda) = \psi(\alpha_i)$. Furthermore, by definition of k, (3) and Lemma 1, k divides a and ξ. It follows that α_0 and α_1 are divisible by k. We also note that every integer of $F(\theta)$ may be written in the form $\lambda = (\alpha_0 + \alpha_1\theta + \alpha_2\theta^2)/\sigma$, where the α_i are integers.

3. It is known * that a second root of (4) is

$$\theta^1 = (1/2\sigma D)[2(3Q - P^2)\theta^2 - (9R - 7PQ + 2P^3 + \sigma D)\theta + 4Q^2 - P^2Q - 3PR - PD\sigma]$$

It follows that $F(\theta)$ is a Galois field. We shall now prove

THEOREM 2. *A rational prime p, not a divisor of D, is the product of three distinct conjugate ideals of $F(\theta)$ or is a prime of $F(\theta)$ according as the congruence*

$$f(x) \equiv 0 \pmod{p}, \tag{7}$$

has a solution or has no solution.

It is well-known that every prime ideal in a cubic Galois field is of the first or third degree. Hence every rational prime, not a divisor of D, satisfies one of the conditions of the theorem. Suppose $x = a$ is a solution of (7). If p were a prime of $F(\theta)$ it would divide one of the three factors of $f(a) = (a - \theta)(a - \theta')(a - \theta'')$. This is impossible in view of the basis of $F(\theta)$ given in Theorem 1. The first part of the theorem follows. Suppose (7) has no solution. If $P \neq p$ were a prime ideal factor of p it would have a canonical basis in the form $[p, a - \theta, b - W]$ where a and b are integers. Then the rational integer $f(a) = (a - \theta)(a - \theta')(a - \theta'')$ would belong to P and hence would be divisible by p, contrary to hypothesis. The theorem follows.

By the last sentence in § 2 and the above expression for the integral algebraic number θ' we find

$$3Q - P^2 = fD, \qquad 9R - PQ = hD, \tag{8}$$

where f and h are rational integers. Let p be a prime, not a divisor of $6D$. In (7) set $3x = z - P$. We obtain by means of (8),

$$z^3 + 3fDz + (3h - 2Pf)D = 0 \pmod{p}. \tag{9}$$

The discriminant of (9) is $3^6\sigma^2D^2$. Then from the well-known formula for the discriminant of a reduced cubic we find

$$-f^3D = (\lambda + 3\sigma\rho)(\lambda + 3\sigma\rho^2) \tag{10}$$

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* Serret, *Cours d'Algèbre Supérieure*, Vol. 2, 6th ed., p. 467.

where ρ is an imaginary cube root of unity and $2\lambda = 2Pf + 3(\sigma - h)$. By Lemma 1, k divides σ and D and contains no square factor. Hence $\lambda = k\lambda'$, λ' an integer. From (10)

$$-f'^3k^2D' = (\lambda' + 3\sigma'\rho)(\lambda' + 3\sigma'\rho^2).$$

By Lemma 1, σ' is prime to $f'kD'$ and hence it is prime to λ'. Then every prime factor $\neq 3$ of the left side of this equation is the product of two conjugate primes of $F(\rho)$, one and only one of which divides $\lambda' + 3\sigma'\rho$. By Lemma 1, k is prime to 3. Let

$$k = \prod_{i=1}^{m} (s_i + t_i\rho)(s_i + t_i\rho^2),$$

$$D' = 3^j \prod_{i=m+1}^{n} (s_i + t_i\rho)(s_i + t_i\rho^2)$$

where every $s_i + t_i\rho$ $(i = 1, 2, \cdots, n.)$ is a primary prime of $F(\rho)$ which divides $\lambda' + 3\sigma'\rho^2$ or $\lambda' + 3\sigma'\rho$ according as $i \leqq m$ or $i > m$. By Lemma 1, $j = 0$ or $j = 2$. Let $-f' = 3^t(A + B\rho)(A + B\rho^2)$ where $A + B\rho$ is the product of the primary prime factors of $F(\rho)$ which divide $\lambda' + 3\sigma'\rho$. Since $3 = (1-\rho)(1-\rho^2) = -\rho^2(1-\rho)^2$, we have

$$\lambda' + 3\sigma'\rho = \pm \rho^e (A + B\rho)^3 (1-\rho)^{j+3t} \times \prod_{i=1}^{m} (s_i + t_i\rho^2)^2 \prod_{i=m+1}^{n} (s_i + t_i\rho), \tag{11}$$

where $e = 0$, 1 or 2. Let $H = D(\lambda + 3\sigma\rho) = k^2D'(\lambda' + 3\sigma'\rho)$. Then

$$H = \rho^{e+2j} (A + B\rho)^3 M^3 \prod_{i=1}^{n} (s_i + t_i\rho)^2 (s_i + t_i\rho^2)$$

where $M = \pm (1-\rho)^{j+t} \prod_{i=1}^{m} (s_i + t_i\rho^2)$. By a result due to Dickson,[*] (9) and hence (7) has a solution if and only if there exists an integer ζ of $F(\rho)$ such that $\zeta^3 \equiv H \pmod{p}$. By hypothesis p is prime to M and hence we may replace H by $H_1 = H/M^3$ in this criterion.

By (8) and the last sentence of Theorem 1, it is readily shown that a and b are divisible by D_1 of Lemma 1. Then if D_1 were divisible by q^2, q a prime $\neq 3$, σ would be divisible by q from the definition of σ. Therefore by (5), b would be divisible by q^3 contrary to (A). Hence D contains no square factor $\neq 9$. We have then (a) and (b) of

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[*] *Bulletin of the American Mathematical Society,* Vol. 13 (1907), p. 2.

LEMMA 2. (a) $D = 9$ *or* $3^j \prod_{i=1}^{n} (s_i + t_i\rho)(s_i + t_i\rho^2)$, *where* $j = 0$ *or* 2 *and the imaginary factors are distinct primary primes of* $F(\rho)$.

(b) *If* p *is a rational prime, not a divisor of* $6D$, (7) *has a solution if and only if there exists an integer* ζ *of* $F(\rho)$ *such that*

$$\zeta^3 \equiv H_1 = \rho^{e+2j} (A + B\rho)^3 \prod_{i=1}^{n} (s_i + t_i\rho)^2 (s_i + t_i\rho^2). \tag{12}$$

(c) $j = 0$ *implies* $e = 0$; $j = 2$ *implies* $e \neq 2$.

To prove (c) write (11) in the form

$$\lambda' + 3\sigma'\rho = \pm \rho^e (1 - \rho)^{j+3t} (S + T\rho). \tag{11'}$$

We have $S + T\rho = \pm 1 \pmod 3$ since it is a product of primary primes of $F(\rho)$. Since σ' is prime to 3 and $(1 - \rho)^4 = 9\rho^2$, it follows from (11′) that $j + 3t \leqq 3$. If $j = t = 0$, the imaginary part of the left side of (11′) is divisible by 3 only if $e = 0$. If $j = 0$, $t = 1$, we find from the definition of λ and (11′) that

$$-h + \sigma(-3)^{1/2} = \pm 2k\rho^{e+1}[S + T + (2T - S)\rho].$$

By (8_2) h is divisible by 9. But it will be found that the real part of the right side of this equation is divisible by 9 only if $e = 0$. The first part of (c) follows. If $j = 2$ then $t = 0$ and

$$\lambda' + 3\sigma'\rho = \mp 3\rho^{e+1}(S + T\rho).$$

Since T is divisible by 3 and σ' is prime to 3, it follows that $e \neq 2$. This proves the lemma.

4. If π is a prime of $F(\rho)$ such that $N(\pi) = \mu \neq 3$ and if η is an integer of $F(\rho)$ not divisible by π, by Fermat's Theorem

$$\eta^{\mu-1} - 1 = (\eta^{(\mu-1)/3} - 1)(\eta^{(\mu-1)/3} - \rho)\ (\eta^{(\mu-1)/3} - \rho^2) \equiv 0 \pmod{\pi}.$$

We shall use Eisenstein's symbol, $[\eta/\pi] = 1$, ρ, or ρ^2 according as π divides the first, second or third of the above factors. It is well known that η is a cubic residue of π if and only if the value of the symbol is unity.

From the definition of the symbol we have

$$[\lambda/\pi]^2 = [\lambda'/\pi'], \quad [\lambda/\pi][\eta/\pi] = [\lambda\eta/\pi], \quad [(\eta + \kappa\pi)/\pi] = [\eta/\pi] \tag{13}$$

where the primed letters represent the conjugates of the corresponding unprimed letters. Furthermore, by the cubic reciprocity law[*] if η and π are primary primes of $F(\rho)$, then $[\eta/\pi] = [\pi/\eta]$.

If π is a prime of $F(\rho)$, not a divisor of $N(H_1)$, by (13_1) we have $[H'_1/\pi] = [H_1/\pi']^2$. Furthermore, since $N(H_1)$ is a cube, $[H'_1/\pi] = [H_1/\pi]^2$. Hence $[H_1/\pi] = [H_1/\pi']$. Then if $p = \pi\pi'$ or π is a rational prime, not a divisor of $6D$, where π is a primary of $F(\rho)$ not a divisor of $A + B\rho$, a necessary and sufficient condition that (12) have a solution is that $[H_1/\pi] = 1$. We may choose the notation so that π does not divide $A + B\rho$. For if p divided $A + B\rho$, by (11) it would divide σ' and λ', whereas we have seen that they are relatively prime. Therefore by Theorem 2, (b) of Lemma 2 and the cubic reciprocity law we have

THEOREM 3. *If p is a rational prime, not a divisor of $6D$, $p = \pi\pi'$ or $p = \pi$ where π is a primary prime of $F(\rho)$ which does not divide $A + B\rho$. Then a necessary and sufficient condition that p be the product of three conjugate ideals of $F(\theta)$ is that*

$$[\pi/(s_1 + t_1\rho)][\pi/(s_2 + t_2\rho)] \cdots [\pi/(s_n + t_n\rho)]$$
$$= \rho^a[\pi/(s_1 + t_1\rho^2)]\,[\pi/(s_2 + t_2\rho^2)] \cdots [\pi/(s_n + t_n\rho^2],$$

where $\alpha = [N(\pi) - 1]\,(e + 2j)/3$ and if $n = 0$ each Eisenstein symbol is taken as unity.

We shall next prove

THEOREM 4. *If p is a rational prime, not a divisor of $6D$, a sufficient condition that p be the product of three conjugate ideals of $F(\theta)$ is that p be a cubic residue of D. If (a) $D = 9$ or $D = q$, a prime, or (b) $F(\theta)$ is of class unity, this condition is also necessary.*

To prove the sufficient condition, suppose that p is a rational prime, not a divisor of $6D$, and a cubic residue of D. Let $p = \pi\pi'$ or $p = \pi$ where π is a primary prime of $F(\rho)$. If D is divisible by 3, p is a cubic residue of 9 and hence is in the form $9u \pm 1$. Then $N(\pi) \equiv 1 \pmod 9$. If D is prime to 3, $j = 0$ and by (c) of Lemma 2, $e = \alpha = 0$. In either case $\rho^a = 1$.

Suppose $p = \pi\pi'$. Since it is a cubic residue of D, by (13_2) we have

$$[\pi/(s_i + t_i\rho)][\pi'/(s_i + t_i\rho)] = [\pi/(s_i + t_i\rho^2)][\pi'/(s_i + t_i\rho^2)] = 1$$
$$(i = 1, 2, \cdots, n).$$

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[*] Bachmann, *Die Lehre von der Kreisteilung*, p. 198.

By (13_1) these equations are equivalent to

$$[\pi/(s_i+t_i\rho)]=[\pi/(s_i+t_i\rho^2)] \quad (i=1,2,\cdots,n).$$

If $p=\pi$ we obtain these equations at once from our hypothesis that p is a cubic residue of D. The sufficient condition of the theorem follows from these equations and Theorem 3.

To prove (a), we note that $n=0$ or 1. We may then retrace our steps in the above sufficiency proof and find that if a prime p is the product of three conjugate ideals of $F(\theta)$ then it is a cubic residue of D.

To prove (b), suppose $F(\theta)$ is of class unity and that $N(\lambda)=p$, where λ is an integer of $F(\theta)$ and p is a prime, not a divisor of $6D$. By the last paragraph in § 2, $\delta^6\sigma^3p=\psi(\alpha_i)$, where the α_i are rational integers, α_0 and α_1 being divisible by k. If D is prime to 3, by (8) and Theorem 1 it may be shown that a and b are divisible by D. By Lemma 1, k^2 divides a and b, and by (a) of Lemma 2, D contains no square factor. Hence a and b are divisible by kD. Then from (6) and the fact that $\alpha_0=\alpha_0'\,k$ and α_1 are divisible by k we find $\delta^6\sigma^3p\equiv\alpha_0^3 \pmod{k^3D}$ or

$$\delta^6\sigma'^3p\equiv\alpha_0'^3 \pmod{D}.$$

By (c) and (e) of Lemma 1, σ' is prime to $D=kD'$, and by hypothesis δ is prime to D. Hence p is a cubic residue of D.

If D is divisible by 3 then $D=9D_1$, and we may repeat the above argument with D replaced by D_1. It remains to show that p is a cubic residue of 9. From $fD=3Q-P^2$ we have $P=0$, $\delta=1$, $Q=a=3a'$, $R=b$. By (a) of Lemma 1, σ is prime to 3 and hence by (5), $4a'^3+b^2\equiv-3 \pmod 9$. Therefore $a=3a'\equiv-3$, $b=\pm1$, (mod 9). Since $\sigma^3p=\psi(\alpha_i)$, we find from these congruences and (6) that

$$\sigma^3p\equiv\xi^3-3\xi\eta^2+\eta^3 \pmod{9},$$

where $\xi=\alpha_0+2\alpha_2$, $\eta=\mp\alpha_1-\alpha_2$. For the case $b\equiv1 \pmod 9$, this congruence is essentially the identity used by Nowlan* who showed that the right member is divisible by 3 or is congruent to ±1, modulo 9, for every ξ, η. Since the left side is prime to 3, it follows that $p\equiv\pm1 \pmod 9$, i. e. p is a cubic residue of 9. This completes the proof of the theorem.

The condition of Theorem 4, i. e. that p be a cubic residue of D, is obviously equivalent to the condition that p occur in one of a certain set of arithmetic progressions, T_i+Dz, where the T_i are rational integers. If D is prime to 3, the α of Theorem 3 is zero for every p and it may be shown

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[*] Ibid., p. 379.

that the condition of Theorem 3 is equivalent to the condition that p occur in one of a certain set of arithmetic progressions of the above type. If D is divisible by 3, we may obtain a similar result; except that since $\alpha \neq 0$ we have three sets of progressions instead of one, the set which is used in applying the criterion to a given p depending on the residue of p, modulo 9.[*]

Westlund [†] treated the problem of determining the degree of the prime ideal factors of a rational prime p in the field defined by a root of

$$\phi(y) = y^3 - 3qy - qA = 0,$$

where q is a prime, $4q = A^2 + 27B^2$, $A \equiv q \equiv 1 \pmod 3$. The discriminant of the field is q^2. He found that if p is a prime, not a divisor of $6qB$, then p is the product of three prime ideals of the field if and only if $\phi(y) \equiv 0$ (mod p) has a solution. This is implied by Theorem 2. However, Theorem 4 gives a simpler criterion, namely that p be a cubic residue of q.

5. We shall now prove

THEOREM 5. *A necessary condition that $F(\theta)$ be of class unity is that $D = 9$ or $D = q$, a prime.*

Suppose $F(\theta)$ is of class unity and that D contains two prime factors, one of which $\neq 3$. Consider first the case where D is prime to 3. By (a) of Lemma 2, the prime factors of D are distinct. We may then determine integers, λ_i, n_i, of $F(\rho)$ which are prime to D and such that

$$\begin{aligned}
[\lambda_1/(s_1 + t_1\rho)] &= [\eta_2/(s_2 + t_2\rho^2)] = [\lambda_2/(s_2 + t_2\rho)]^2 \\
&= [\eta_1/(s_i + t_i\rho^2)] = \rho, \\
[\lambda_i/(s_i + t_i\rho] &= [\eta_i/(s_i + t_i\rho^2)] = 1 \qquad (i = 3, \cdots, n.),
\end{aligned}$$

where the last two equations are ignored if $n = 2$. Let δ be an integer of $F(\rho)$ such that

$$\delta \equiv \lambda_i \pmod{s_i + t_i\rho}, \quad \delta \equiv \eta_i \pmod{s_i + t_i\rho^2} \qquad (i = 1, 2, \cdots, n.).$$

After adding proper multiples of D to δ we may assume $\delta \equiv 2 \pmod 9$. Then δ is prime to $9D$. By a theorem due to Fanta [‡] there is an infinitude of primes of $F(\rho)$ in the form $\delta + 9Dz$, where z is an integer of $F(\rho)$.

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[*] Cf. Bauer, *Archiv der Mathematik und Physik*, (3), Vol. 6 (1903), pp. 221-4; Bulletin, National Research Council, Number 28, Vol. 5 (1923), *Algebraic Numbers*, p. 49.

[†] *Deutsche Mathematiker Vereinigung*, Vol. 22 (1913), pp. 135-40; Bulletin, National Research Council, *Algebraic Numbers*, pp. 57-8.

[‡] *Monatshefte für Mathematische Physik*, Vol. 12 (1901), pp. 1-44; Bulletin National Research Council, *Algebraic Numbers*, p. 21.

Let π be such a prime which does not divide $6f'D$. Since $\pi \equiv \delta \equiv -1$, (mod 3), π is primary. We have

$$[\pi/(s_i + t_i\rho)] = [\lambda_i/(s_i + t_i\rho)], \quad [\pi/(s_i + t_i\rho^2)] = [\eta_i/(s_i + t_i\rho^2)] \qquad (i = 1, 2, \cdots, n).$$

Then by Theorem 3 and the definition of the λ_i, η_i it follows that the rational prime $p = \pi$ or $N(\pi)$ is the product of three conjugate ideals of $F(\theta)$. But

$$\rho = [\pi/(s_1 + t_1\rho)] = [\pi/(s_1 + t_1\rho^2)]^2 = [\pi'/(s_1 + t_1\rho)],$$

and therefore p is a cubic non-residue of D, contrary to (b) of Theorem 4. This proves the theorem if D is prime to 3.

Consider the case where $D = 9D_1 \neq 9$. Let λ_i, η_i be integers of $F(\rho)$ prime to D and such that

$$[\lambda_1/(s_1 + t_1\rho)] = \rho^{e+1}, \quad [\lambda_i/(s_i + t_i\rho)] = [\eta_j/(s_j + t_j\rho^2)] = 1 \qquad (i = 2, \cdots, n;\ j = 1, 2, \cdots, n).$$

Let δ and π be defined as before, except that D is replaced by D_1. Since $\delta \equiv 2 \pmod 9$ it follows that $N(\pi) - 1 \equiv 3 \pmod 9$ and hence $\rho^{\alpha} = \rho^{e+1}$. By (c) of Lemma 2, $e \neq 2$. We then obtain as before a contradiction of (b) of Theorem 4. This completes the proof of the theorem.

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On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve.

By Oscar Zariski.

I. *Introduction and Statement of the Problem.*

1. The problem which we investigate in this paper is a direct generalization of the well-known existence problem in the theory of algebraic functions of one variable to the case of two independent variables, and may be formulated as follows:

> *Does an algebraic function z of x and y exist, possessing a preassigned curve f as branch curve?*

This problem was first considered by F. Enriques in a paper published in 1923.[*] Enriques finds that the assigned substitutions on the branches of the function z must satisfy certain conditions, which he calls *conditions of invariance.* In conversations, from which this investigation arose, it was pointed out to me by S. Lefschetz that the above problem is intimately bound up with the Poincaré *group* of f relative to its carrying complex projective plane, the conditions given by Enriques merely expressing the fact that the assigned substitutions must satisfy the generating relations of the fundamental group of the curve f. By making use of the theorem of existence of algebraic functions of one variable, possessing given branch points, Enriques proves that these conditions are also sufficient, and hence announces his existence theorem for algebraic functions of two variables.

Enriques' result, important as a principle, is nevertheless incomplete, since we do not know anything about the fundamental group of an algebraic curve. The complete solution of the existence problem depends upon the solution of the following purely topological problem:

> *Given an algebraic curve, to find its fundamental group.*

In this paper we attempt to throw some light upon the structure of the fundamental group. Our investigation is far from giving the complete solution of the proposed problem. However, the theorems established below settle the problem in the most simple cases, and suggest a possible direction for a deeper investigation.

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[*] "Sulla costruzione delle funzioni algebriche di due variabili possedenti una data curva di diramazione," *Annali di matematica pura ed applicata,* Ser. 4, Vol. 1 (November 1923), pp. 185-198.

2. Riemann announced the following existence-theorem:[*] There exists an algebraic function $y(x)$ (determined to within a birational transformation), which possesses in the plane of the complex variable x given critical points $a_1, a_2, \cdots, a_m$, and the branches of which—with respect to a given system of consecutive non-intersecting loops Oa_i—are permuted according to assigned substitutions $S_1, S_2, \cdots, S_m$, provided they satisfy the following two conditions:

(a) $S_1 S_2 \cdots S_m = 1$;
(b) the substitutions S_i generate a transitive group.

In the case of two independent variables, we have an algebraic function z of x and y defined by an irreducible algebraic equation

$$\phi(x, y, z) = 0. \tag{1}$$

The *branch curve* f of the function z in the plane of x and y can be defined as the locus, as x varies, of the branch points of z, considered as a function of y alone, by letting $x = \text{const.}$ The equation of f,

$$f(x, y) = 0, \tag{2}$$

is obtained by eliminating z between equation (1) and

$$\partial\phi/\partial z = 0,$$

and by neglecting the *apparent* branch components, which arise from ordinary multiple curves of the surface (1).

If the curve f is given, and if we look for an algebraic function z, which should possess f as branch curve, we must first assign the substitutions on the branches of the hypothetical function z. This is done by considering the *fundamental group*, introduced by Poincaré, for the curve f, which is a two-dimensional Riemann surface immersed in the four-dimensional real space S_4 of the two complex variables x and y, or, in more exact terms: the fundamental group of the residual space $S_4 - f$.

We recall that the fundamental group of f, which we will denote by G, is defined as follows:[†]

The elements of G are all possible one-dimensional oriented circuits g in the residual space $S_4 - f$, which start from and end at a given point O. Two circuits are said to be equivalent if one can be carried into the other by a continuous deformation in $S_4 - f$, the point O remaining fixed. Equivalent circuits are not considered as essentially distinct and correspond to one and the same element of the fundamental group. In particular, any circuit g, which can be shrunk into the point O, by a continuous deformation as above, is represented in G by the identity 1, and we write $g = 1$. The product of two elements g_1, g_2 is the circuit obtained by describing first the circuit g_1 and then the circuit g_2. Generally $g_1 g_2 \neq g_2 g_1$. From the definition it follows that g and g^{-1} refer to one and the same circuit with opposite orientations. The choice of the point O has no effect on the structure of the fundamental group.

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[*] "Theorie der Abelschen Functionen," §§ 3-5, *Crelle's Journal*, Vol. 54 (1857).

[†] See, for instance, O. Veblen, "Analysis Situs," pp. 132-133, *The Cambridge Colloquium* 1916, (New York, 1922).

A finite set of generators of G can easily be constructed. It can be shown,* that any circuit g is equivalent to a circuit g' belonging to a generic "line" l, $\alpha x + \beta y + \gamma = 0$, through O (a two-dimensional manifold, homeomorphic to a sphere). Let $a_1, a_2, \cdots, a_n$ be the intersections of f with l, where n is the order of f. In l, the circuit g' is equivalent to a product of elementary non-intersecting circuits (loops) $g_1, g_2, \cdots, g_n$, which all start at O and surround the points $a_1, a_2, \cdots, a_n$ respectively. Hence the *loops* $g_i (i = 1, 2, \cdots, n)$ *can be taken as generators of* G.

The generators g_i satisfy several relations, called *generating relations.* We will study these relations in the next section. Here it may be observed that evidently

$$g_1 g_2 \cdots g_n = 1, \tag{3}$$

and that other relations are obtained by letting the line l vary in the pencil of lines through O. Any generating relation is of the form:

$$g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n} g_1^{l_1} g_2^{l_2} \cdots g_n^{l_n} \cdots g_1^{m_1} g_2^{m_2} \cdots g_n^{m_n} = 1.$$

Let us write down all the generating relations of G:

$$\phi_1(g_1, g_2, \cdots, g_n) = 1, \quad \phi_2(g_1, g_2, \cdots, g_n) = 1, \cdots, \quad \phi_r(g_1, g_2, \cdots, g_n) = 1. \tag{4}$$

In assigning the substitutions on the branches of an algebraic function z of x and y, it is sufficient to assign the substitutions $s_1, s_2, \cdots, s_n$ relative to the generators g_i. However, these generating substitutions of the monodromie group of the function z cannot be given arbitrarily. By (4), they must satisfy the relations

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* S. Lefschetz, L'analysis Situs et la géometrie algébrique, p. 33. Paris, Gauthiers-Villars, 1924.

$$(4') \qquad \phi_1(s_1, s_2, \cdots, s_n) = 1, \quad \phi_2(s_1, s_2, \cdots, s_n) = 1, \cdots, \quad \phi_r(s_1, s_2, \cdots, s_n) = 1,$$

analogous to (4).

A knowledge of the generating relations (4), and hence of the fundamental group G, is thus essential for the solution of the existence problem, which we are now in a position to formulate in an exact manner. It consists of two parts:

1. *Given an algebraic curve f, to determine its fundamental group.*
2. *Does an algebraic function z of x and y exist, which possesses a given branch curve f, and the values of which are permuted—with respect to the above constructed set of generators g_i—according to given substitutions s_i, satisfying the relations* (4′).

The second question is answered in the affirmative by F. Enriques (*loc. cit.*). The possibility of constructing the function, satisfying the above conditions, is, as was shown by Enriques, a consequence of the theorem of existence of algebraic functions of one variable. This paper is hence dedicated exclusively to the investigation of the fundamental group G.

II. *General properties of the group G.*

3. Let

$$f(x, y) = 0$$

be an algebraic curve of order n—irreducible or not. We consider the pencil of lines

$$x = \text{const.}$$

Let $x = a$ be a definite line in this pencil, and let

$$y_i = b_i, \quad (i = 1, 2, \cdots, n),$$

be the intersections of the considered line with the curve f. We mark in the plane of the complex variable y the points b_i and we construct a set of non-intersecting loops $g_i = Ob_i$, where by O we mean the point $y = \infty$, center of the considered pencil. As we said in the preceding section, the loops g_i form a set of generators of the group G.

Let x traverse in its plane a closed path, starting from its initial position $x = a$ and returning to it. The set of points b_i will move continuously in the plane y, and will return to its initial position, though the points b_i may be permuted among themselves. The variation of the loops g_i is completely determined by the paths described by the points b_i and by the condition that they should never intersect. Thus, at the end of the above cyclic variation of the point x, the set of the original loops g_i is transformed into a new set of non-intersecting loops g_i'. More precisely, if the point b_i is carried into b_j then the loop g_i is transformed into some loop g_i' which surrounds the point b_j alone. The two loops g_i' and g_j do not coincide necessarily, but since g_i' surrounds the same point b_j as g_j, g_i' is evidently the transform of g_j by some element g of G, i. e. g_i' and g_j are conjugate elements of G:

$$g_i' = g^{-1} g_j g.$$

The loops g_i and g_i' being equivalent, we obtain the following generating relation:

$$g_i = g^{-1} g_j g.$$

Hence we have the following theorem:

THEOREM 1. *If b_i can be carried into b_j by a cyclic variation of the point x, then the generators g_i and g_j are conjugate elements of the group G.*

If the curve is *irreducible*, then it is always possible to carry any of the points b_i into any other point b_j. Hence, by theorem 1 we deduce:

THEOREM 2. *If the curve f is irreducible, then the n generators g_i are conjugate elements of the group G.*

Let us suppose that the group G is abelian. Then by means of theorem 2, we deduce

$$g_1 = g_2 = \cdots = g_n.$$

The relation $g_1 g_2 \cdots g_n = 1$ becomes in this case

$$g_1^n = 1.$$

On the other hand, no lower power of g_1 is equal to the identity. In fact, if g_1 was of order $n_1 < n$, then the only existing algebraic functions z admitting f as branch curve would be n_1-valued functions (or n_1'-valued, where n_1' is a divisor of n_1), possessing a cyclic monodromie group. But this is impossible, since *any* curve f of order n is the branch curve of an n-valued function, namely of the function z defined by the equation [*]

$$z^n = f(x, y).$$

Hence, we have the following theorem:

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[*] If $z^n = f(x, y)$, then the branch curve of z is the curve f and the line at infinity of the plane x, y, *unless the order of f is a multiple of n.* In this case the line at infinity does not belong to the branch curve of z.

THEOREM 3. *If the curve f of order n is irreducible, and if it is known that the group G is abelian, then G is necessarily the cyclic group of order n.*

Theorem 3 shows that it is impossible to construct upon an *irreducible* branch curve an algebraic function possessing an abelian non-cyclic monodromie group.

4. *The elementary generating relations.* Let us mark in the plane of the complex variable x the critical points $\alpha_1, \alpha_2, \cdots, \alpha_n$ of the function y. We include among the points α_i also those values of x, which correspond to ordinary multiple points of the curve f (apparent branch points). Let $a\alpha_1, a\alpha_2, \cdots, a\alpha_n$ be a set of non-intersecting loops in the plane x. Under the simplest conditions—but quite general and sufficient for the theory of algebraic surfaces from the point of view of birational transformations—the curve f will only possess ordinary double points and cusps. If the axes of x and y are not specially related to the curve f, then at each critical point α_i two and only two values of y coincide. The investigation of the generating relations which are obtained when x traverses a loop $a\alpha_i$ is due to Enriques. His results can be stated as follows:

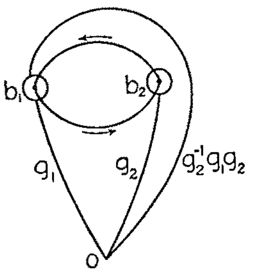

FIG. 1

(a) If α_i corresponds to a simple tangent of the curve f, and if $y_1 = b_1$, $y_2 = b_2$ are the two values of y which tend to coincide when x tends to α_i along the loop $a\alpha_i$, then two corresponding loops g_1, g_2 in the y-plane can be constructed, such that, when x traverses the loop $a\alpha_i$, g_1 is transformed into g_2 and g_2 is transformed into $g_2^{-1}g_1g_2$, while the remaining $n-2$ loops remain invariant (see fig. 1).

This leads to the generating relation:

$$g_1 = g_2. \tag{5}$$

(b) A critical value α_i, which corresponds to an ordinary double point, can be thought of as being the limit of two simple branch points α_i', α_i'', at which the same two branches y_1 and y_2 are permuted and which come together. The loop $a\alpha_i$ is the limit of a circuit, which surrounds the infinitely near branch points α_i', α_i'' and which produces the identical substitution $(b_1b_2)^2$. Therefore, the transformation of the loops in the y-plane, which arises when x traverses the loop $a\alpha_i$, is the square of the transformation mentioned above with reference to a simple tangent, i. e., the following:

$$g_1' = g_2^{-1} g_1 g_2, \qquad g_2' = g_2^{-1} g_1^{-1} g_2 g_1 g_2.$$

The corresponding generating relation in this case is the following:

$$g_1 g_2 = g_2 g_1. \tag{6}$$

(c) Finally, if α_i corresponds to a cusp, it is the limit of three simple branch points, and the resulting transformation of the loops must be the cube of the transformation mentioned above, i. e., the following:

$$g_1' = g_2^{-1} g_1^{-1} g_2 g_1 g_2, \qquad g_2' = g_2^{-1} g_1^{-1} g_2^{-1} g_1 g_2 g_1 g_2.$$

In this case the corresponding generating relation becomes:

$$g_1 g_2 g_1 = g_2 g_1 g_2. \tag{7}$$

Remark. The above transformation of the loops can also be deduced rigorously (see Enriques, quoted paper), by a careful inspection of the expansion of the function y in the three considered cases.

The above considerations show that when a curve acquires an ordinary double point, an equality (5) between two generators is replaced by a permutability relation (6). The appearance of a cusp destroys a permutability relation, and replaces it by a more complicated relation (7). The effect of the singularities of an algebraic curve upon its fundamental group is thus explicitly revealed.

We will call the above generating relations (5), (6) and (7), which correspond to the loops $a\alpha_i$, *the elementary generating relations.*

The form of the above stated elementary generating relations is closely connected with the topological structure of the neighborhood of a multiple point of an algebraic curve. To such a multiple point there corresponds a closed curve, or a set of closed curves, in the ordinary 3-dimensional space, by means of the following construction: Let us consider a neighborhood (a 4-cell) of a multiple point P. The boundary S of the 4-cell (3-dimensional sphere) will meet the curve f along a closed one-dimensional circuit, or along a set of such circuits, l. By a stereographic projection of S on to the ordinary 3-dimensional space the circuit l is projected into a closed curve l', or into a set l' of such curves, in that space. If P is a simple point of f, then l' is a single unknotted curve. If P is an ordinary double point of f, then l' is composed of two unknotted linked curves.* If P is a cusp, then l' is the well-known trefoil knot. The above elementary relations are merely the generating relations of the fundamental group of the curve l'.

The following theorem is an implicit consequence of the existence theorem, as it is stated by Enriques:

THEOREM 4. *The elementary generating relations together with the relation* $g_1 g_2 \cdots g_n = 1$, *form a complete set of generating relations,* i. e. *every relation between the generators is a consequence of them.*

In fact, Enriques shows that there always exists an algebraic function z, if the substitutions on the branches of z satisfy the elementary generating relations and the relation (3) of section 2. This proves the non-existence of other independent relations between the generators.

5. *The fundamental group of a curve* $\bar{f}$, *limit of a variable curve* f. An important consequence of the above considerations concerns the fundamental group of a curve $\bar{f}$ which is the limit of a variable curve f. If $\bar{f}$ has the same singularities as f, then f and $\bar{f}$ are isotopic, and hence possess the same fundamental group. Suppose now that when f tends to $\bar{f}$, a new multiple point is acquired. For simplicity, let us suppose that when f tends to $\bar{f}$, two simple critical points α_1 and α_2 of the variable function y come together, so that $\bar{f}$ acquires a new ordinary double point. What happens to the generating relations? It is evident that any elementary generating relation, which corresponds to a critical point α_j different from α_1 and α_2, will remain a true relation between the generators of the fundamental group of the limit curve $\bar{f}$. The only relations which are destroyed are those relative to the critical points α_1 and α_2, namely the one relation $g_1 = g_2$, since at the limit, when α_1 and α_2 come together at a point α, the two loops g_1 and g_2 pass through α. However, we obtain the relation $g_1 g_2 = g_2 g_1$ between the generators of the group G of the variable curve f, when x traverses a closed path surrounding both critical points α_1 and α_2, and this relation is *not* destroyed, when α_1 and α_2 coincide at α: it becomes the elementary generating relation corresponding to the loop $a\alpha$. Thus in this case, one generating relation ($g_1 = g_2$) is destroyed, and is replaced by the relation $g_1 g_2 = g_2 g_1$, which is a *consequence* of the old relation.

This conclusion holds in the most general case, of any acquired new multiple point, namely: if several critical points $\alpha_1, \alpha_2, \cdots, \alpha_r$ come together to form a new singularity, or a singularity of higher order, of the limit curve $\bar{f}$, then r corresponding elementary relations are destroyed, and are replaced by *one* elementary relation, which on the variable curve f corresponds to a closed path surrounding all the r mentioned points α, and which is therefore a consequence of the destroyed r relations. We have hence the following theorem:

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* This is a consequence of a theorem of the Kronecker index of two algebraic curves on a surface, Lefschetz, *loc. cit.*, p. 19.

THEOREM 5. *If a variable curve f tends to a limit curve $\bar{f}$, then several relations between the generators of the group G of f are abolished, but no new relation arises which is not a consequence of the old relations.*

We may express this theorem also in the following form:

> *All the generating relations of the fundamental group of the limit curve are also generating relations of the fundamental group of the variable curve.*

This last theorem still holds, even if the limit curve $\bar{f}$ degenerates, provided that all its components are simple. If, however, $\bar{f}$ possesses multiple components (so that the number of acquired singularities is infinite), the above considerations cannot be applied without modification.

We see that the fundamental group of the limit curve is simply more general than the fundamental group of the variable curve. Hence, the more numerous and the more complicated the singularities of an algebraic curve f, the larger the class of birationally distinct algebraic functions which can be constructed upon f as branch curve.

6. *The continuous system* $\{f\}$. Let f_0 be a definite curve of order n with assigned singularities, and let us consider a complete irreducible continuous system of curves f of order n, possessing the same singularities as f_0, to which f_0 belongs. We leave aside the question, as yet unsolved in general, whether the system is unique.[*] If f_0 belongs to several continuous systems, then we consider one definite system. We denote it by $\{f\}$. The *general* curve f of $\{f\}$ is isotopic to f_0 and hence possesses the same fundamental group

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[*] This question is solved for the case in which the assigned singularities are ordinary double points. It is even proved that *all curves of given order and genus form an irreducible continuous system.*

See F. Enriques and O. Chisini, *Courbes et fonctions algébriques d'une variable*, Chap. III, p. 365 (Paris, Gauthier-Villars, 1926); F. Severi, *Vorlesungen über algebraische Geometrie*, Anhang F., pp. 342-349 (Leipzig, Teubner, 1921).

as f_0. We consider again a definite line l, say $x=a$, and a set of loops $g_1, g_2, \cdots, g_n$ in the y-plane. Instead of keeping the curve f fixed and letting the line l vary in a pencil, as we did before, we keep the line l fixed and we let the curve f vary in the continuous system $\{f\}$, starting from and returning to its initial position f_0. Let

$$\phi(g_1, g_2, \cdots, g_n) = 1 \tag{8}$$

be a definite generating relation of the fundamental group of the original curve f_0. When f varies in $\{f\}$ and returns to its original position f_0, the original loops g_i will be transformed into new loops g_i'. The relation (8), with which we started, will become

$$\phi(g_1', g_2', \cdots, g_n') = 1, \tag{8'}$$

and this is necessarily a true relation between the old generators g_i, possibly different from that with which we started. Thus we have a means of obtaining several new generating relations, by transforming a given relation in the above mentioned way.

Let us make the following hypotheses on the continuous system $\{f\}$.

(a). *The general curve f of $\{f\}$ cuts out on a generic line l a general set of n points, i. e. the dimension of the series cut out by the curves of the system $\{f\}$ on l is exactly n.*

(b). *Those curves of $\{f\}$ which intersect the line l in n given points still form an irreducible continuous system.*

Under those conditions it can be proved that the fundamental group of the general curve of the system $\{f\}$ is necessarily cyclic of order n.

Indeed, by hypothesis (a), we can assign in advance arbitrary paths for the intersections b_i $(i=1, 2, \cdots, n)$ of the variable curve f of $\{f\}$ with the line l. We start with the intersections b_i^0 of the original curve f_0 with l, and we consider an arbitrary set of paths which carries the set of points b_i^0 into its original position, the points b_i^0 being possibly permuted. The continuous succession of the intermediate positions of the variable curve f, determined (to within a certain degree of freedom, which can be eliminated by putting further linear conditions on the coefficients of f) by the intermediate positions of the points b_i, may carry us possibly to a curve $\bar{f}_0$, different from the original curve f_0. But if hypothesis (b) holds, then it is possible to carry $\bar{f}_0$ into f_0, keeping the intersections b_i^0 fixed. In conclusion, we have that it is always possible to find a cyclic variation of the curve f, starting from f_0, such that when f returns to its initial position f_0 the points b_i have traversed an arbitrary set of paths, given in advance. Since the most general transformation of the loops g_i can be obtained by letting the set of points b_i move along conveniently chosen paths, which carry the set back to its original position, we deduce that if $\phi(g_1, g_2, \cdots, g_n) = 1$ is a generating relation, then *any* transformation of the loops can be applied to ϕ in order to obtain a new generating relation.

Let us take in particular an elementary generating relation $g_1 = g_2$, corresponding to a tangent line to the curve f_0, and let us consider the following elementary transformation T_i $(i = 1, 2, \cdots, n-1)$ of the loops:

$$(9) \qquad T_i: g_i' = g_{i+1}; \qquad g'_{i+1} = g_{i+1}^{-1} g_i g_{i+1}; \qquad g_j' = g_j \qquad (j \neq i, i+1).$$

The transformation T_i is obtained by interchanging b_i^0 and b^0_{i+1} along conveniently chosen paths (see Fig. 1, sec. 4). The transformation T_2, applied to the relation $g_1 = g_2$, leads to the relation $g_1 = g_3$. If we apply to this relation the transformation T_3, we obtain $g_1 = g_4$. By applying successively the transformations T_i, we obtain

$$g_1 = g_2 = \cdots = g_n.$$

Hence the group G is cyclic, and due to relation (3) it is necessarily of order n, q. e. d.

Hence, we have the following theorem:

THEOREM 6. *If the general curve of an irreducible complete continuous system* $\{f\}$ *possesses a non-cyclic fundamental group, then at least one of the following two conditions must be satisfied:* (a) *either the curves of the system cut out on a generic line special sets of* n *points* (for which, say, some invariant vanishes), (b) *or the subsystem of curves of* $\{f\}$, *which pass through* n *given points on a line, breaks up into several irreducible continuous systems.*

Thus the study of the series of sets of points cut out by the curves of $\{f\}$ on a generic line of the plane, is of prime importance for the determination of the fundamental group of the general curve f of the system. In several cases which we study below, the knowledge of this series leads directly to the determination of the group.

III. *Investigation of Some Important Particular Cases.*

7. *The fundamental group of a curve possessing ordinary double points only.* In this section we prove the following theorem:

THEOREM 7. *The fundamental group of an irreducible curve f of order n, possessing ordinary double points only, is cyclic of order n.*

First we consider separately the case in which the curve f has no singularities at all. Since two curves of the same order and without singularities are isotopic and hence have one and the same fundamental group, it is enough to prove the theorem for some special curve of order n. We consider a curve of order n, which possesses a flex of order $n-2$, at which the n intersections of the flex tangent with the curve coincide. For instance, we may consider the curve

$$y^n = (x+1)^n - 1,$$

which has no singularities at all, and which possesses a flex of order $n-2$ at the origin.

The development of y in the neighborhood of $x=0$ starts with the term $(nx)^{1/n}$. It follows that when x traverses in its plane a small circle of center $x=0$, the n values of y: y_1, $y_2 = y_1 \cdot e^{2\pi i/n}$, $y_3 = y_2 \cdot e^{2\pi i/n}, \cdots, y_n = y_{n-1} \cdot e^{2\pi i/n}$, move along a simple closed curve C surrounding the point $y=0$, and that the argument of each y_i is increased by $2\pi/n$. We start with the initial value $x=-1$, and we construct in the y-plane a set of loops g_i, in the way indicated by the figure 2 (the point O, being the point at infinity, must be taken outside the curve C). (Fig. 2.) When x moves along the circle $|x|=1$ in the positive direction and comes back to the initial value $x=-1$ the loops g_i are transformed as follows:

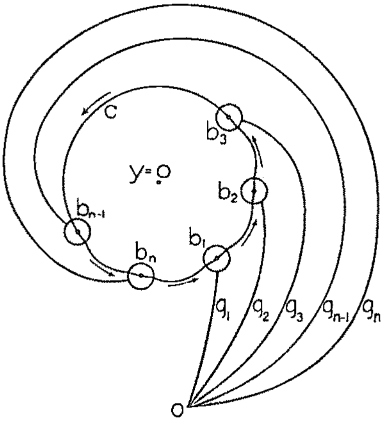

FIG. 2

$$g_1' = g_2; \qquad g_2' = g_3; \; \cdots; \qquad g'_{n-1} = g_n;$$
$$g_n' = g_n^{-1} g_{n-1}^{-1} \cdots g_2^{-1} g_1 g_2 \cdots g_{n-1} g_n.$$

It follows immediately that

$$g_1 = g_2 = g_3 = \cdots = g_n,$$

hence the group G is cyclic. Relation (3) shows that it is cyclic of order n.

Since the preceding proof is based exclusively upon the fact that the general curve of order n can acquire a flex of order $n-2$ at a simple point of the curve, it applies to every irreducible curve possessing this property, no matter what are the singularities of the curve. It is even sufficient to know that a given curve possesses (or can acquire, by varying in a continuous system) a flex of order $n-3$, in order to deduce that its fundamental group is cyclic of order n. Indeed, if a curve f possesses a flex of order $n-3$, at which $n-1$ intersections with the tangent coincide, then, by exactly the same reasoning as above, we can show that the corresponding $n-1$ generators $g_1, g_2, \cdots, g_{n-1}$ are equal. Relation (3) shows that $g_n = g_1^{-(n-1)}$, hence the group is cyclic. By theorem 3, sec. 3, it follows then that the order of the group G is n. If the curve f itself does not possess a flex of order $n-3$, but if it can be deformed into a curve $\bar{f}$ possessing a flex of this order, then, since the fundamental group of $\bar{f}$ is cyclic of order n, we deduce, by theorem 5, sec. 5, that the group G of f is also cyclic of order n. Applying this result to curves of order 3 and 4 we have the following theorem:

THEOREM 8. *The fundamental group of any irreducible cubic or quartic curve is cyclic of order* 3 *and* 4 *respectively, with the only exception of the three-cuspidal quartic curve.*

The three-cuspidal quartic does not possess flexes, and in fact its fundamental group is not cyclic, as we will show later.

To prove the theorem 7, we prove first the following lemma:

LEMMA. *The fundamental group of n lines in arbitrary position is abelian.*

Let $L_1, L_2, \cdots, L_n$ be n lines in arbitrary position. Let be: A the intersection of L_1 and L_2, l a line through a fixed point O, very near to the line OA, and b_i the intersection of L_i and l. When l tends to the line OA, the two intersections b_1 and b_2 tend to coincide at the *double point* A of the curve $L = L_1 + L_2 + \cdots + L_n$. Hence in l a set of generators $g_1, g_2, \cdots, g_n$ can be constructed, such that the following relation holds:

$$(10) \qquad g_1 g_2 = g_2 g_1.$$

Now, let the curve L vary and come back to its initial position, the lines L_i being possibly permuted. Since the L_i's are arbitrary lines, we may arrange the variation in such a way that the intersections b_i with the fixed line l shall follow arbitrarily prescribed paths. Hence (see sec. 6) any transformation of the loops, if applied to (10), will lead to a true relation. We apply successively the transformations T_i, defined by the formulas (9) of sec. 6, and we obtain:

$$(11) \qquad g_1g_2 = g_2g_1; \qquad g_1g_3 = g_3g_1; \cdots; \qquad g_1g_n = g_ng_1.$$

Transforming each of the relations (11) by T_1 we obtain:

$$(12) \qquad g_2g_1 = g_1g_2; \qquad g_2g_3 = g_3g_2; \cdots; \qquad g_2g_n = g_ng_2.$$

Transforming each of the relations (12) by T_2 we obtain:

$$g_3g_1 = g_1g_3; \qquad g_3g_2 = g_2g_3; \cdots; \qquad g_3g_n = g_ng_3.$$

Transforming successively by $T_3, T_4, \cdots, T_{n-1}$, we deduce that any two generators are permutable, and hence that the group G is abelian.

Now let f be an irreducible curve of order n with ordinary double points only. The continuous system $\{f\}$, in this case uniquely determined (see footnote on p. 313), contains in particular curves which degenerate into n arbitrary lines.* Now, if f, varying continuously, degenerates into n lines, then the fundamental group becomes abelian, by the Lemma proved above. But in this case, the group G must originally have been abelian, since by theorem 5, sec. 5, the generating relations of the group of the limit curve (in this case: $g_ig_j = g_jg_i$) hold also for the generators of the group of the variable curve. Hence the fundamental group of f is abelian, and by the theorem 3 sec. 3 it is then necessarily the cyclic group of order n.

The previous considerations are applicable without modification to a reducible curve f of order n, which breaks up into s simple irreducible curves $f_1, f_2 \cdots f_s$, possessing ordinary double points only and having simple mutual intersections only. It is still possible to let f degenerate into n lines, and hence the fundamental group of f is still abelian. *But it is not cyclic any more.* Let n_i be the order of f_i ($n = n_1 + n_2 + \cdots + n_s$), and let $g_{r_i}^{(i)}$

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* This follows from a noted principle, announced by F. Enriques in 1904, "Sulla proprietà caratteristica delle superficie algebriche irregolari," *Rendiconto delle sessioni della R. Accademia delle Science dell'Instituto di Bologna*, nuova serie, Vol. 9 (1904-1905), pp. 5-13, and completed later by F. Severi, *Vorlesungen über algebraische Geometrie:* Anhang F, No. 7. Leipzig, Teubner, 1921. In that part which interests us, this principle says that: *a reducible curve of order n, composed of s simple irreducible curves of genus* $p_1, p_2, \cdots, p_s$ *respectively, can always be considered as the limit of an irreducible curve of the same order n and of genus p, provided that* $p \geq \Sigma p_i$. In our case each p_i is $= 0$, hence the general irreducible curve of given genus p and order n can always be degenerated into n lines.

$(i=1,2,\cdots,s;\ r_i=1,2,\cdots,n_i)$ be a set of non-intersecting loops, the loops $g_1^{(i)}, g_2^{(i)},\cdots, g_{n_i}^{(i)}$ corresponding to the intersections of the component f_i with a fixed line l. The curve f_i, being irreducible, it follows (theorem 1, sec. 3) that the generators $g_{r_i}^{(i)}$, i being fixed, are conjugate elements of the group G, and since G is abelian, it follows that

$$g_1^{(i)}=g_2^{(i)}=\cdots=g_{n_i}^{(i)}. \qquad (i=1,2,\cdots,s).$$

Thus the group G is generated by s generators $g_1, g_2,\cdots, g_s$, where $g_i=g_1^{(i)}=g_2^{(i)}=\cdots=g_{n_i}^{(i)}$. Relation (3) becomes in this case

$$g_1^{n_1}g_2^{n_2}\cdots g_s^{n_s}=1. \tag{13}$$

Hence we have the theorem:

THEOREM 9. *The fundamental group of a reducible curve f, possessing ordinary double points only and composed of s irreducible curves $f_1, f_2,\cdots, f_s$ of orders $n_1, n_2,\cdots, n_s$ respectively, is abelian, and is generated by s generators $g_1, g_2,\cdots, g_s$, connected by one generating relation* (13).

By introducing a new set of generators the relation (13) can be simplified as follows: It is always possible [*] to reduce the matrix

$$\| n_1, n_2,\cdots, n_s \|$$

to its normal form:

$$\| t, 0,\cdots, 0 \|,$$

where t is the H. C. F. of $n_1, n_2,\cdots, n_s$, by a unimodular linear integral transformation of the generators. For the new set of generators, which we still denote by g_i, the relation (13) becomes simply:

$$g_1^t=1.$$

It is thus seen that G *is an abelian group, generated by* $s-1$ *independent elements and by one element of order* t.

Corollary: The monodromie group of an algebraic function z constructed upon f as branch curve must be abelian and must be generated by one substitution of order t (or a divisor of t) and by $s-1$ other arbitrary substitutions. The order of the function z can be arbitrarily great, except when $s=1$.

8. *The fundamental group of the branch curve of the general cubic surface.*

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[*] Frobenius, "Theorie der linearen Formen," *Journal für Mathematik*, Vol. 86 (1879).

Let

$$F(x, y, z) = 0 \tag{14}$$

be the equation of the cubic surface. The branch curve f of the function z is a sextic with 6 cusps on a conic, the equation of which has the following form:

$$f(x, y) = \phi_2^3(x, y) + \phi_3^2(x, y) = 0, \tag{15}$$

where $\phi_2 = 0$ and $\phi_3 = 0$ represent a conic and a cubic curve respectively. By letting ϕ_2 and ϕ_3 vary, we obtain a continuous system $\{f\}$, ∞^{15}, of sextic curves, possessing 6 cusps on a conic. The condition that a plane sextic curve should possess a cusp in non-assigned position is expressed by two relations between the coefficients of the curve. If we admit, that the 12 relations, which express the condition that a sextic curve should possess 6 cusps in non-assigned position, are *independent,* then it follows that these sextic curves form a system ∞^{15}. The above system $\{f\}$ is hence complete. But the curves of this system possess 6 cusps in particular position: they lie on a conic. Hence it seems that there must exist a second continuous system, also ∞^{15}, of sextics having 6 cusps not on a conic, since it is highly improbable that 6 cusps of an irreducible sextic *must* lie on a conic.

We have raised this question (quite unessential for the further investigation) only because of the opportunity which it offers to raise and solve another important question, concerning the fundamental groups of two curves having the same singularities. Let us suppose that there exist two complete continuous systems, both ∞^{15}, of sextics possessing 6 cusps. Then the question arises, whether the fundamental group of a sextic f whose 6 cusps are on a conic, is the same as the fundamental group of a sextic ψ, whose 6 cusps are not on a conic. They would be the same, of course, if it were possible to pass from f to ψ by an isotopic deformation. But such a deformation is *impossible,* at least in the algebraic field, i. e. if the intermediate positions of the deformed curve are *algebraic* curves. In fact, to pass from f to ψ it is necessary to pass through a curve $\bar{f}(=\bar{\psi})$ common to both systems $\{f\}$ and $\{\psi\}$. If the deformation is isotopic, then $\bar{f}$ must not possess new acquired singularities. This is impossible, since it can be easily proved that a sextic with only 6 cusps cannot belong to both systems $\{f\}$ and $\{\psi\}$.

Now, to cut short the discussion, we prove that *the fundamental groups of the above two types of sextics are necessarily distinct (non-isomorphic).*

In fact, let us suppose for a moment that the fundamental group of the sextic ψ is the same as the fundamental group of f. Under this hypothesis, any set of generating substitutions of the monodromie group of an algebraic function z, possessing f as branch curve, will also correspond to an actual function Z, of the same order as z, having ψ as branch curve. In particular, since f is the branch curve of a three-valued function, defined by equation (14), it is possible to construct a three-valued function Z, defined by an equation of the third degree in Z,

$$\bar{F}(x, y, Z) = 0, \tag{16}$$

and possessing ψ as simple branch curve. The surface $\bar{F}$ possesses a net of elliptic curves C, cut out by the planes $\alpha x + \beta y + \gamma = 0$. Hence, by a well known theorem of Castelnuovo-Enriques,[*] $\bar{F}$ is a rational surface or a ruled surface. Now, $\bar{F}$ cannot be a ruled surface of genus $p > 0$, because it must be a regular surface. Indeed, the irregularity of an algebraic surface depends upon the following elements only: the order of the branch curve, its fundamental group and the assigned substitutions. These elements are the same for the surface $\bar{F}$ as for the cubic surface F. Since F is regular, it follows that $\bar{F}$ is also regular, hence rational. By another well-known theorem on the reduction of complete linear systems of elliptic plane curves,[†] it is possible to transform birationally the *rational* surface $\bar{F}$ into a plane in such a way, that the curves C of the above net (which is of the degree 3) be transformed into cubics through 6 points. By mapping the plane on a cubic surface

$$F(x', y', z') = 0,$$

whose ∞^3 plane sections correspond to the ∞^3 cubics through the above 6 points, we will have a birational correspondence between the two surfaces $\bar{F}$ and F which transforms the plane sections

$$\alpha x + \beta y + \gamma = 0$$

of $\bar{F}$ into the plane sections of F, cut out by the planes through a point, which we may suppose to be the planes

$$\alpha' x' + \beta' y' + \gamma' = 0.$$

It follows that the branch sextic ψ of the function Z is transformed by a collineation into the branch sextic f of the function z', which is impossible, since the 6 cusps of f lie on a conic.

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[*] G. Castelnuovo and F. Enriques, "Sopra alcune questioni fondamentali nella teoria delle superficie algebriche," *Annali di Matematica pura ed applicata*, Vol. 6 (series 3), 1900, sec. 17.

[†] See, for instance, F. Enriques and O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, Vol. III, p. 195 (Bologna, N. Zanichelli). French edition (quoted above), Chap. II, p. 281.

Thus we see, that the fundamental groups of two curves having the same order and the same singularities may well be distinct, provided they belong to two distinct complete continuous systems.

We pass now to the determination of the fundamental group of the sextic f, given by the equation (15).

The set Γ_6 of the intersections of f with an arbitrary line, say with the line $y=0$, is given by the roots of the polynomial

$$f_6(x)=f(x,0)=\phi_2^3(x,0)+\phi_3^2(x,0).$$

If we assign the set Γ_6 and if we look for the sextic curves f which pass through Γ_6, we are led in the first place to the problem of representing a given polynomial f_6 of 6th degree as a sum of a square and a cube. This problem was treated by Clebsch,* who showed its connection with the problem of the trisection of the periods in the case of the hyperelliptic curve of genus 2

$$y^2=f_6(x).$$

Clebsch proves that every polynomial f_6 admits 40 different representations of the above type, and that each of the 40 representations corresponds to two third-period characteristics $(\epsilon)_3$, $(-\epsilon)_3$. From this it follows that: 1) there always exist an infinite number of irreducible sextic curves f in the continuous system $\{f\}$, which pass through a given Γ_6 on the line $y=0$, and that in assigning Γ_6 we only fix the intersections of the variable conic ϕ_2 and cubic ϕ_3 with $y=0$; 2) since the intersections of ϕ_2 and ϕ_3 with $y=0$, when Γ_6 is given, can be determined in 40 different ways, the sextic curves f which pass through Γ_6 belong to 40 distinct continuous systems. Thus hypothesis a) of sec. 6 holds in this case, while hypothesis b) does not hold.

The 40 above continuous systems are correlated to the 40 above mentioned pairs of period-characteristics. It is clear that, if we start with a given sextic curve f, belonging to one of these systems, and let f vary in $\{f\}$, until we come back to a curve $\bar{f}$ belonging to the *same* system, while Γ_6 returns to its initial position, then the transformation of the periods, produced by this cyclic variation of the set Γ_6, must leave *invariant* the pair of the corresponding period-characteristics. Conversely it is evident that any circulation of the Γ_6 which leaves invariant a given pair $(\epsilon)_3$ and $(-\epsilon)_3$ of period-characteristics will always carry a curve f of the corresponding system into a curve $\bar{f}$ of the same system, and then $\bar{f}$ can be carried into f by keeping the Γ_6 fixed. We conclude that it is permissible to transform any generating relation of the fundamental group of f by those and only those transformations of the loops g_i (i $\cdots$, 6), which leave invariant the corresponding pair of period-characteri

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* *Zur Theorie der binären Formen sechster Ordnung*, 1869.

We write below the formulas for the loop transformations and th corresponding transformations of the period-characteristics in the hyperel case of any genus p:

$$y^2=(x-a_1)(x-a_2)\cdots(x-a_{2p+2}). \tag{17}$$

Let L be a simple closed curve passing through the points a_i, and let 0 be a point inside the region Q bounded by L. We construct a set of non-int ing loops $g_i=0a_i$, meeting L at the respective points a_i alone. As in se 4, we denote by T_i that transformation of the loops which arises when carried into a_{i+1} along a path situated in the interior of Q and a_{i+1} is ca into a_i along a path situated in the exterior of Q. Then, for $i=1, 2,$ $2p-1$, we have:

$$T_i: g'_i=g_{i+1};\ \ g'_{i+1}=g_{i+1}^{-1}g_ig_{i+1};\ \ g'_j=g_j,\ (j\neq i,\ i+1).$$

We construct the two-sheeted Riemann surface R of the curve (17) by jo sheets along the $p+1$ cuts: a_1a_2; a_3a_4; $\cdots$; $a_{2p+1}a_{2p+2}$, made alon corresponding arcs of the curve L. We fix then a set of $2p$ circuits b_i, a_i ($i=1, 2, \cdots, p$) in the usual manner: let b_i be a circuit in the upper surrounding the cut $a_{2i-1}a_{2i}$ alone, and let a_i be a circuit which crosse cuts $a_{2i-1}a_{2i}$, $a_{2p+1}a_{2p+2}$ alone. Then, on R,

$$b_i=g_{2i-1}-g_{2i};\ a_i=g_{2i}-g_{2i+1}+g_{2i+2}-\cdots+g_{2p}-g_{2p+1}. \tag{18}$$

Let

$$\|\,0,0,\cdots,\ 0,\pi i,0,\cdots 0;\ \ a_{j1},a_{j2},\cdots,a_{jp}\,\|,\ a_{ij}=a_{ji},$$

be the normalized period matrix defined by the chosen set of circuits b_i and Let $g_1, g_2, \cdots, g_p; h_1, h_2, \cdots, h_p$* be the elements of a period characteristic $(g;h)$. We denote by R_i and Q_i the transformation of the circuits b_i, a_i and the transformation of the characteristic $(g;h)$ respectively, which are involv by the transformation T_i of the loops. We have then, by means of the formulas (18), the following formulas for the transformations R_i:

$$R_{2i-1}: \qquad \begin{cases} b_k'=b_k, & (k=1,2,\cdots,p); \\ a_i'=a_i-b_i;\ \ a_k'=a_k, & (k\neq i). \end{cases}$$
$$(i=1,2,\cdots,p)$$

$$R_{2i}: \qquad \begin{cases} b_i'=b_i+a_i-a_{i+1};\ \ b'_{i+1}=b_{i+1}-a_i-a_{i+1}; \\ \qquad\qquad b_k'=b_k\ \ (k\neq i,\ i+1), \\ a_k'=a_k, \qquad (k=1,2,\cdots,p). \end{cases}$$
$$(i=1,2,\cdots,p-1)$$

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* The reader will not confuse the elements g of the characteristic with the generators g of the fundamental group.

$$R_{2p}: \quad \begin{cases} b_p' = b_p + a_p;\ b'_k = b_k, \quad (k \neq p); \\ a_k' = a_k, \quad (k = 1, 2, \cdots, p). \end{cases}$$

$$R_{2p+1}: \begin{cases} b_k' = b_k; \\ a_k' = a_k - b_1 - b_2 - \cdots - b_p. \qquad (k = 1, 2, \cdots, p). \end{cases}$$

Consequently, the corresponding transformations Q_i of the period-characteristics are:[*]

$$\begin{matrix} Q_{2i-1}: \\ (i = 1, 2, \cdots, p) \end{matrix} \quad \begin{cases} g_k' = g_k, \quad (k = 1, 2, \cdots, p); \\ h_i' = h_i - g_i;\ h_k' = h_k, \quad (k \neq i). \end{cases}$$

$$\begin{matrix} Q_{2i}: \\ (i = 1, 2, \cdots, p-1) \end{matrix} \quad \begin{cases} g_i' = g_i + h_i - h_{i+1};\ g'_{i+1} = g_{i+1} - h_i + h_{i+1}; \\ \qquad\qquad g_k' = g_k, \quad (k \neq i, i+1); \\ h_k' = h_k, \quad (k = 1, 2, \cdots, p-1). \end{cases}$$

$$Q_{2p}: \begin{cases} g_p' = g_p + h_p; \quad g_k' = g_k, \quad (k \neq p); \\ h_k' = h_k, \quad (k = 1, 2, \cdots, p). \end{cases}$$

$$Q_{2p+1}: \begin{cases} g_k' = g_k; \qquad (k = 1, 2, \cdots, p). \\ h_k' = h_k - g_1 - g_2 - \cdots - g_p. \end{cases}$$

We start with a generating relation corresponding to a simple tangent to the sextic curve f. This relation establishes the equality between two generators, and we may suppose that the equal generators are g_1 and g_2:

$$g_1 = g_2. \tag{19}$$

Moreover it is unessential with what period-characteristic we start. We may suppose that the curve f corresponds to the characteristic:

$$(0, 0; 1/3, 1/3). \tag{20}$$

However, it must be pointed out explicitly that the generating relation (19) implies that the points a_1 and a_2 can be interchanged by means of the transformation T_1. Hence a condition upon the characteristic is introduced, namely, that it be invariant under Q_1. The above chosen characteristic (20) satisfies this condition.

The characteristic (20) is left invariant by the transformations Q_2, Q_3

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[*] See A. Krazer, *Lehrbuch der Thetafunktionen*, Leipzig, Teubner, 1903.

and Q_3. Transforming the relation (19) by the transformation T_2 and by T_3 we obtain

$$g_1 = g_2 = g_3 = g_4.$$

Since the cube of Q_4 leaves any characteristic invariant modulo 3, we transform the relation $g_1 = g_4$ by the cube of T_4 and we obtain the following relation:

$$g_1 g_5 g_1 = g_5 g_1 g_5. \tag{21}$$

Finally we transform the last relation by T_5 and we obtain

$$g_1 g_6 g_1 = g_6 g_1 g_6. \tag{22}$$

The relation $g_1 g_2 \cdot\cdot\cdot g_6 = 1$ becomes in this case

$$g_1^4 g_5 g_6 = 1.$$

Hence $g_6 = g_5^{-1} g_1^{-4}$, and substituting in (22), we obtain, by making use of (21) after simple transformations, the following relation:

$$(g_1 g_5 g_1)^2 = 1. \tag{23}$$

Since, by (21)

$$g_1 g_5 g_1 \cdot g_1 g_5 g_1 = g_1 g_5 g_1 \cdot g_5 g_1 g_5 = (g_1 g_5)^3,$$

it follows that

$$(g_1 g_5)^3 = 1. \tag{24}$$

Hence (23) and (24) are the generating relations of the group G, which is generated by the two elements g_1 and g_5. We may take $u = g_1 g_5 g_1$, $v = g_1 g_5$ as new generators, and we can state our result as follows:

The fundamental group of a sextic curve f possessing 6 cusps on a conic (branch curve of the general cubic surface) is generated by two elements of orders 2 and 3 respectively.

9. *The fundamental group of the three-cuspidal quartic.* If the cubic surface, considered above, acquires a double line, then the branch sextic curve degenerates into a line $\lambda_1 = 0$ counted twice and into a three-cuspidal quartic

$$f_4(x, y) = 0. \tag{25}$$

The last one is of special interest, inasmuch as it is the irreducible curve of lowest order, whose fundamental group is not cyclic (see Theorem 8, sec. 3).

The set Γ_6 of 6 points on the x-axis is made up in this case of a double point, say $x = \xi$, and of 4 simple points—intersections of $y = 0$ with the quartic (25). We have then

$$(26) \qquad f(x, 0) = f_4(x, 0)(x - \xi)^2 = \phi_2^3(x, 0) + \phi_3^2(x, 0).$$

Here, in general, i. e. for an arbitrary choice of the line $y = 0$, $\phi_2(x, 0)$ and $\phi_3(x, 0)$ do not contain $x - \xi$ as a factor. Indeed, in the contrary case it would follow that the conic ϕ_2 and the cubic ϕ_3 degenerate into the line λ_1 and into residual components. The equation of the quartic curve (25) would be in this case

$$\lambda_1\mu_1^3 + \lambda_2^2 = 0,$$

where $\mu_1 = 0$ is a line and $\lambda_2 = 0$ is a conic. But this equation represents a quartic with two cusps only.* We will call the representation (26) *non-degenerate,* if $x - \xi$ is not a factor of $\phi_2(x, 0)$ and $\phi_3(x, 0)$. It can be easily proved in a manner similar to that adopted by Clebsch for the general case, that if $f_4(x, 0)$ is given, then ξ may be taken arbitrarily, and once ξ is fixed, the problem of finding the non-degenerate representation (26) is connected with the problem of the trisection of the periods in the case of the elliptic curve

$$y^2 = f_4(x, 0).$$

There exist namely 4 different non-degenerate representations of the polynomial $f_4(x, 0)(x - \xi)^2$ as a sum of a square and a cube, each representation corresponding to two third-period characteristics $(\epsilon)_3$, $(-\epsilon)_3$.

We deduce that the three-cuspidal quartics which pass through 4 given points on a line belong to 4 different continuous systems. As in the preceding section we find the fundamental group of the quartic by starting from the relation $g_1 = g_2$, and by transforming it by those transformations of the loops which leave a conveniently chosen characteristic invariant, say the characteristic (0; 1/3). We obtain thus the following generating relations:

(a) $g_1 = g_2$;

(b) $g_1g_3g_1 = g_3g_1g_3$;

(c) $g_1g_4g_1 = g_4g_1g_4$;

(d) $g_1^2g_3g_4 = 1$.

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* The statement in the text can be proved also by observing that if the cubic surface possesses a double line, the polar plane of a generic point of the space does not pass through the double line.

Substituting in (c) the expression of g_2, determined by (d), we find

$$g_1g_3g_1g_3^{-1}g_1g_3 = 1,$$

and, by (b), this relation becomes

$$\text{(e)} \qquad g_1^2g_3^2 = 1.$$

We have

$$(g_1g_3)^3 = g_1g_3g_1\cdot g_3g_1g_3 = g_1g_3g_1^2g_3g_1 = g_1^2,$$

and moreover,

$$(g_1g_3)^3 = g_3g_1g_3\cdot g_3g_1g_3 = g_3^2.$$

Hence $g_1^2 = g_3^2$, and, by (e), $g_1^4 = g_3^4 = 1$.
We find the following complete set of generating relations:

$$\text{(27)} \qquad g_1^2 = g_3^2; \qquad g_1^4 = 1; \qquad (g_1g_3)^3 = g_1^2.$$

The group defined by the relations (27) *is a finite group of order* 12. It is isomorphic with the *regular* transitive group of degree 12, generated by the following substitutions:

$$\text{(28)} \qquad \begin{aligned} s_1 &= (z_1z_2z_3z_4)(z_5z_6z_7z_8)(z_9z_{10}z_{11}z_{12});\\ s_3 &= (z_1z_5z_3z_7)(z_2z_9z_4z_{11})(z_6z_{10}z_8z_{12}). \end{aligned}$$

The problem of finding all types of algebraic functions which can be constructed upon the three-cuspidal quartic as branch curve, is equivalent to the problem of finding all transitive groups of substitutions, which can be generated by two substitutions s_1 and s_3, satisfying the above relations (27). Since the abstract group G is of order 12, we may have, as possible groups of substitutions, outside the regular group of degree 12, mentioned above, groups of degrees 6, 4, 3 and 2. It is easily shown that the only possible group of degree 4 is here the cyclic group, generated by the substitution

$$s_1 = s_3 = (z_1z_2z_3z_4).$$

We find a transitive *regular* group of degree 6, in a (2, 1) isomorphism with the abstract group G_{12}, by putting:

$$\text{(29)} \qquad s_1 = (z_1z_2)(z_3z_4)(z_5z_6); \qquad s_3 = (z_1z_3)(z_2z_5)(z_4z_6).$$

Finally we obtain the total group of degree 3, by putting

$$\text{(30)} \qquad s_1 = (z_1z_2), \qquad s_3 = (z_1z_3).$$

Hence, we have the following theorem:

Excluding the obvious cases of the four-valued and of the two-valued functions possessing a cyclic monodromie group, there are only three types of birationally distinct irreducible algebraic functions, which admit the three cuspidal quartic as branch curve, one of order 3 *(and this gives rise to the cubic surface with a double line), one of order* 6 *(which is defined by the resolvent of the cubic equation in the previous case), and finally one of order* 12. *The monodromie group of each of these functions is generated by two substitutions, indicated by the formulas* (30), (29) *and* (28) *respectively.*

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A Class of Polynomials and Rational Functions in Four Variables.

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1. *Introduction.* Let (w, x, y, z) be the coordinates of the

$$q \equiv w + ix + jy + kz, \tag{1.1}$$

where w, x, y, z are independent variables, real or complex, and let n teger. Then q^n is a quaternion; let its coordinates be $(W_n, X_n,$

$$T_m = T_m(w, x, y, z) \qquad (T = W, X, Y, Z). \tag{1.2}$$

If $m \geqq 0$, T_m is a homogeneous polynomial of degree m in w, x, y, z. s an integer $\geqq 0$, we shall see that

$$W_{-n} = W_n/N^n, \qquad T_{-n} = -T_n/N^n \qquad (T = X, Y, Z) \tag{1.3}$$

where N is the norm of q,

$$\begin{aligned} N &= N(w, x, y, z) = w^2 + x^2 + y^2 + z^2, \\ N^n &= W_n^2 + X_n^2 + Y_n^2 + Z_n^2; \end{aligned} \tag{1.4}$$

and hence, if $m < 0$, $T_m (T = W, X, Y, Z)$ is a rational function of and homogeneous. Thus T_m in all cases is homogeneous of degree where m is an integer $\gtreqqless 0$.

We shall see also that

$$T_{m+2} - 2wT_{m+1} + NT_m = 0 \qquad (T = W, X, Y, Z) \tag{1.5}$$

for all integers m. Let S be the tensor of q; then

$$S \equiv S(w, x, y, z) \equiv x^2 + y^2 + z^2. \tag{1.6}$$

Let $\alpha, \beta,$

$$\alpha = \alpha(w, x, y, z), \qquad \beta \equiv \beta(w, x, y, z) \tag{1.7}$$

be the roots of

$$\theta^2 - 2w\theta + N = 0. \tag{1.8}$$

Then we may take

$$\alpha = w + iS^{1/2}, \qquad \beta = w - iS^{1/2}, \tag{1.9}$$

where i is the imaginary unit of algebra [there can be no confusion between this and i as in (1.1)], and any solution of (1.5) is of the form

$$(1.10) \qquad T_n = a\alpha^n + b\beta^n \qquad (n = 0, \pm 1, \pm 2, \cdots),$$

where a, b are arbitrary constants. The particular pair U_n, V_n of solutions

$$(1.11) \qquad U_n \equiv (\alpha^n - \beta^n)/(\alpha - \beta), \qquad V_n \equiv \alpha^n + \beta^n,$$

whose initial values are $(U_0, U_1) = (0, 1)$, $(V_0, V_1) = (2, 2w)$, will be required, as in terms of them the T_n are simply expressible. The U_n, V_n are the well known "singly periodic numerical functions" of E. Lucas,[*] for the *parameter* $(2w, N)$; that is, his P, Q are our $2w$, N respectively. From (1.8), (1.11) we have

$$(1.12) \qquad U_{-n} = -N_n/N^n, \qquad V_{-n} = V_n/N^n,$$

for all integers n. When necessary we shall indicate the variables w, x, y, z, of which U_n, V_n are functions, by writing

$$(1.13) \qquad T_n \equiv T_n(w, x, y, z) \qquad (T = U, V).$$

The polynomials and rational functions defined in (1.2) are those to be discussed. They have many interesting properties. Thus, their close connection with the Lucas functions for the parameter $(2w, N)$ enables us to transfer directly a great part of Lucas' theory to the T_n. Again, the T_m, $m > 0$, are Appell polynomials in w, but not in x, y, or z; with respect to a properly chosen argument they can be exhibited as Tchebycheff polynomials; they are resolvable, in an elegant way, into factors of the first degree; with respect to w, x, y, z the $T_n (n \gtreqless 0)$ are solutions of an interesting set of self-adjoint linear differential equations of the second order. When $N = 1$, there is an isomorphism between the algebraic properties of the $T_n (T = W, X, Y, Z)$ and the σ, ζ, $\wp$ of Weierstrass.

There is an immediate generalization of the T_n, obtained by replacing n in (1.11) by a real or complex variable. Practically all of the algebra developed here for the T_n goes over with but slight changes, if any, to the generalization; but as the complete development requires a revision of certain parts of Lucas' theory, we shall defer discussion of it to another occasion, and restrict m in (1.2) to be an integer. A similar generalization is obviously feasible from the differential equations for the polynomials, precisely as in passing from the Bessel coefficients to the Bessel functions.

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[*] *American Journal of Mathematics*, Vol. 1 (1878), pp. 184, 289.

2. *The* T_n $(T = W, X, Y, Z)$. Throughout this section n is an integer $\geqq 0$ and q, q' are quaternions. Let the coordinates of q, q' be (w, x, y, z), (w', x', y', z') respectively, and let the coordinates of qq' be (W, X, Y, Z). Then

$$(2.1)\qquad \begin{aligned} W &= ww' - xx' - yy' - zz', \\ X &= wx' + xw' + yz' - zy', \\ Y &= wy' + yw' + zx' - xz', \\ Z &= wz' + zw' + xy' - yx'. \end{aligned}$$

Let the coordinates of q^n be (W_n, X_n, Y_n, Z_n). Then, multiplication of quaternions being associative, $q^n q = qq^n = q^{n+1}$, and hence, by (2.1), we have

$$(2.2)\qquad zY_n - yZ_n = xZ_n - zX_n = yX_n - xY_n = 0,$$

and

$$(2.3)\qquad \begin{aligned} W_{n+1} &= wW_n - xX_n - yY_n - zZ_n, \\ X_{n+1} &= xW_n + wX_n \qquad\qquad , \\ Y_{n+1} &= yW_n \qquad + wY_n \qquad , \\ Z_{n+1} &= zW_n \qquad\qquad + wZ_n. \end{aligned}$$

From (2.2), (2.3), by a short calculation which need not be reproduced, we have

$$(2.4)\qquad T_{n+2} - 2wT_{n+1} + NT_n = 0 \qquad (T = W, X, Y, Z)$$

and from q^0, q, we see that the pairs of initial values which, with (2.4), completely define the T_n for $n = 0, 1, 2, \cdots$, are

(2.5)

$$\begin{aligned} (W_0, W_1) &= (1, w), & (X_0, X_1) &= (0, x), \\ (Y_0, Y_1) &= (0, y), & (Z_0, Z_1) &= (0, z). \end{aligned}$$

It is clear that T_n is a homogeneous polynomial of degree n in w, x, y, z.

Denote the reciprocal of q by q^{-1}, so that $qq^{-1} = q^{-1}q = 1$.

$$(2.6)\qquad (q^{-1})^n q^n = q^n (q^{-1})^n = (qq^{-1})^n = 1.$$

Let the coordinates of q^{-1} be (w', x', y', z'). Then

$$(2.7)\qquad (w', x', y', z') = (w/N, -x/N, -y/N, -z/N).$$

Define the $T_{-n} \equiv T_{-n}(w, x, y, z)$ by

$$(2.8)\qquad (q^{-1})^n = W_{-n} + iX_{-n} + jY_{-n} + kZ_{-n};$$

namely, the coordinates of $(q^{-1})^n$ =are $(W_{-n}, X_{-n}, Y_{-n}, Z_{-n})$. Write

$$(2.9)\qquad T_n' \equiv T(w', x', y', z') \qquad (T = W, X, Y, Z).$$

Then $T_n' = T_{-n}$, or in full,

$$T_n(w/N, -x/N, -y/N, -z/N) = T_{-n}(w, x, y, z). \tag{2.10}$$

From (2.6) we readily infer by (2.1) that

$$W_{-n} = W_n/N^n, \qquad T_{-n} = -T_n/N^n \qquad (T = X, Y, Z). \tag{2.11}$$

Solving (2.11) for W_n, T_n, we see that the result could be obtained by changing the sign of n in (2.11). Hence the theorem (1.3) is proved.

The theorem (1.5) follows at once from (1.3) and (2.4).

By (1.3) we may remove from the integer n in (2.2) the restriction that it be $\geqq 0$; n in (2.2) is therefore now any integer. Similarly for (2.3) and (2.4).

Hence finally the integer $n \geqq 0$ in all formulas concerning T_n $(T = W, X, Y, Z)$ in this section can be replaced by the integer $m \gtreqless 0$.

3. *The functions* U_n, V_n $(n \gtreqless 0)$. As already defined in (1.11), (1.13), U_n, V_n are the Lucas functions for the parameter $(2w, N)$. If now Θ_n $(n = 0, 1, 2, \cdots)$ is any solution of

$$\Theta_{n+2} - 2w\Theta_{n+1} + N\Theta_n = 0, \tag{3.1}$$

we can express Θ_n as a linear homogeneous function of the linearly independent solutions of U_n, V_n. We find

$$\Theta_n = (\Theta_1 - w\Theta_0)U_n + \tfrac{1}{2}\Theta_0 V_n. \tag{3.2}$$

Hence, by (2.5), we have

$$W_n = \tfrac{1}{2}V_n, \qquad X_n = xU_n, \qquad Y_n = yU_n, \qquad Z_n = zU_n, \tag{3.3}$$

for all integers n.

Thus in any formula or theorem concerning the Lucas functions U_n, V_n for the parameter $(2w, N)$, we may replace U_n by any one of X_n/x, Y_n/y, Z_n/z, and V_n by $2W_n$, to obtain a formula or theorem for the T_n $(T = W, X, Y, Z)$.

From (1.2), (1.6), (1.7) we have the following parity properties of the functions α, β, U_n, V_n,

$$\begin{aligned} \alpha(-w, x, y, z) &= -\beta(w, x, y, z), \\ \alpha(w, -x, -y, -z) &= \alpha(w, x, y, z), \\ \beta(w, -x, -y, -z) &= \beta(w, x, y, z); \end{aligned} \tag{3.4}$$

$$\begin{aligned} U_n(-w, x, y, z) &= (-1)^{n-1}U_n(w, x, y, z), \\ V_n(-w, x, y, z) &= (-1)^n V_n(w, x, y, z); \\ U_n(w, -x, -y, -z) &= U_n(w, x, y, z), \\ V_n(w, -x, -y, -z) &= V_n(w, x, y, z); \end{aligned} \tag{3.5}$$

and therefore by (3.3),

$$W_n(-w, x, y, z) = (-1)^n W_n(w, x, y, z),\quad W_n(w, -x, -y, -z) = W_n(w, x, y, z); \tag{3.6}$$

$$T_n(-w, x, y, z) = (-1)^{n-1} T_n(w, x, y, z),\quad T_n(w, -x, -y, -z) = -T(w, x, y, z),\quad (T = X, Y, Z). \tag{3.7}$$

4. *Explicit Forms.* Let n be an integer $\geqq 0$, and define

$$\phi_n \equiv \phi_n(w\ x, y, z), \qquad \psi_n \equiv \psi_n(w, x, y, z) \tag{4.1}$$

by

$$\phi_{2n} = \psi_{2n+1} = 0, \qquad \phi_{2n+1} = \psi_{2n} = (-S)^n, \tag{4.2}$$

where S is as in (1.6). Then, from (1.11), (1.9) we have

$$U_n = (w + \phi)^n, \qquad V_n = 2(w + \psi)^n; \tag{4.3}$$

and therefore, by (3.3),

$$W_n = (w + \psi)^n, \quad X_n = x(w + \phi)^n, \quad Y_n = y(w + \phi)^n, \quad Z_n = z(w + \phi)^n, \tag{4.4}$$

in which, after expansion by the binomial theorem ψ^r, ϕ^r $(r = 0, \cdots, n)$ are to be replaced by ψ_r, ϕ_r respectively, as always in the symbolic or umbral calculus. Together, (4.4) and (1.3) give the explicit forms of the T_m for all integers m.

Following Lucas (*loc. cit.*, p. 189), we can express U_n, V_n in terms of circular or hyperbolic functions of an appropriate argument. Principal determinations of all irrationals are to be understood. We find

$$U_n = (N^n/S)^{\frac{1}{2}} \sin\left\{ ni \log\left(\frac{w - iS^{\frac{1}{2}}}{N^{\frac{1}{2}}} \right)\right\},\quad V_n = 2N^{\frac{1}{2}} \cos\left\{ ni \log\left(\frac{w + iS^{\frac{1}{2}}}{N^{\frac{1}{2}}} \right)\right\}. \tag{4.5}$$

From (3.3) we may write down by (4.5) the trigonometric forms of the T_n. It is unnecessary to transcribe them. The above have been reduced. By means of the following, equivalent to (4.5) and its consequences, the whole of trigonometry can be transposed to relations between the T_n. Write

$$\lambda \equiv \lambda(w, x, y, z) \equiv (i/2) \log(\alpha/\beta). \tag{4.6}$$

Then, from (4.5) we have

$$\sin n\lambda = -(S/N^n)^{\frac{1}{2}} U_n, \qquad \cos n\lambda = \tfrac{1}{2} N^{-n/2} V_n; \tag{4.7}$$

$$(4.8) \qquad \sin n\lambda = -(S/N^n)^{1/2}\cdot T_n/t, \qquad \cos n\lambda = (1/N^n)^{1/2}\, W_n,$$
$$(T,t) = (X,x),\ (Y,y),\ (Z,z).$$

As in previous sections it follows at once that the restriction $n \geqq 0$ can be removed from (4.5)-(4.8), and that these results are valid for all integers n.

The connection with Tchebycheff polynomials is now evident. As we have already used T, U, the customary letters for these polynomials, we shall define the Tchebycheff polynomials $L_n(u)$, $M_n(u)$, first for n an integer $\geqq 0$, (L, M here replace T, U in the usual notation) by

$$(4.9) \qquad \cos\lambda \equiv u, \qquad L_n(u) \equiv \cos n\lambda, \qquad \sin\lambda = (1-u^2)^{1/2},$$
$$M_n(u) \equiv \frac{1}{n+1}\ \frac{d}{du}\ L_{n+1}(u) = \frac{\sin(n+1)\lambda}{\sin\lambda},$$

so that $L_m(u)$ is a polynomial of degree m in u in which the coefficient of u^m is 2^{m-1}, and $M_m(u)$ is a polynomial of degree m in u, with leading coefficient 2^m $(m = 1, 2, \cdots)$. From (4.7)-(4.9) we have

$$(4.10) \qquad T_{n+1} = -t\,[N^{n+1}S^{-1}(1-u^2)]^{1/2}\, M_n(u),$$
$$(T,t) = (U,1),\ (X,x),\ (Y,y),\ (Z,z);$$

$$(4.11) \qquad T_n = tN^{n/2}L_n(u), \qquad (T,t) = (V,2),\ (W,1),$$

which, so far, have been proved only for $n = 0, 1, \cdots$. But, taking in (4.9) only the second form of the definition of $M_n(u)$, we see that the definitions,

$$(4.12) \qquad L_{-n}(u) \equiv L_n(u), \qquad M_{-n}(u) \equiv -M_{n-2}(u),$$

the second of which implies $M_{-1}(u) = 0$, are consistent with everything that precedes; so that finally, as in previous extensions from $n \geqq 0$ to $n \underset{<}{\geqq} 0$, we see that (4.10), (4.11) hold for all integers $n \underset{<}{\geqq} 0$.

5. *Algebraic relations between the* T_n. An indefinite number of algebraic relations between the T_n $(T = W, X, Y, Z)$ can be developed isomorphically with the theory of the circular or the hyperbolic functions, either from Lucas' functions or directly from (4.8) or its equivalent in terms of hyperbolic functions. Write

$$(5.1) \qquad \mu \equiv \mu(w, x, y, z) = -i\lambda,$$

with λ as in (4.6). Then

$$(5.2) \qquad \operatorname{sh} n\mu = i(S/N^n)^{1/2}U_n, \qquad \operatorname{ch} n\mu = \tfrac{1}{2}N^{-n/2}V_n;$$

$$(5.3)\qquad \operatorname{sh} n\mu = i(S/N^n)^{\frac{1}{2}}\cdot T_n/t, \qquad \operatorname{ch} n\mu = (1/N^n)^{\frac{1}{2}} W_n,$$
$$(T,t) = (X,x),\ (Y,y),\ (Z,z),$$

for all integers n.

The number of relations obtainable being unlimited, there is no point in writing out an extensive selection, as all follow, without computations, by the means indicated. We may, however, notice a few, some of which will be found useful in verifying an algebraic isomorphism with the elliptic and allied functions of Weierstrass. In what follows, m, n are arbitrary integers.

Corresponding to $ch^2\theta + sh^2\theta = 1$, we have

$$(5.4)\qquad t^2W_n{}^2 + ST_n{}^2 = t^2N^n,$$

for the same (T,t) as in (5.3). The addition and subtraction theorems for the T_n ($T = W, X, Y, Z$) with respect to n are

$$(5.5)\qquad \begin{aligned} T_{m+n} &= T_mW_n + W_mT_n,\\ N^nT_{m-n} &= T_mW_n - W_mT_n, \end{aligned}$$

for (T,t) as in (5.3); for the same (T,t),

$$(5.6)\qquad \begin{aligned} t^2W_{m+n} &= t^2W_mW_n - ST_mT_n,\\ T^2N^nW_{m-n} &= t^2W_mW_n + ST_mT_n. \end{aligned}$$

The addition theorems can of course be obtained from $q^mq^n = q^{m+n}$ and (2.1), (2.2); the subtraction theorems then follow by (1.3). If this method be used, we find

$$W_{m+n} = W_mW_n - X_mX_n - Y_mY_n - Z_mZ_n,$$

which, by (3.3), is equivalent to the first of (5.6).

From (5.5) we get an important identity, valid only when $N = 1$, $T = X, Y, Z$:

$$(5.7)\qquad (W_m/T_m)^2 - (W_n/T_n)^2 = -(T_{m+n}T_{m-n}/T_m{}^2T_n{}^2).$$

The equivalent of De Moivre's theorem has many interesting consequences, of which we record only the following multiplication theorems. Let r be an integer $\geqq 0$. Then, from (3.3) and

$$2\alpha^n = V_n + 2iS^{\frac{1}{2}}U_n,$$

we infer, for $(T,t) = (X,x),\ (Y,y),\ (Z,z)$, that

$$(tW_n + iS^{\frac{1}{2}}T_n)^m = t^{m-1}(tW_{nm} + iS^{\frac{1}{2}}T_{nm}),$$

for all integers n, m. Take $m = r$. Then we find at once

(5.8) $$t^r W_{nr} = (tW_n + \psi T_n)^r, \qquad t^{r-1}T_{nr} = (tW_n + \phi T_n)^r,$$

where ϕ, ψ are as in (4.2), valid for all integers n and for all integers $r \geqq 0$. Note that the zeroth power of a binomial involving ϕ or ψ is ϕ_0 or ψ_0.

We shall pass over the numerous algebraic properties that parallel those of Lucas' functions, such as the expression of the polynomials as recurrents, or as terminating continued fractions, their remarkable divisibility properties, and note only two arithmetical theorems. From identities such as (5.4) we infer parametric solutions of diophantine equations. Thus, if in (5.4), n be restricted to be $\geqq 0$, and if k is an integer > 0 not of the form $4^h(8s + 7)$, h, $s \geqq 0$, we find immediately at least 12 $E(k)$ sets $(\xi, \eta, \zeta, \theta)$, of solutions, where $E(k)$ is the usual class number function, of

$$\xi^2 + k\eta^2 = \zeta^2\theta^n,$$

where k, n are given constant integers of the kinds prescribed. Again, if p is a prime > 1 of the form $4h + 1$, the quotient $4T_{pr}/T_r$ $(T = X, Y, Z)$, where r is an integer > 0, is expressible in the form $P^2 - pQ^2$, where P, Q are polynomials in N and functions $2W_j$, for determinate values of $j \geqq 0$, with integer coefficients. This is but one of several of a similar kind. Thus, for primes $p = 4h + 3$, the form is of the type $-4SP^2 + pQ^2$. These follow from Lucas' expression (*loc. cit.*, p. 227) of $4U_{pr}/U_r$ as a cyclotomic quadratic form.

6. *Linear relations.* The following are useful in computations with the T_n. Let n be an integer $\gtreqless 0$, unless otherwise stated. The general solution Θ_n of (3.1) may be written in the form

(6.1) $$(\alpha - \beta)\Theta_n = (\Theta_1 - \beta\Theta_0)\alpha^n + (\alpha\Theta_0 - \Theta_1)\beta^n,$$

where Θ_0, Θ_1 are arbitrary constants. Hence

(6.2) $$(\alpha - \beta)N^n\Theta_{-n} = (\alpha\Theta_0 - \Theta_1)\alpha^n + (\Theta_1 - \beta\Theta_0)\beta^n,$$

and therefore, on multiplication and reduction,

(6.3) $$\begin{aligned} 4SN^n\Theta_n\Theta_{-n} = {} & (\Theta_1^2 - 2w\Theta_1\Theta_0 + N\Theta_0^2)V_{2n} \\ & - 2\,[\Theta_1^2 - 2w\Theta_0\Theta_1 + (w^2 - S)\Theta_0^2]\,N^n; \end{aligned}$$

also, by addition,

(6.4) $$\Theta_n + N^n\Theta_{-n} = \Theta_0 V_n.$$

Hence, if $\Theta_0 = 0$,

(6.5) $$\Theta_{-n} = -\Theta_n/N^n \qquad (\Theta_0 = 0),$$

which includes the results previously stated for U, X, Y, Z; that for W is included in (6.4), by (3.3).

Let Θ_n, Φ_n be linearly independent solutions of (3.1), so that

$$D = \Theta_0\Phi_1 - \Phi_0\Theta_1 \neq 0. \tag{6.6}$$

Then we can express any solution A_n as a linear homogeneous function of Θ_n, Φ_n. We find

$$DA_n = (A_0\Phi_1 - A_1\Phi_0)\,\Theta_n + (A_1\Theta_0 - A_0\Theta_1)\,\Phi_n. \tag{6.7}$$

The necessary initial values for (6.7) and subsequent equations are

$$\begin{array}{lll} U_0 = 0, & U_1 = 1, & U_2 = 2w; \\ V_0 = 2, & V_1 = 2w, & V_2 = 2(w^2 - S); \\ W_0 = 1, & W_1 = w, & W_2 = w^2 - S; \\ T_0 = 0, & T_1 = t, & T_2 = 2wt, \end{array} \tag{6.8}$$
$$(T, t) = (X, x),\ (Y, y),\ (Z, z).$$

Hence the condition (6.6) is satisfied only for the eight pairs (Θ, Φ) as follows,

$$\begin{array}{llll} (U, V), & (U, W), & (V, X), & (V, Y), \\ (V, Z), & (X, W), & (Y, W), & (Z, W). \end{array}$$

Since $V_n = 2W_n$, it suffices to consider only V_n or W_n in the sequences of (6.7), (6.8), say W_n. Then, expressing any solution A_n of (3.1) in terms of the W_n, X_n, Y_n, Z_n, we find

$$A_n = (A_1 - wA_0)\,U_n + A_0W_n; \tag{6.9}$$

$$tA_n = (A_1 - wA_0)\,T_n + tA_0W_n, \quad (T, t) = (X, x),\ (Y, y),\ (Z, z), \tag{6.10}$$

either of which implies the other by (3.3). Thus there is but one [illegible] necessary, say (6.9). By (6.8) we readily check (6.9) on (3.3).

A useful alternative to (6.9) is the erpression of A_{n+1} as a linear homogeneous function of Θ_{n+1}, Θ_n, where A_n, Θ_n are any solutions of (3.1):

$$(\Theta_1^2 - \Theta_0\Theta_2)A_{n+1} = (A_1\Theta_1 - A_2\Theta_0)\,\Theta_{n+1} + (A_2\Theta_1 - A_1\Theta_2)\,\Theta_n. \tag{6.11}$$

Hence, by (6.8), we have

$$tA_n = A_1T_n + (A_2 - 2wA_1)\,T_{n-1}, \tag{6.12}$$
$$(T, t) = (U, 1),\ (X, x),\ (Y, y),\ (Z, z);$$

$$tSA_n = (wA_1 - A_2)\,T_n + [wA_2 - (w^2 - S)A_1]\,T_{n-1}, \tag{6.13}$$
$$(T, t) = (W, 1),\ (V, 2).$$

These give the following, which, with (3.3), are all of the types (6.12), (6.13) for the W, X, Y, Z,

$$
\begin{aligned}
&(6.14) && ST_n = -t(wW_n - NW_{n-1}) = t(wW_n - W_{n+1}),\\
& && tW_n = wT_n - NT_{n-1} = T_{n+1} - wT_n,\\
& && (T, t) = (X, x),\ (Y, y),\ (Z, z).
\end{aligned}
$$

Similarly, and using (3.1) with $\Theta = U,\ V$, we find

$$(6.15) \qquad 2SU_n = -wV_n + NV_{n-1} = wV_n - V_{n+1},$$

$$(6.16) \qquad V_n = 2(wU_n - NU_{n-1}) = 2(-wU_n + U_{n+1}).$$

7. *Derivatives.* It will be evident that the total derivatives in all that follows, may, if desired, be replaced by partial derivatives. There is no gain in generality if partials be adopted. From (1.6), (1.9) we have

$$(7.1) \qquad (d\alpha/dw) = (d\beta/dw) = 1,$$

$$(7.2) \qquad S^{1/2}(d\alpha/dt) = -S^{1/2}(d\beta/dt) = it \qquad (t = x,\ y,\ z).$$

Since $\alpha - \beta = 2iS$ is independent of w, it follows from (1.11), (3.3) that

$$(7.3) \qquad (dT_n/dw) = nT_{n-1} \qquad (T = U,\ V,\ W,\ X,\ Y,\ Z),$$

for all integers n. Hence all the above six T_n are Appell polynomials in w. Otherwise, for $n > 0$, (7.3) is obvious from (4.3), (4.4); and for $n < 0$, (7.3) then follows from the same source on using (1.3).

From (7.1), (7.2), (1.11) we see that, for all integers n, and $t = x, y, z$,

$$(7.4) \qquad 2S(dU_n/dt) = t(nV_{n-1} - 2U_n),$$

$$(7.5) \qquad (dV_n/dt) = -2ntU_{n-1};$$

and hence, by (6.15), (6.16),

$$(7.6) \qquad S(dU_n/dt) = t[(n-1)U_n - nwU_{n-1}],$$

$$(7.7) \qquad S(dT_n/dt) = nt(T_n - wT_{n-1}) \qquad (T = V,\ W).$$

The following are immediate from (3.3), (7.6):

$$
\begin{aligned}
&(7.8) && S(dT_n/dt) = t[(n-1)T_n - nwT_{n-1}],\\
& && (T, t) = (X, y),\ (X, z),\ (Y, z),\ (Y, x),\ (Z, x),\ (Z, y),
\end{aligned}
$$

valid for all integers n. There remain to be calculated only the dT_n/dt for $(T, t) = (X, x),\ (Y, y),\ (Z, z)$. For these (T, t) we have $T_n = tU_n$. Hence, by (7.6),

$$tS(dT_n/dt) = [(n-1)t^2 + S]\,T_n - nwt^2T_{n-1}, \tag{7.9}$$
$$(T, t) = (X, x),\ (Y, y),\ (Z, z).$$

All 24 derivatives of the T_n $(T = U, V, W, X, Y, Z)$ with respect to w, x, y, z are thus given by (7.3), (7.6)-(7.9), valid for all integers n.

8. *Differential Equations.* For the moment let accents denote derivatives with respect to w. Then, from (7.3), for T as there,

$$T_n' = nT_{n-1}, \qquad T_n'' = n(n-1)T_{n-2}.$$

We have also

$$T_n - 2wT_{n-1} + NT_n = 0.$$

Eliminating T_{n-2}, T_{n-1} between these three equations we find

$$N(d^2T_n/dt^2) - 2(n-1)t\,(dT_n/dt) + n(n-1)T = 0, \tag{8.1}$$
$$(t = w;\ T = U, V, W, X, Y, Z).$$

Proceed similarly with (7.6), (7.8), which are of the same form. If for a moment we write

$$Q \equiv \frac{S}{t}\,\frac{dT_n}{dt}, \qquad P \equiv \frac{S}{t}\,\frac{dQ}{dt},$$

the elimination of T_{n-2}, T_{n-1} (as in the preceding derivation) gives

$$\begin{vmatrix} P - (n-1)^2T_n & n(2n-3) & -n(n-1)w^2 \\ Q - (n-1)T_n & n & 0 \\ T_n & -2 & w^2 + S \end{vmatrix} = 0,$$

which is satisfied for all the (T, t) to which (7.6), (7.8) refer. Hence, upon reduction we find, after a short calculation,

$$tNS(d^2T_n/dt^2) - A(dT_n/dt) + (n-1)(n-2)t^3T_n = 0, \tag{8.2}$$

where

$$A \equiv NS + t^2\,[2(n-1)S - 3N],$$
$$\begin{array}{rlll} (T,t) = & (U,x), & (U,y), & (U,z), \\ & & (X,y), & (X,z), \\ & (Y,x), & & (Y,z), \\ & (Z,x), & (Z,y) & . \end{array}$$

Similarly, from (7.9) we get

$$(8.3)\qquad t^2NS(d^2T_n/dt^2)-tB(dT_n/dt)+CT_n=0,$$

where

$$B\equiv 3NS+t^2[2(n-1)S-3N],\qquad C\equiv B+(n-1)(n-2)t^4,$$
$$(T,t)=(X,x),\quad (Y,y),\quad (Z,z).$$

Finally, (7.7) gives in the same way,

$$(8.4)\qquad tNS(d^2T_n/dt^2)-D(dT_n/dt)+n(n-1)t^3T_n=0,$$

where

$$D\equiv NS+t^2[2(n-1)S-N],$$
$$(T,t)=(V,x),\quad (V,y),\quad (V,z),\quad (W,x),\quad (W,y),\quad (W,z).$$

Thus (8.1)-(8.4) are the 24 possible differential equations for the six T_n $(T=U,\ V,\ W,\ X,\ Y,\ Z)$ with respect to the four independent variables $w,\ x,\ y,\ z$. The equations hold for all integers n.

The self-adjoint equivalents of the foregoing equations are sufficiently simple to be noticed. Write $T_n\equiv\theta$. Then, corresponding to (8.1)-(8.4) respectively, we have their self-adjoint equivalents (8.11)-(8.41),

$$(8.j1)\qquad (d/dt)\,[K_j(d\theta/dt)]-G_j\theta=0\qquad (j=1,\cdots,4),$$

where

$$K_1\equiv N^{1-n}\quad,\qquad G_1\equiv n(1-n)N^{-n};$$
$$K_2\equiv S^{3/2}N^{1-n}t^{-1},\qquad G_2\equiv(n-2)(1-n)tS^{1/2}N^{-n};$$
$$K_3\equiv S^{3/2}N^{1-n}t^{-3},\qquad -G_3\equiv Ct^{-5}S^{1/2}N^{-n};$$
$$K_4\equiv S^{1/2}N^{1-n}t^{-1},\qquad G_4\equiv n(1-n)tS^{-1/2}N^{-n}.$$

In (8.j1) the same values (and only those) as in (8.j) of (T,t) are of course to be understood.

9. *Resolutions into factors; zeros.* Unless otherwise stated, let n be an integer >0. Referring to (4.9), we observe that $M_n(u)$ has only the n zeros $\mu_{n,j}$, all distinct, where

$$(9.1)\qquad \mu_{n,j}\equiv\cos[j\pi/(n+1)]\qquad (j=1,2,\cdots,n),$$

and that $L_n(u)$ has only the n zeros $\lambda_{n,j}$, all distinct where

$$(9.2)\qquad \lambda_{n,j}\equiv\cos[(2j-1)\pi/2n]\qquad (j=1,2,\cdots,n).$$

Hence, from (4.10), (4.11), noting the leading coefficients in $M_n(u)$, $L_n(u)$ as stated after (4.9), and taking the value of u as given by (4.9), (4.7), we have the following resolutions into factors,

$$(9.3) \qquad T_{n+1} = 2^n t \prod_{j=1}^{n} (w - \mu_{n,j} N^{1/2}),$$

$$(T, t) = (U, 1), \quad (X, x), \quad (Y, y), \quad (Z, z);$$

and

$$(9.4) \qquad T_n = 2^{n-1} t \prod_{j=1}^{n} (w - \lambda_{n,j} N^{1/2}),$$

$$(T, t) = (V, 2), \quad (W, 1).$$

If $m = -n < 0$, the resolutions of the T_m are written down from (9.[illegible] by (1.3), (1.12).

The above resolutions can readily be expressed rationally with respect to the independent variables. For, from (9.1), (9.2) we have

$$\mu_{n,j} = -\mu_{n,n-j+1}, \qquad \lambda_{n,j} = -\lambda_{n,n-j+1} \qquad (j = 1, \cdots, n);$$

Hence, from (9.3), we have

$$(9.5) \qquad T_{2n} = 2^{2n-2} t w \prod_{j=1}^{n-1} (w^2 - \mu^2_{2n-1,j} N),$$

$$T_{2n+1} = 2^{2n} t \prod_{j=1}^{n} (w^2 - \mu^2_{2n,j} N),$$

$$(T, t) = (U, 1), \quad (X, x), \quad (Y, y), \quad (Z, z);$$

and from (9.4),

$$(9.6) \qquad T_{2n-1} = 2^{2n-2} t w \prod_{j=1}^{n-1} (w^2 - \lambda^2_{2n-1,j} N),$$

$$T_{2n} = 2^{2n-1} t \prod_{j=1}^{n} (w^2 - \lambda^2_{2n,j} N),$$

$$(T, t) = (V, 2), \quad (W, 1).$$

Each pair (9.5), (9.6) can be compressed into a single formula by using the greatest integer function $[n/2]$; there is but slight advantage in so doing.

Generalizing the concept of the zeros of a function of one variable to functions $P = P(u_1, u_2, \cdots, u_r)$ of $r > 1$ independent variables, real or complex, we define a zero of P to be a set $[u_1', u_2', \cdots, u_r']$ of values of $u_1, u_2, \cdots, u_r$ such that $P(u_1', \cdots, u_r') = 0$. We proceed to determine all the zeros $[w', x', y', z']$ of $T_n(w, x, y, z)$ $(T = U, V, W, X, Y, Z)$.

The unit sphere, in real or imaginary space of 3 dimensions,

$$(9.7) \qquad \xi^2 + \eta^2 + \zeta^2 = 1,$$

is uniformized, among other ways, by the transformation to spherical coordinates,

$$\xi = \sin\theta\cos\phi,\quad \eta = \sin\theta\sin\phi,\quad \zeta = \theta, \tag{9.,8}$$

which will suffice for our purpose. In (9.8), θ, ϕ are independent complex variables; the restriction, already imposed, to principal determinations of all irrations, may be repeated here for emphasis.

Let a, b, c, d be arbitrary numbers, real or complex, and write

$$\begin{gathered}\alpha_{n,j} \equiv \tan(j\pi/2n),\quad \beta_{n,j} \equiv \tan\,[(j\pi/(2n+1)],\\ \gamma_{n,j} \equiv \tan\,[(2j-1)\pi/(4n-2)],\quad \delta_{n,j} \equiv \tan\,[(2j-1)\pi/4n],\end{gathered} \tag{9.9}$$

where j is an integer. Then, with ξ, η, ζ as in (9.8), we see from (9.5) that T_{2n}, for $T = U, X, Y, Z$, has the zeros

$$[a,\ a\xi\alpha_{n,j},\ a\eta\alpha_{n,j},\ a\zeta\alpha_{n,j}]\ (j = 1, 2, \cdots, n-1); \tag{9.10}$$

T_{2n+1}, for $T = U, X, Y, Z$, has the zeros

$$[a,\ a\xi\beta_{n,j},\ a\eta\beta_{n,j},\ a\zeta\beta_{n,j}]\ (j = 1, 2, \cdots, n); \tag{9.11}$$

and from (9.6), T_{2n-1}, for $T = V, W$, has the zeros

$$[a,\ a\xi\gamma_{n,j},\ a\eta\gamma_{n,j},\ a\zeta\gamma_{n,j}]\ \ (j = 1, 2, \cdots, n-1); \tag{9.12}$$

T_{2n}, for $T = V, W$, has the zeros

$$[a,\ a\xi\delta_{n,j},\ a\eta\delta_{n,j},\ a\zeta\delta_{n,j}]\ (j = 1, 2, \cdots, n). \tag{9.13}$$

In addition to the zeros (9.10), T_{2n} has the zeros $(0, b, c, d)$, for $T = U, X, Y, Z$, and X_{2n}, Y_{2n}, Z_{2n} have also, respectively, the zeros

$$[a, 0, c, d],\ [a, b, 0, d],\ [a, b, c, 0]. \tag{9.14}$$

In addition to the zeros (9.11), T_{2n+1} has the zeros (9.14) for $T = X, Y, Z$, respectively. In addition to the zeros (9.12), T_{2n-1} has the zeros $[0, b, c, d]$, for $T = V, W$. It is evident that all the zeros of T_n $(T = U, V, W, X, Y, Z)$ have been stated in what precedes.

10. *Analogies with Weierstrass' σ, ζ, p.* Throughout this section, $N = 1$, and therefore $S = 1 - w^2$. Hence, by (4.2), (4.3), each of U_n, V_n is now a function of but one variable, w. We have now (10.1) $\phi_{2n} = \psi_{2n+1} = 0$, $\phi_{2n+1} = \psi_{2n} = (w^2 - 1)^n$, in place of (4.2). Thus the explicit forms of all

[illegible] T are known from (4.3), (4.4) with ϕ, ψ as in (10.1). Each of T [illegible] X, Y, Z) is now a function of precisely two independent variables, namely (w, t) $(t = x, y, z)$, respectively, and W_n is a function of w alone. In a note to appear shortly elsewhere,[*] I have shown that Lucas' U_n, V_n possess, up to a certain point, an isomorphism with the functions of Weierstrass provided the Lucas functions have the parameter $(h, 1)$, where h is arbitrary. It follows from the above that the like holds for the six T_a. As further elaboration of this isomorphism would be merely a reproduction of the note [illegible], with a few obvious modifications, we may take it as accomplished by reference to the note. As in the reference, the point of departure is [illegible], from which follows, in an obvious manner, a "relation of three terms" of precisely the same form as that for the sigma function.

11. *Generalizations.* One way in which the preceding admits of generalization has been pointed out in § 1. We note now that N is a norm in a linear algebra, and that the product of a sum of s squares by a sum of s squares is a sum of s squares only when $s = 2, 4, 8$. The case $s = 2$ gives the theory of the circular, hyperbolic, or Lucas functions; $s = 4$ has been treated above; the discussion of $s = 8$ will appear elsewhere. It may be stated here that the case $s = 8$ parallels $s = 4$ with but a few changes in definition and notation. Thus the 8 functions concerned are again Lucas functions; the [illegible] are rationally resolvable into factors of degree $\leqq 2$, etc., precisely as for $s = 4$. The case $s = 8$ includes $s = 4$. Again, in immediate sequence to the ideas of this paper, are those arising from the functions defined by powers of elements of an algebraic number field. I have elaborated the last in considerable detail. [illegible], the identity of Eisenstein (Crelle, 27, 105), is in an entirely different category; it does not belong to linear algebra.

* * * * * * * *

The following considerations, not discussed in the body of this paper, may be of interest to those who would prefer the geometrical approach to the algebraic methods followed exclusively here. First, it will be clear that the fundamental formulas can be derived at once from simple considerations concerning rotations as interpreted in quaternions. This approach has been avoided for the following reason. The difference equation of the second order with constant coefficients which is satisfied by the functions and the polynomials is of course that whose associated quadratic equation is the characteristic equation satisfied by the general quaternion. Precisely the same difference equation appears in the discussion of the polynomials and rational functions in eight variables which arise from a discussion of Cayley's eight-square imaginaries, which follows step by step the developments of this paper. In fact, the algebra of both cases could be treated in one presentation; they are abstractly identical. It would seem to follow then that the true geometrical interpretation of the quaternion case should apply equally well to that of the eight-square. Not being acquainted with any such interpretation of the eight-square imaginaries as rotations or other geometrical concepts, I have chosen the algebraic approach. It is significant also that the cases of four and eight squares exhaust the possibilities so far as polynomials and rational functions of the kind considered are concerned. In a geometrical approach this fact also should be accounted for.

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[*] Bulletin of the American Mathematical Society.

Representations in the Form $xy+yz+zx$.

BY WALTER H. GAGE.

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1. It has been shown by Bell* that the number of representations [illegible] integer n in the form

$$xy+yz+zx$$

[illegible] which x, y, z are integers > 0 is $3[G(n)-1]$ when and only when [illegible] In this paper we obtain by the same methods and formulae the [illegible] any positive integer n for which x, y, z are integers $\geqq 0$. By another [illegible] Mordell proved† that in this case the number of representations [illegible] provided we adopt the artificial convention that any solution [illegible] of the unknowns is zero be reckoned as $\frac{1}{2}$ instead of 1.

2. We first outline the notation and method and those results of Bell [illegible] which we need.

The $F(n), F_1(n)$ have their usual class number significance [illegible] $(n), G(n)$ are respectively equal to $F(n)-3F_1(n), F(n)+F_1(n)$. [illegible] symbol $\zeta(n)$ denotes the number of divisors of n; $\zeta'(n)$ denotes the number of odd divisors of n; and $\xi(n)$ is the number of divisors of n less than [illegible] the opposite parity to their conjugates; $(-1\,|\,n)$ is the Legendre symbol [illegible] $\epsilon(n)$ equals 1 or 0 according as n is or is not the square of an integer. [illegible] define $\psi_1(m), \psi_2(m)$ by the properties: $\psi_1(m)=0$ for $m\equiv 1$ (mod [illegible] for $m\equiv 3$ (mod 4) $\psi_1(m)$ is defined to be the number of divisors [illegible] less than their conjugates; $\psi_2(m)$ has the same properties if the congruences [illegible] interchanged. Throughout the paper m denotes an odd integer > 0.

Let $f(x)$ be an even function of x so that $f(x)=f(-x)$, which [illegible] entirely arbitrary except that $f(x)$ takes a single definite value for each [illegible] integer $\gtreqless 0$. If

$$a_0+\sum_{i=1}^{r} a_i\cos n_i x=0 \qquad (2)$$

[illegible] identity in x, the a_i, n_i being integers, then

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* *Tôhoku Mathematical Journal*, Vol. 19, 1921.

† *American Journal of Mathematics*, Vol. 45, 1923.

$$a_0 + \sum_{i=1}^{r} a_i f(n_i) = 0. \tag{3}$$

We may take $f(x) = \phi(x)$ where

$$\phi(0) = 1, \qquad \phi(x) = 0, \qquad\qquad | x | > 0, \tag{4}$$

whence we get

$$a_0 + \sum_{i=1}^{r} a_i \phi(n_i) = 0. \tag{5}$$

From the theta-function identities

$$\vartheta_0 \, \frac{\vartheta_2(x)\vartheta_3(x)}{\vartheta_1(x)} \times \vartheta_0\vartheta_3 \, \frac{\vartheta_2(x)}{\vartheta_1(x)} = \vartheta_0{}^2\vartheta_3 \, \frac{\vartheta_2{}^2(x)\vartheta_3(x)}{\vartheta_1{}^2(x)} \, ,$$

$$\vartheta_2 \, \frac{\vartheta_0(x)\vartheta_3(x)}{\vartheta_1(x)} \times \vartheta_2\vartheta_3 \, \frac{\vartheta_0(x)}{\vartheta_1(x)} = \vartheta_2{}^2\vartheta_3 \, \frac{\vartheta_0{}^2(x)\vartheta_3(x)}{\vartheta_1{}^2(x)} \, ,$$

by expanding in powers of q, equating the coefficients of like powers of q, and applying the above theorem with $\phi(x)$ as in (4) we obtain the following.

I. Let $N_J(m)$ be the number of representations of m in the form (1) for which

$$0 < y, z \equiv 1 \pmod 2) ; \; y \lessgtr z ; \; 0 < x \equiv J \pmod 2).$$

Then

$$N_0(m) - N_1(m) = I(m) - \tfrac{1}{2}[1 + (-1 \mid m)]\zeta(n) + \epsilon(m)(-1 \mid m)/2.$$

II. Let $M(n)$, n an arbitrary integer, denote the number of representations of n in the form (1) for which

$$0 < x, y, z ; \; x + y \equiv 1 \pmod 2).$$

Then

$$M(n) = 2F(n) - \zeta'(n) - \tfrac{1}{2}[1 + (-1)^n]\xi(n).$$

The number of representations of n in the form (1) when two of $x, y, z > 0$ are equal is obviously deducible from the number of positive integral solutions for n fixed of $n = u^2 + 2uv$, which is the number of divisors of n of the same parity as but less than their conjugates. The number of representations of n in the form (1) with one of x, y, z zero depends upon the number of positive integral solutions of $n = uv$ and is $3\zeta(n)$.

3. Let n be odd, and replace it by m in II. When n is odd x, y, z

must be either all odd and $m \equiv 3 \bmod 4$, or two odd and one even with $m \equiv 1 \bmod 4$. In the first case the number $N_1'(m)$ of representations with $x, y, z \geqq 0$ is

$$N_1'(m) = N_1(m) + \psi_1(m). \tag{6}$$

In the second case the number $N_0'(m)$ of representations is

$$N_0'(m) = 3F(m) - (3/2)\zeta(m) + 3\zeta(m) = 3[F(m) + \tfrac{1}{2}\zeta(m)]. \tag{7}$$

From the definitions of $N_0'(m)$, $N_0(m)$ we have

$$\tfrac{1}{3}N_0'(m) - \psi_2(m) - \zeta(m) = N_0(m). \tag{8}$$

Substitute in (8) the value of $N_0'(m)$ obtained from (7) and put the resulting value of $N_0(m)$ in I, thus determining $N_1(m)$. Hence from (6) we get $N_1'(m)$. The total number $N(m)$ of representations for which x, y, z is the sum of $N_0'(m)$ and $N_1'(m)$, which are now known. Using the identities $\zeta(m) = 2\psi_1(m) = 2\psi_2(m) + \epsilon(m)$ we find

$$N(m) = 3[G(m) + \tfrac{1}{2}\zeta(m)]. \tag{9}$$

Mordell's result follows when, adopting his convention, we subtract $(3/2)\zeta(m)$, which is half the number of representations of m for which one of x, y, z is zero.

Next, let $n = 2^{\alpha}m$, $\alpha > 0$, m odd. It is easily seen that x, y, z must be all even, or only two even. Let the total number of representations in the two cases, for which $x, y, z \geqq 0$, be $N_0(n)$, $N_1(n)$ respectively. In the latter case it follows from § 2 that for one of x, y, z zero the number of representations is $3\zeta(m)$, since the factors u, v must be of opposite parities. Hence from I

$$N_1(2^{\alpha}m) = 3F(2^{\alpha}m) + 3\zeta(m). \tag{10}$$

Suppose now that x, y, z are all even. We then divide out the factor 4 and consider the number of representations of $2^{\alpha-2}m$ in the form (1). Some of the solutions are again given by (10) where 2^{α} is replaced by $2^{\alpha-2}$. To find the remaining representations again divide out a factor 4 and proceed as above. Continue this process until $2m$ or m is reached, according as α is odd or even. The number of representations of $2m$ is $N_1(2m)$, of m it is $3[G(m) + \tfrac{1}{2}\zeta(m)]$. Hence if the total number of representations is $N(n)$ we have finally

α odd: $N(2^{\alpha}m) = 3[F(2^{\alpha}m) + F(2^{\alpha-2}m) + \cdots + F(2m) + (\alpha+1)/2\,\zeta(m$

(11)

α even: $N(2^{\alpha}m) = 3[F(2^{\alpha}m) + F(2^{\alpha-2}m) + \cdots + G(m) + (\alpha + 1/2)\zeta(m)]$.

Both reduce by the class number relations $F(4n) = 2F(n)$, $E(4n) = E(n)$ to

$$N(2^{\alpha}m) = 3[G(2^{\alpha}m) + (\alpha + 1/2)\zeta(m)]. \tag{12}$$

The result in this form holds true for $\alpha \geqq 0$ and hence for all positive integers. Adopting Mordell's convention we get his result immediately.

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The Resultant of Two Power Series in Two Variables.

By L. L. Dines.

For a system of r power series

$$f_i(x_1, x_2, \cdots x_r) \qquad (i = 1, 2, \cdots, r),$$

each without constant term and converging in some neighborhood of the origin there is in general a neighborhood of the origin in which the system of simultaneous equations $f_1 = 0$, $f_2 = 0, \cdots, f_r = 0$ admits no solution except the trivial one $x_1 = x_2 = \cdots = x_r = 0$.[*]

In order that this system of equations shall admit a nontrivial solution in every neighborhood of the origin, it is obvious then that the coefficients of the series f_i must satisfy special conditions. In the very particular case in which each series is merely a homogeneous polynomial, the condition is known to be the vanishing of a well defined rational integral function of the coefficients—the resultant of the homogeneous polynomials.[†]

The generalization to power series has not been made. A direct method which suggests itself for obtaining the desired condition is to eliminate $r - 1$ of the variables from the r equations. If we assume the series f_i to be formal power series (with literal coefficients) of positive order, this elimination can be effected (theoretically at least) by either of two methods: (1) the general elimination algorithm for power series as developed by Kistler,[‡] (2) the beautiful elimination theorem due to Bliss.[§]

Each of these methods leads to a power series in a single variable, the vanishing of whose coefficients appears to be the condition sought. The difficulty lies in the fact that the coefficients of this resultant series, though rational in the coefficients of the given series, are not *integral* functions. Hence the resultant series becomes meaningless if the literal coefficients of the given series be replaced by numerical values which cause denominators in the resultant series to vanish.

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[*] Cf. Osgood, *Lehrbuch der Funktionentheorie II*^{1}, p. 104: "The second theorem of Weierstrass."

[†] Cf. J. Konig, *Einleitung in die allgemeine Theorie der algebraischen Gröszen*, p. 295.

[‡] *Über Funktionen von mehreren komplexen Veränderlichen*, Dissertation, Göttingen (1905).

[§] A generalization of Weierstrass' Preparation Theorem for a power series in several variables, *Transactions of the American Mathematical Society*, Vol. 13, p. 133.

This difficulty can be overcome at least in the simplest case $r = 2$ and it is the object of the present paper to develop a theory of the resultant for that case.

In § 12 an application is made to the theory of implicit functions.

1. *Statement of some properties of the resultant.* We shall consider two *formal* power series of orders m and n respectively. These we shall write in the form

$$\begin{aligned} f(x,y) &\equiv f_m + f_{m+1} + \cdots \\ g(x,y) &\equiv g_n + g_{n+1} + \cdots \end{aligned}, \tag{1}$$

where f_j and g_j are homogeneous polynomials (forms) of degree j in the two variables x, y, with literal coefficients. The coefficients in f will be denoted generically by a and those in g by b. Relative to these two series we shall establish the following:

THEOREM: *There exists an infinite sequence*

$$r \equiv (R_{mn}, R_{mn+1}, R_{mn+2}, \cdots)$$

of well defined functions of the coefficients a and b, having the following properties:

(a) If the coefficients a and b in the formal series be assigned numerical values such that the two series retain their respective orders m and n and converge in a neighborhood of the origin, then the two equations $f = 0$, $g = 0$ will admit a common solution $(x,y) \neq (0,0)$ in every neighborhood of the origin if and only if the sequence r vanishes identically.

(b) Each element R_{mn+k} of the sequence r is a rational function of the coefficients a and b and of four auxilliary symbols u_{11}, u_{12}, u_{21}, u_{22} (generically u). The only denominators occurring in these rational functions are products of powers of the two forms $f_m(u_{11}, u_{21})$ and $g_n(u_{11}, u_{21})$.

(c) The first element R_{mn} of the sequence is the product of two factors of which one is the resultant of the two leading forms f_m, g_n, and the other is $| u_{ij} |^{mn}$.

(d) Each element R_{mn+k} is rationally homogeneous[*] in the coefficients

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[*] The terms *rationally homogeneous* and *rationally isobaric* are to be understood in the following sense. A rational function of a set of symbols is said to be rationally homogeneous if it is a quotient of two homogeneous polynomials. Its degree is the

a and *b* and the auxilliary symbols *u*. Its degree is *n* in *a*, *m* in *b* and $2mn - k$ in *u*.

(c) Each element $R_{mn,k}$ is rationally isobaric* of weight $2mn - k$ in the combined set of coefficients *a*, *b*, the weight of a coefficient being the degree of the form f, or g, in which it occurs.

The sequence *r* will be called the *resultant* of the two power series *f* and *g*. Its derivation is described in § 2. Property (*a*) is proven in § 3, the remaining properties in later sections after a prerequisite formal study of the important Weierstrass Preparation Theorem.

2. *Derivation of the sequence r.* To the variables *x*, *y* of the two formal series (1) we apply the linear homogeneous transformation

$$x = u_{11}\xi + u_{12}\eta \tag{2}$$
$$y = u_{21}\xi + u_{22}\eta$$

the coefficients *u* being entirely arbitrary except for the restriction $|u_{ij}| \neq 0$, and consider the resulting series in the new variables ξ, η. These may be written

$$\phi(\xi, \eta) \equiv \phi_m + \phi_{m+1} + \cdots \tag{3}$$
$$\psi(\xi, \eta) \equiv \psi_n + \psi_{n+1} + \cdots$$

where ϕ_j and ψ_j denote homogeneous polynomials in ξ, η, with coefficients homogeneous of degree *j* in *u*. The coefficients in φ and ψ will be denoted generically by α and β respectively. In particular, the coefficient of ξ^m in ϕ_m and the coefficient of ξ^n in ψ_n will be denoted by α_{m0} and β_{n0} respectively. They are easily seen to have the forms

$$\alpha_{m0} = f_m(u_{11}, u_{21}), \qquad \beta_{n0} = g_n(u_{11}, u_{21}). \tag{4}$$

To the two series $\phi(\xi, \eta)$, $\psi(\xi, \eta)$ we now apply formally the Preparation Theorem of Weierstrass.* We thus obtain two formal identities

$$\mu\phi = p, \qquad \nu\psi = q, \tag{5}$$

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degree of the numerator minus the degree of the denominator. If with each symbol of the set there is associated, by means of any definite law, a positive integer called the weight of that symbol, then the weight of any product of symbols of the set is defined to be the sum of the weights of the factors. A polynomial in symbols of the set is isobaric of weight *w* if each of its terms is of weight *w*. A rational function of the symbols is rationally isobaric if it is a quotient of isobaric polynomials, its weight is the weight of the numerator minus the weight of the denominator.

* Cf. Bliss, *Princeton Colloquium Lectures*, p. 50.

where the multipliers μ and ν are power series in ξ, η with constant term unity, and p and q are polynomials in ξ of degrees m and n respectively. These polynomials may be written

$$(6)\qquad \begin{aligned} p &\equiv \alpha_0\xi^m + \alpha_1\xi^{m-1} + \cdots + \alpha_m \\ q &\equiv \beta_0\xi^n + \beta_1\xi^{n-1} + \cdots + \beta_n \end{aligned}$$

the coefficients α_0 and β_0 being equal to α_{m0} and β_{n0} respectively, while the remaining coefficients α_j and β_j are power series in η without constant terms. The multipliers and polynomials are uniquely determined by these conditions.

We next form the algebraic resultant of the two polynomials p, q. This resultant is a well defined rational integral function of the coefficients of the two polynomials and hence is expressible as a power series in η, which we shall call the resultant series and denote by $r(\eta)$. *The sequence of coefficients of the series $r(\eta)$ is the sequence*

$$r \equiv (R_{mn}, R_{mn+1}, \cdots)$$

which we have called the resultant of the two series f and g.

3. *Proof of property (a), and an equivalent statement.* Suppose that the literal coefficients of the series $f(x, y)$ and $g(x, y)$ are assigned numerical values such that the two series preserve their respective orders m and n, and converge in a neighborhood of the origin.

From the nature of the transformation (2) it follows that for every set of finite values which may be assigned to the symbols u, the two series $\phi(\xi, \eta)$ and $\psi(\xi, \eta)$ will converge in some neighborhood of $(\xi, \eta) = (0, 0)$. Furthermore, since $|u_{ij}| \neq 0$, the existence of a common solution $(x, y) \neq (0, 0)$ of the two equations $f = 0$, $g = 0$ in every neighborhood of the origin is equivalent to the existence of a common solution $(\xi, \eta) \neq (0, 0)$ of the two equations $\phi = 0$, $\psi = 0$ in every neighborhood of the origin.

If the root-systems of the homogeneous equations $f_m = 0$, $g_n = 0$ be excluded from the possible values of the auxiliary symbols u_{11}, u_{21}, then the coefficients (4) will be different from zero, and the identities (5) will be valid in a neighborhood of $(\xi, \eta) = (0, 0)$. Since μ and ν have constant terms different from zero, the existence of a common non-trivial solution of $\phi = 0$, $\psi = 0$ in every neighborhood of the origin is equivalent to the existence of a like solution of $p = 0$, $q = 0$ in every neighborhood of the origin. But the criterion for the latter condition is the identical vanishing of the resultant series $r(\eta)$, that is the identical vanishing of the sequence r. This establishes property (a).

Equivalent to property (a) is the following:

Property (a'): Under the same hypotheses as in (a), the vanishing of the resultant r is necessary and sufficient for the existence of a non-trivial common factor of the series $f(x,y)$ and $g(x,y)$; in other words, for the existence of identities

$$f(x,y) \equiv f'(x,y)\,d(x,y), \quad g(x,y) \equiv g'(x,y)\,d(x,y), \tag{7}$$

where f', g', and d are power series and d has no constant term.

First, suppose the resultant r vanishes. Then the two polynomials have a common factor polynomial in ξ.* We may express this by identities

$$p \equiv p'\delta, \quad q \equiv q'\delta \tag{8}$$

where p', q', and δ are polynomials of the same type as p and q.

Substituting (8) in (5), we obtain identities which may be written

$$\phi \equiv p''\delta, \quad q \equiv q''\delta, \tag{9}$$

where $p'' = p'/\mu$ and $q'' = q'/\nu$.

Now assigning any non-singular numerical values to the auxiliary [illegible] and applying the inverse of the substitution (2) to (9), [illegible] identities of the specified form (7).

Conversely, suppose the given series satisfy identities of the form (7). Then due to the common factor $d(x,y)$, the equations $f = 0$, $g = 0$ [illegible] common non-trivial solution in every neighborhood of the origin, and the resultant must vanish by property (a).

4. *The discriminant of a single series in two variables.* Consider the formal power series

$$f(x,y) = f_m + f_{m+1} + \cdots,$$

and the series

$$\phi(\xi,\eta) = \phi_m + \phi_{m+1} + \cdots$$

obtained by the substitution (2).

To the related polynomial

$$p = \alpha_0\,\xi^m + \alpha_1\,\xi^{m-1} + \cdots + \alpha_m,$$

obtained by the Weierstrass Preparation Theorem we may apply the formal differentiation process, and obtain a polynomial

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* Cf. Osgood, *loc. cit.*, p. 84.

$$p_\xi = m\,\alpha_0\,\xi^{m-1} + (m-1)\alpha_1\,\xi^{m-2} + \cdots + \alpha_{m-1}.$$

The algebraic resultant of the two polynomials p and p_ξ is a series which we may call the discriminant series of the series $f(x, y)$ and denote by $d(\eta)$. The sequence of coefficients constitutes the discriminant d of the series $f(x, y)$.

If the literal coefficients in $f(x, y)$ be assigned numerical values such that the series converges and retains its order, then the identical vanishing of the discriminant d is necessary and sufficient for the existence of a repeated polynomial factor of the polynomial p,* and hence for the existence of a repeated non-trivial series factor of the series $f(x, y)$.

5. *Formal properties of the Weierstrass Preparation Identity.* The properties (b)—(e) of § 1 are formal properties of the resultant r which depend upon certain formal properties inherent in the Weierstrass Preparation Identity. These latter properties will be established in the present section.

According to the Weierstrass theorem, a formal series $\phi(\xi, \eta)$ of order m satisfies an identity

$$\mu\phi = p \tag{10}$$

where μ and p are power series in ξ and η with certain special properties; namely, μ has a constant term unity while p involves no powers of ξ higher than the $(m-1)th$ except the single term $\alpha_{m0}\xi^m$.

Adopting as generic notations for coefficients in ϕ, μ and p the respective letters α, γ, ρ, and taking account of the above mentioned properties, we may assume the following expressions:

$$\begin{aligned}
\phi &\equiv \phi_m + \phi_{m+1} + \cdots \\
\mu &\equiv 1 - \mu_1 - \mu_2 - \cdots \\
p &\equiv p_m + p_{m+1} + \cdots \\
\phi_j &\equiv \alpha_{j0}\xi^j + \alpha_{j-1,1}\xi^{j-1}\eta + \cdots + \alpha_{1,j-1}\xi\eta^{j-1} + \alpha_{0j}\eta^j \qquad (j \geqq m) \\
\mu_k &\equiv \gamma_{k0}\xi^k + \gamma_{k-1,1}\xi^{k-1}\eta + \cdots + \gamma_{1,k-1}\xi\eta^{k-1} + \gamma_{0k}\eta^k \qquad (k \geqq 1) \\
p_m &\equiv \rho_{m0}\xi^m + \rho_{m-1,1}\xi^{m-1}\eta + \cdots + \rho_{1,m-1}\xi\eta^{m-1} + \rho_{0m}\eta^m \\
p_j &\equiv \rho_{m-1,\,j-m+1}\,\xi^{m-1}\,\eta^{j-m+1} + \cdots + \rho_{1,\,j-1}\xi\eta^{j-1} + \rho_{0j}\,\eta^j \qquad (j > m)
\end{aligned}$$

Substituting the expressions for ϕ, μ, and p in (10), and equating forms of the same degree on the two sides of the resulting identity, we obtain a system of identities which may be written

$$p_m = \phi_m \tag*{$(11)_0$}$$

$$\phi_m\mu_1 + p_{m+1} = \phi_{m+1} \tag*{$(11)_1$}$$

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* Cf. Osgood, *loc. cit.*, p. 88.

$$\phi_m \mu_k + p_{m+k} = \phi_{m+k} - \sum_{i=1}^{k-1} \mu_i \phi_{m+k-i} \qquad (k = 2, 3, \cdots). \tag*{(11)$_k$}$$

It is almost obvious that these identities determine the coefficients of μ_k and p_{m+k} recursively, but it will be useful to note some of the details of the determination.

The identity $(11)_0$ determines the coefficients of p_m, viz.

$$\rho_{m0} = \alpha_{m0}, \cdots, \rho_{m-j,j} = \alpha_{m-j,j}, \cdots, \rho_{0m} = \alpha_{0m}. \tag{12}$$

The identity $(11)_1$ determines the coefficients of p_{m+1} and μ_1. For substituting the explicit forms in $(11)_1$, and equating coefficients of like power products of x, y, we obtain the system of equations

$$\begin{array}{llllll}
\alpha_{m0}\gamma_{10} & & & & & = \alpha_{m+1,0} \\
\alpha_{m-1,1}\gamma_{10} & + \alpha_{m0}\gamma_{01} & & & & = \alpha_{m1} \\
\alpha_{m-2,2}\gamma_{10} & + \alpha_{m-1,1}\gamma_{01} & + \rho_{m-1,2} & & & = \alpha_{m-1,2} \\
\alpha_{m-3,3}\gamma_{10} & + \alpha_{m-2,2}\gamma_{01} & & + \rho_{m-2,3} & & = \alpha_{m-2,3} \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\
\alpha_{0m}\gamma_{10} & + \alpha_{1,m-1}\gamma_{01} & & & + \rho_{1m} & = \alpha_{1m} \\
 & \alpha_{0m}\gamma_{01} & & & + \rho_{0,m+1} & = \alpha_{0,m+1}.
\end{array} \tag{13}$$

This system of $m + 2$ linear equations suffices to determine the $m + 2$ coefficients $\gamma_{10}, \gamma_{01}, \rho_{m-1,2}, \cdots, \rho_{0,m+1}$, the determinant of the system being equal to α^2_{m0}.

In a similar manner, each identity $(11)_k$ leads to a system of linear equations which uniquely determines the coefficients of μ_k and p_{m+k} in terms of the coefficients of μ_j and p_{m+j} $(j < k)$.

We are now in a position to note certain properties of homogeneity and isobarity of the coefficients γ and ρ. The terms *rationally homogeneous* and *rationally isobaric* have been defined in a footnote under § 1. Defining the *weight* of a coefficient α to be the degree k of the form ϕ_k in which it appears, we shall use the notation

$$[\alpha]^d_w$$

to denote generically a function of the coefficients α which is rationally homogeneous of degree d and rationally isobaric of weight w in these coefficients. Then we shall see that

(i) Each coefficient γ in μ_k has the character $[\alpha]^0_k$.

(ii) Each coefficient ρ in p_k has the character $[\alpha]^1_k$.

(iii) The only denominators occurring in the coefficients γ and ρ are powers of α_{m0}.

These properties are obviously true for the coefficients of p_m in (12). They can be verified easily for the coefficients of μ_1 and p_{m+1} from the form of the system of equations (13). Their general validity follows by mathematical induction from a consideration of the system of linear equations arising from the identity $(11)_k$. The matrix of coefficients of this system will be similar to that in (3). There are $k+1$ columns headed by α_{m0}, and there are m elements 1 in the principal diagonal. The value of the determinant is therefore $\alpha_{m0}{}^{k+1}$. The right side of each equation has the character $[\alpha]^1{}_{m+k}$ under the assumption that the properties hold for the coefficients of μ_i $(i < k)$. The Cramer rule for the solution of a system of equations then makes the properties (i), (ii), and (iii) apparent.

6. *Formal properties of $p(\xi, \eta)$ as a polynomial in ξ.* The series $p(\xi, \eta)$ appearing in the identity (10), when considered as a polynomial in ξ, may be written

$$p(\xi, \eta) \equiv \alpha_0 \xi^m + \alpha_1 \xi^{m-1} + \cdots + \alpha_m.$$

Since, as we have seen, $p(\xi, \eta)$ is a series of order m in ξ, η, the coefficient α_j is a series of order j in η:

$$\alpha_j(\eta) \equiv \rho_{m-j,j}\eta^j + \rho_{m-j,j+1}\eta^{j+1} + \rho_{m-j,j+2}\eta^{j+2} + \cdots. \tag{14}$$

Making use of property (ii) of the preceding section, we see that each series $\alpha_j(\eta)$ has as coefficients a sequence of functions which have the respective characters

$$[\alpha]^1{}_m,\ [\alpha]^1{}_{m+1},\ [\alpha]^1{}_{m+2},\ \cdots.$$

7. *Formal properties of the resultant r.* The resultant r of two series $f(x, y)$ and $g(x, y)$ has been defined in § 2 as the sequence of coefficients of a certain power series which represents the algebraic resultant of two polynomials

$$p \equiv \alpha_0 \xi^m + \alpha_1 \xi^{m-1} + \cdots + \alpha_m$$
$$q \equiv \beta_0 \xi^n + \beta_1 \xi^{n-1} + \cdots + \beta_n.$$

The coefficients α_0 and β_0 are equal to α_{m0} and β_{n0} respectively. For $j > 0$, the coefficients α_j and β_j are power series in η, the formal characters of which are indicated by

$$\begin{aligned} \alpha_j&: \ [\alpha]^1{}_m\eta^j + [\alpha]^1{}_{m+1}\eta^{j+1} + [\alpha]^1{}_{m+2}\eta^{j+2} + \cdots \\ \beta_j&: \ [\beta]^1{}_n\eta^j + [\beta]^1{}_{n+1}\eta^{j+1} + [\beta]^1{}_{n+2}\eta^{j+2} + \cdots. \end{aligned} \tag{15}$$

The only denominators occurring in the coefficients of α_j are powers of α_{m0}, and the only denominators in the coefficients of β_j are powers of β_{n0}.

Now the formal properties of algebraic resultants are well kn[illegible] According to the algebraic theory, the resultant of the two polynomials [illegible] rational integral function of the coefficients, and has the follo[illegible] itional properties: (1) It is homogeneous of degree n in the coefficien[illegible] p, and homogeneous of degree m in the coefficients of q; (2) it is isob[illegible] weight mn in the combined set of coefficients, the weight of α_j and β_j [illegible] [illegible]ned to be j.

From the second of these properties and the expressions (15) for α [illegible] it follows that the resultant of p and q is a power series of order mn [illegible] write it

$$r(\eta) = R_{mn}\eta^{mn} + R_{m,n+1}\eta^{mn+1} + \cdots .$$

From the properties (1) and (2) and the expressions for α and β [illegible] also the following formal properties relative to the coefficients R. [illegible] designations (b)-(e) are used to correlate these properties with the [illegible] similarly designated in § 1.)

(b) Each coefficient $R_{m,n+k}$ is a rational function of the coefficients α, [illegible] and the only denominators appearing are products of powers of α [illegible]

(c) The first coefficient R_{mn} is the algebraic resultant of the lead[illegible] nomials $\phi_m(\xi,\eta)$ and $\psi_n(\xi,\eta)$ of the series ϕ and ψ. This follows [illegible] fact that the leading coefficient of the series $\alpha_j(\eta)$ is $\alpha_{m-j,j}$ (in view [illegible] and (12)), and the leading coefficient of the series $\beta_j(\eta)$ is β [illegible]

(d) Each coefficient $R_{m,n+k}$ is rationally homogeneous of degree n [illegible] coefficients α and rationally homogeneous of degree m in the coefficients β.

(e) Each coefficient $R_{m,n+k}$ is rationally isobaric of weight $2mn+k$ [illegible] combined set of coefficients α, β.

8. *Proof of properties* (b) . . . (e) *of* § 1.

In § 2, the given series $f(x,y)$ and $g(x,y)$ were replaced by two series $\phi(\xi,\eta)$ and $\psi(\xi,\eta)$ by means of the transformation

$$(2)\qquad \begin{aligned} x &= u_{11}\xi + u_{12}\eta \\ y &= u_{21}\xi + u_{22}\eta \end{aligned}$$

[illegible] 7 formal properties of the resultant r have been obtained in terms of [illegible] [illegible]cients α, β of the series ϕ and ψ. It is easy to translate these propert[illegible] [illegible] terms of the coefficients a and b.

From the nature of the transformation (2) it follows that each coeffici[illegible]

α of the polynomial $\phi_k(\xi, \eta)$ is homogeneous in the coefficients a of the polynomial $f_k(x, y)$ and in the auxilliary symbols u. It is linear in the former and of degree k in the latter. An analagous statement can be made relative to each coefficient β in $\psi_k(\xi, \eta)$.

In view of these facts the properties (b)-(e) of § 1 follow from the similarly designated properties in § 7. In verifying (b) reference may be made to formulas (4). The factorization of R_{mn} indicated in (c) is a consequence of the well known invariant property of algebraic resultants.*

9. *The Product Theorem for resultants of power series.* Let $f'(x, y)$ and $g'(x, y)$ be a pair of formal power series of orders m' and n' with resultant

$$r' \equiv (R'_{m'n'},\ R'_{m'n'+1}, \cdots),$$

and let $f''(x, y)$ and $g''(x, y)$ be a second pair of formal power series of orders m'' and n'' with resultant

$$r'' \equiv (R''_{m''n''},\ R''_{m''n''+1}, \cdots),$$

Then *the resultant of the two product series*

$$f \equiv f'f'' \text{ and } g \equiv g'g''$$

is

$$r \equiv (R_{mn},\ R_{mn+1}, \cdots)$$

where

$$R_{mn} = R'_{m''n''}\ \ R''_{m''n''}$$

$$R_{mn+k} = \sum_{i=0}^{k}\ R'_{m'n'+i}\, R''_{m''n''+k-i}, \qquad (k = 1, 2, 3, \cdots).$$

To prove the theorem, we apply the substitution (2) to each of the four factor series and to the two product series, obtaining six series

$$\phi'(\xi, \eta),\ \psi'(\xi, \eta),\ \phi''(\xi, \eta),\ \psi''(\xi, \eta),\ \phi(\xi, \eta),\ \psi(\xi, \eta),$$

satisfying the two identities

$$\phi'(\xi, \eta)\cdot\phi''(\xi, \eta) = \phi(\xi, \eta);\ \psi'(\xi, \eta)\psi''(\xi, \eta) = \psi(\xi, \eta).$$

To each of the series ϕ', ϕ'', ψ', ψ'', ϕ, ψ, we apply the Weierstrass Preparation Theorem, obtaining six identities

$$\mu'\phi' = p',\ \mu''\phi'' = p'',\ \nu'\psi' = q',\ \nu''\psi'' = q'',\ \mu\phi = p,\ \nu\psi = q.$$

Equating the products of corresponding sides of the first two of these identities we obtain

$$\mu'\mu''\phi'\phi'' = p'p''.$$

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* Cf. J. Konig, *loc. cit.*, pp. 293-4.

Comparing this with the identity

$$\mu\,\phi = p$$

we conclude that

$$p = p'p''. \tag{16}$$

But $\phi'\phi'' = \phi$, and $\mu'\mu''$ is a series with constant term unity. Hence the uniqueness of the Weierstrass identity necessitates (16). Similarly we see that

$$q = q'q''. \tag{17}$$

The three resultants r', r'', r, are the sequences of coefficients of the constant series $r'(\eta)$, $r''(\eta)$, $r(\eta)$, which are by definition the algebraic resultants of the three pairs of polynomials (p', q'), (p'', q''), and (p, q) respectively.

Hence it follows from (16), (17), and the product theorem[*] for algebraic resultants that $r(\eta) = r'(\eta) \cdot r''(\eta)$.

The resulting relations between the coefficients of the three series yield the theorem.

10. *The effect of a linear transformation of variables.* If two series $f(x, y)$ and $g(x, y)$ with resultant r be subjected to a linear homogeneous transformation of variables

$$\begin{aligned} x &= c_{11}x' + c_{12}y' \\ y &= c_{21}x' + c_{22}y' \end{aligned} \qquad |c_{ij}| \neq 0, \tag{18}$$

the resulting pair of series $f'(x', y')$, $g'(x', y')$ will have a resultant which we may denote by r'. The relation between the two resultants r and r' is easily determined.

The resultant r is obtained by a definite procedure from two series $\phi(\xi, \eta)$, $\psi(\xi, \eta)$ which are the transforms of $f(x, y)$ and $g(x, y)$ under the transformation

$$\begin{aligned} x &= u_{11}\xi + u_{12}\eta \\ y &= u_{21}\xi + u_{22}\eta. \end{aligned} \tag{19}$$

The resultant r' is obtained by the same procedure from two series $\phi'(\xi, \eta)$, $\psi'(\xi, \eta)$ which are the transforms of $f'(x', y')$ and $g'(x', y')$ under the transformation

$$\begin{aligned} x' &= u_{11}\xi + u_{12}\eta \\ y' &= u_{21}\xi + u_{22}\eta. \end{aligned} \tag{20}$$

Cf. J. Konig, *loc. cit.*, p. 288.

two series $\phi'(\xi,\eta)$, $\psi'(\xi,\eta)$ can of course be obtained directly from $g(x,y)$ by a single transformation which is the product of (18) and (19), that is by the transformation

$$x = u'_{11}\xi + u'_{12}\eta \qquad (2')$$
$$y = u'_{21}\xi + u'_{22}\eta$$

where the coefficients u' are given by the matrix equation

$$\begin{pmatrix} u'_{11}u'_{12} \\ u'_{21}u'_{22} \end{pmatrix} = \begin{pmatrix} c_{11}c_{12} \\ c_{21}c_{22} \end{pmatrix}\begin{pmatrix} u_{11}u_{12} \\ u_{21}u_{22} \end{pmatrix}. \qquad (20)$$

Since the series $f(x,y)$, $g(x,y)$ are transformed by (2) into $\phi(\xi,\eta)$, $\psi(\xi,\eta)$ and by (2′) into $\phi'(\xi,\eta)$, $\Psi'(\xi,\eta)$, it follows that the resultants r and r' differ only in the auxiliary symbols. More explicitly, r' may be obtained from r by a transformation of auxiliary symbols indicated by the matrix identity (20).

If in particular we take for (18) the very simple transformation $x = y'$, $y = x'$, we note that the only effect upon the resultant is an interchange of auxiliary symbols.

11. *The order of the resultant of two series with numerical coefficients.* For two formal series $f(x,y)$, $g(x,y)$ of orders m and n, the resultant is, as we have seen, a sequence

$$r \equiv (R_{mn},\ R_{mn+1},\ \cdots).$$

If numerical values be assigned to the coefficients of f and g, this sequence may vanish identically, or some of its terms may vanish while others do not. If the first non-vanishing element of the sequence r is R_{mn+k}, the resultant of the two series f and g will be said to be of order $mn + k$.

From the developments of the preceding section it follows that the order of the resultant is invariant under transformations of the form (18). This invariant may be interpreted geometrically with reference to the intersection of the two surfaces $z = f(x,y)$ and $z = g(x,y)$ in the neighborhood of the origin. Of somewhat more general significance is the application indicated in the next section.

12. *Application of the resultant to the theory of implicit functions.* A fundamental question in the theory of implicit functions has to do with the number and character of the solutions of a system of r equations

$$F_i(x_1, x_2, \cdots, x_r;\ z_1, z_2, \cdots, z_s) = 0, \qquad (21)$$
$$(i = 1, 2, \cdots, r),$$

dependent variables $x_1, \cdots, x_r$ in the neighborhood of a point at which the functions F_i vanish.

Suppose the functions F_i are power series in the indicated variables, convergent in a neighborhood of the origin, at which point all of the series vanish. Suppose further that the following condition is satisfied.

Condition 0. The r equations

$$F_i(x_1, x_2, \cdots, x_r; 0, \cdots, 0) \qquad (i = 1, 2, \cdots$$

admit no common non-trivial solution in the neighborhood of the origin.

Then a characterization of the solutions of the system (21 ghborhood of the origin may be made by means of a related polynomial

$$E(\eta; z_1, z_2, \cdots, z_s) = \eta^N + e_1\eta^{N-1} + \cdots +$$

where η is a suitably chosen linear combination of the variables $x_1, \cdots,$ the coefficients e_i are power series in $z_1, z_2, \cdots, z_s$ vanishing with these variables.*

The relationship of this polynomial $E(\eta)$ to the system of equations (such that it may properly be called an *eliminating polynomial* for dependent variables x_i and its degree N may be taken as defining the number of solutions in the neighborhood of the origin.

The degree N of the eliminating polynomial is equal to the product of the degrees of the leading homogeneous polynomials of the series $F_i(x_1, x_2, \cdots, x_r; 0, \cdots, 0)$ *provided the algebraic resultant of these homogeneous polynomials is not zero.*†

Relative to the case $r = 2$, we are now able to state the *Theorem*.

$$F(x, y; z_1, \cdots, z_s), \qquad G(x, y; z_1, \cdots, z_s) \tag{22}$$

power series in the indicated variables, converging in a neighborhood of the origin and vanishing at that point, then the degree of the eliminating polynomial with respect to the dependent variables x, y is equal to the order of the resultant of the two series

$$\begin{aligned} f(x, y) &\equiv F(x, y; 0, \cdots, 0) \\ g(x, y) &\equiv G(x, y; 0, \cdots, 0). \end{aligned} \tag{23}$$

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[*] Cf. Osgood, *Madison Colloquium Lectures* (1913), p. 194; and *Lehrbuch der Funktionentheorie II*, p. 107.

[†] Cf. Bliss, "A generalization of Weierstrass' Preparation Theorem for a power series in several variables," *Transactions of American Mathematical Society*, Vol. 13 (1912), p. 133.

The proof is essentially an adaptation of the method used by Professor Bliss[*] to *obtain* the eliminating polynomial, and we shall give only those details which have to do with the *degree* in question.

We first replace the series F and G by two formal series in the same variables, then delete certain terms so that the formal series corresponding to (23) will have the same orders respectively as $f(x, y)$ and $g(x, y)$, say m and n.

We now have two formal series which by proper specialization of coefficients will reduce to the series (22). To these two formal series we apply the transformation

$$x = u_{11}\xi + u_{12}\eta$$
$$y = u_{21}\xi + u_{22}\eta,$$

and to the two resulting series apply the Weierstrass Preparation Theorem, obtaining two polynomials

$$P \equiv A_0\xi^m + A_1\xi^{m-1} + \cdots + A_m$$
$$Q \equiv B_0\xi^n + B_1\xi^{n-1} + \cdots + B_n.$$

The coefficients A_0 and B_0 are independent of $\xi, \eta, z_1, \cdots, z_s$, while A_i and $B_i (i > 0)$ are power series in $z, z_1, \cdots, z_s$, vanishing with these variables.

The algebraic resultant of the polynomials P and Q is a power series which we may denote by $R\ (\eta; z_1, \cdots, z_s)$. From the way in which this series was derived it follows that $R(\eta; 0, \cdots, 0)$ is precisely the resultant series $r(\eta)$ of the two formal series from which (23) are obtained by specialization of the coefficients.

If we assign to all the literal coefficients in $R(\eta; z_1, \cdots, z_s)$ the values they had in the special series (22), and to the resulting series apply the Weierstrass Preparation Theorem, we obtain a polynomial

$$E(\eta) \equiv \eta^N + e_1\eta^{N-1} + \cdots + e_N,$$

which is an eliminating polynomial for the two series (22). The degree N will be the order of the series $R(\eta; 0, \cdots, 0)$ under the specialization of coefficients, that is will be the order of the resultant of the series (23).

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[*] *Princeton Colloquium Lectures*, p. 70 et seq.

An Analysis of Logical Substitution.

BY H. B. CURRY.

Contents.[*]

I. Preliminary Discussion of the Nature of Mathematical Logic.
II. Logical Substitution; its Relation to a Combinatory Problem.
III. Solution of the Combinatory Problem.

I.

Mathematical Logic has been defined as an application of the formal methods of mathematics to the domain of Logic.[†] Logic, on the other hand, is the analysis and criticism of thought.[‡] In accordance with these definitions, the essential purpose of mathematical logic is the construction of an abstract (or strictly formalized) theory, such that when its fundamental notions are properly interpreted, there ensues an analysis of those universal principles in accordance with which valid thinking goes on. The term analysis here means that a certain rather complicated body of knowledge is exhibited as deriveable from a much simpler body assumed at the beginning. Evidently the simpler this initial knowledge, and the more explicitly and carefully it is set forth, the more profound and satisfactory is the analysis concerned.

In the present paper I propose to take some preliminary steps toward a theory of logic in which the assumed initial knowledge is simpler than in any existing theory with which I am acquainted. Before this is done, however, it is necessary to consider somewhat in detail what is meant by the phrase "abstract theory," and what is the significance of such a theory for the analysis of thought. The object of this discussion is to see just how the assumed knowledge enters into the theory; for this purpose we shall need to be explicit, even at the risk of repeating what has already been better said by others.

Certain ideas concerning the nature of an abstract theory can be disposed of at once. In the first place the naive notion that such a theory consists of a set of primitive ideas and propositions together with their consequences by the laws of pure logic, must be dismissed on the ground of its circularity. Again it is said that an abstract theory is one from which all meaning has been abstracted. This requires that the sense of the term meaning be explained. If we take the term meaning, as applied to objects, to signify the totality of properties (of those objects) which are directly apprehensible to our intuition, then every object presented to the mind has meaning, and a meaningless theory is a contradiction in terms. Even a symbol cannot be meaningless in this sense; for either it denotes some object, or else it is itself the object, and so has meaning. If we use the word meaning in some other sense, then it loses its significance as related to the assumed initial knowledge of our theory. Consequently the idea of a meaningless theory must be subjected to further scrutiny.

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[*] The three parts of this paper are to a certain extent independent of one another. However, certain definitions needed in Part III are given in the last six paragraphs of Part II.

[†] Hilbert, D., and Ackermann, W., *Grundzüge der theoretischen Logik*, 1928, p. 1.

[‡] Johnson, W. E., *Logic*, Part I, Cambridge (1921), p. xiii.

Let us use the word meaning, as applied to concepts, in the sense of the preceding paragraph. Then, relative to a given theory, we may distinguish two kinds of meanings, which we shall call natural and conventional meanings respectively. Natural meanings are those which are comprehensible a priori in terms of our previous knowledge; conventional meanings those based on relations to the theory itself. Natural meanings we may further subdivide into essential and accidental: essential meanings are those on which the deduction of the theory depends; accidental meanings those which are non-essential. The distinction between these three kinds of meaning is important in what follows.

The distinction between natural and conventional meanings has a counterpart in that between statements of fact and statements of convention. By a statement of fact I mean something of which truth or falsehood can significantly be predicated; by a statement of convention a declaration of intention, definition, or the like. The former corresponds to an act of judgment, the latter to one of volition. Common sense and grammar have long recognized both of these types; yet logicians seem to belittle the latter in that they define the proposition so as to exclude it.* Both these kinds of statement, however, are equally intelligible to a rational mind; in this sense it is false to say that one of them is less significant than the other. As examples of statements of convention we have of course the definitions of technical terms; but not all statements of convention are verbal—for instance the rules of chess, which, by a sufficient amount of circumlocution, may be stated without defining any new terms whatever. The postulates of any branch of mathematics are of this character.

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* See Johnson, W. E., *l. c.*, p. 1. Johnson's definition of the proposition is what I have given as the definition of a statement of fact.

Let us now return to the abstract theory. I suggest that such a theory is characterized by the following: 1) the explicit indication of all essential meanings; 2) the absence, or at least omission from consideration, of accidental meanings; 3) the circumstance that the statements with which the theory begins are conventional, and are, furthermore, sufficiently detailed so that all the acts necessary to the deduction are specified.

To be yet more precise, an abstract theory begins with a set of primitive notions, which, taken collectively, we shall call the *primitive frame*, as [illegible]

NON-FORMAL PRIMITIVE IDEAS.*

A set of ideas to each of which a certain amount of essential meaning is [illegible], although they need not coincide with any ideas previously [illegible].† For example:

1. *Entities.* In order for an object to be considered in the theory at all it must have some property; this fact we may express by saying it is an entity of one sort or another. These properties must then be among the primitive ideas of the theory; and they must have essential meaning, for they are predicates. In the simplified theory only one such is necessary; but in the more complicated ones there are several; e. g. in Principia Mathematica there are individual, proposition, function, etc., and also of various orders and types.

2. *Modes of Combination.* I. e. processes by means of which entities are combined to get new entities. These have essential meaning in that they are combinations. It must be specified by rules that the results of combination are entities. In the simple cases only one such notion is necessary, [illegible] a dyadic one; in the more complicated cases the various processes of substitution are of this nature.

3. *Assertions.* An assertion is a kind of entity, which is of some importance because the object of deduction is to derive new assertions. The idea of assertion has essential meaning only in that it is a predicate applicable to certain entities. Ordinarily an assertion is interpreted as a statement to which belief attaches, but this meaning is accidental.

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* The term idea is used here to denote an object, not a process of thought.

† *I. e.*, their meaning may be partly conventional.

II. FORMAL PRIMITIVE IDEAS.

Ideas which have no essential meanings (except that they are concepts). They must of course be entities and their relations to other parts of the primitive frame will give them conventional meanings.

III. POSTULATES.

All propositions of the theory are statements that certain particular entities are assertions; the postulates are the propositions, if any, which are assumed at the beginning. They are purely conventional.

IV. RULES.

Statements of the processes by means of which new entities * or new propositions may be constructed. Such statements are of course conventional; moreover they are universal statements (involving the notion of "every" or its equivalent †). They thus differ from propositions not only in that they involve intuitive ideas from which the propositions are free, but also in that they form the methods of transition, rather than the stopping places in the theory. A typical example is the "rule of inference" which may be stated thus: whenever p and $p \supseteq q$ are assertions, then q shall also be an assertion.

In addition to the above notions there are yet to be considered those associated with the use of symbolism. Whether these are to be regarded as a part of the theory or as something superposed upon it, is a question which I prefer to leave to the reader to adopt such views as seem best to him. However he may decide, certainly language is necessary in order that the theory may be communicated. The use of this language may involve intuitive operations other than those we have mentioned; it is desirable that these, too, be specified by rules; because otherwise it is not certain that intuitive, knowledge, other than that expressly mentioned, does not creep into the theory. We shall call such rules *symbolic conventions.*

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* Strictly speaking we should consider in the theory not only statements that an entity is an assertion, but also statements that such and such combinations are entities. But the latter are, in simple cases at least, of so trivial a nature that it is not necessary to give them special prominence.

† Otherwise the rule would make possible the addition of only a finite number of constituents, and these could just as well be added explicitly to the preceding categories of the primitive frame.

So much for the primitive frame. The abstract theory itself may be defined as the doctrine built upon such a primitive frame by means of the following processes: 1) the derivation of new propositions, each of which [illegible] the form that such and such an entity is an assertion, by means of the rules; 2) the addition of new ideas by definitions. The latter process may be regarded either as a symbolic matter, governed by symbolic conventions, or as the introduction of a new idea along with postulates and rules to the effect that it is identical with some already existing entity. It is worth [illegible] that since statements that entities are not assertions do not occur [illegible] the propositions, such a theory can never lead to a contradiction there.

The importance of such a theory for the analysis of thought lies in the definiteness with which the intuitive knowledge entering into it is set forth. Indeed, so far as the abstract theory itself is concerned, the only knowledge assumed is the appreciation of the essential meanings and conventional statements appearing in the primitive frame. When the theory is interpreted, the additional knowledge that must be brought to bear consists of the following: that the concepts which we substitute for the primitive ideas have the necessary essential meanings, and that the conventional statements in the primitive frame correspond to facts. In both cases the required information is precisely specified.

On the philosophic nature of such a theory, its relations to the symbols used in its expression, and to the various concrete theories obtained by interpreting it, it suffices to say that such questions are largely metaphysical, and therefore irrelevant to the present discussion. It is by no means self-evident that the best interests of science are served by adopting any one theory to the exclusion of all others; any more than it is desirable that two persons following the same argument should have the same mental imagery.[5]

The next point to which I wish to direct the reader's attention is the crucial importance of the rules in any abstract theory related to logic. For the amount of initial knowledge which enters into the first three categories of the primitive frame is slight. In the rules, however, such knowledge is [illegible] every step of the construction; for we have to pass judgment as to whether the contemplated act is or is not according to Hoyle. These judgments, however, are the only ones which are required. The rules, therefore, form the point of entry of intelligence; and since nothing can be done without them

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In writing the foregoing account I have naturally made use of any ideas I may have gleaned from reading the literature. The writings of Hilbert [illegible] fundamental in this connection. I hope that I have added clearness to [illegible] points where the existing treatments are obscure.

they represent the atoms of thought, so to speak, into which the reasoning can be decomposed. It follows that in constructing such a theory it is not sufficient merely to reduce the postulates and primitive ideas to their lowest terms; it is even more important to so chose the rules that they involve, in their application, only the simplest actions of the human mind.

Now although the rule of inference, stated above, is simple enough, yet in all current mathematical logics there exist rules which are highly complex. The presence of these complex rules raises the question whether it is possible to formulate a theory which is—1) adequate for the whole of logic, 2) based on a finite number of primitive ideas, postulates, and rules, the last of the same order of complexity as the rule of inference. I believe that it is; indeed steps in that direction have already been taken.* As a preliminary to treating this general problem, I shall discuss in the rest of this paper a special one connected with it; viz., the analysis of the process of substitution. The latter process is one of those complicated rules which occur in practically every logical theory to-day.

The reader will observe that in the theory which results from the analysis the formulas are more complicated, and the deductions required to produce them more lengthy, than would be the case in the older theory. This is inevitable. Indeed if we are to dissect the reasoning into microscopic pieces it is but natural that more of them should be necessary to bring about a given result. Consequently we must adopt a point of view suggested by Hilbert. With each theory there is associated a metatheory in which we reason intuitively about the theory. In this metatheory we can derive more and more complicated rules by showing, in general terms, how any particular consequence of the derived rules can actually be deduced from the primitive ones. The aim of mathematical logic is, in fact, not to reduce mathematics to a formalism, à la *Principia,* in which all steps explicitly appear; but rather to analyze logic with a view to obtaining a greater command over its use, and a more profound understanding of its nature. In this paper we shall adopt this metatheoretic point of view.

II.

The process of substitution referred to above is the insertion of a constant entity for one or more of the variables in a propositional function. The complexity of this process is manifest. For not only is a function of n variables a distinct concept for every value of n, but the constant may be inserted in one of the n places, and each such insertion is a distinct act; [illegible] [illegible], in connection with the universal and existential prefixes, we [illegible] process which virtually amounts to the simultaneous substitution of an ent[illegible] [illegible] two or more distinct places, as a procedure distinct from any of the [illegible] [illegible]g. The process of substitution is therefore compound in that it is [illegible] however, but many. Moreover, when these acts of substitution are [illegible] [illegible] in succession, there are many equivalences between the different po[illegible]ties. To take the simplest example: suppose in a given function [illegible] substitute a for x, and in the result, which is $\phi(a, y)$, we substitute [illegible] the proposition $\phi(a, b)$; on the other hand, if we first substitute [illegible] and then a for x, we obtain the very same proposition. Yet these [illegible]ses are in no sense identical. The substitution process is therefore [illegible] compound but complex, in the sense that it has structure. Thus there is [illegible] considerable amount of information presupposed by the process; [illegible] involving it cannot have the maximum possible simplicity.

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* See the paper of Schönfinkel cited below.

A notion closely related to substitution is that of transformation [illegible]. Suppose we regard a function as having inherent in its definition [illegible] order of its variables. Then permuting these variables in a[illegible] [illegible]ing two or more of them alike, will produce new functions re[illegible]; let us call them transforms of the original function, and the [illegible] by which they are produced transformations. If we number the [illegible] consecutively 1, 2, 3, ⋯, then the transforms for a function of [illegible] will be

$$\phi(1, 2), \phi(2, 1), \phi(1, 1).$$

[illegible] variables there will be 13 transforms, for four variables 75, [illegible] 541, etc. It is clear that the process of substituting a series [illegible] an arbitrary manner (such that the total number of entities [illegible] [illegible]ions is n) into the original function is equivalent to the substitution [illegible] same entities in a prescribed manner (viz., the first entity into the place [illegible] first variable, the second into that of the second, etc.) into one of [illegible] transforms. The study of substitution is thus to a certain degree equivalent to the study of these transformations.

An important step toward the analysis of this situation was made by M. Schonfinkel.[illegible] Starting, apparently, from the fact that every logical [illegible] is a combination of constants—the variables being only apparent [illegible] shows that neither the notions of propositional function (of various orde[illegible]

[illegible] "Ueber die Bausteine der mathematischen Logik," *Mathematische Annalen*, Vol. 92 (1924), pp. 305-316.

nor that of substitution need be assumed as primitive; his formulation of logic is such that variables, real or apparent, do not appear explicitly. His primitive frame is essentially as follows:[*]

I. NON-FORMAL PRIMITIVE IDEAS.

1. *Entity*—not mentioned by Schönfinkel, but to be understood essentially as a single notion of the sort mentioned in the general description of an abstract theory above.

2. *Application*—a mode of combination, the only one in the theory. Two entities x and y combine to give a third entity called the application of x to y and denoted by (xy). The *interpretation* of this is as follows: if x is a function, then (xy) is the result of substituting y for the first variable in x; thus if f denote a function of one variable, (fx) denotes what is ordinarily written $f(x)$, if f is a function of two variables, $((fx)y)$ denotes what is ordinarily written $f(x, y)$, etc. Nothing is said concerning the interpretation of (xy) when x is not a function; if the reader is disturbed over this lack, he may invent one arbitrarily, *e. g.* (xy) may then be equal to x.

3. *Assertion.* To be understood as in the general description of an abstract theory. Not denoted by any particular symbol; but when a symb of the form $(((=)x)y)$ (or $x = y$) where x and y may be quite complicated, stands out by itself like an equation in algebra, then the proposition that the corresponding entity is an assertion is to be understood.

II. FORMAL PRIMITIVE IDEAS.

Three, denoted by $(=)$, S and K. In the interpretation $(=)$ is to correspond with identity; S and K are operations in the sense defined by the rules.

Symbolic Conventions.

1. If x and y are any entities whatever, then instead of $(((=)x)y)$ we may write $(x = y)$.

2. If $x_1, x_2, \cdots, x_n$ are any entities, then instead of

$$((\cdots(((x_1 x_2)x_3)x_4)\cdots)x_n)$$

we may write $(x_1 x_2 x_3 \cdots x_n)$.

3. The outside parentheses may be left off in the case of a symbol standing by itself or on either side of the sign $=$.

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[*] In this presentation I have changed Schönfinkel's formulation in some matters of detail.

III. POSTULATES.

None.

IV. RULES.

0. If x and y are entities, then (xy) shall be an entity.

1. $(=)$ shall have the properties of identity. These properties are specified by a few simple rules; but in this treatment we shall not go into that detail. We shall treat $(=)$ as if it were precisely the intuitive idea of equality.

2. If x and y are any entities, then

$$Kxy = x$$

3. If x, y, z are entities, then

$$Sxyz = xz(yz)$$

4. If X and Y are combinations of S and K, and if there exists an integer n such that by application of the preceding rules we can reduce the expressions $Xx_1 x_2 \cdots x_n$ and $Yx_1 x_2 \cdots x_n$ to combinations of $x_1, x_2 \cdots x_n$ which have the same structure, then $X = Y$.

If the above primitive frame were a part of a general theory of logic, then an entity would include not only the various combinations of S and K, but the notions of logic as well. In the sequel we shall accordingly speak of the application of combinations S and K to various logical notions, and of applying notions to each other, just as if these notions had been adjoined to the above frame.

The *raison d'être* of the theory based on this frame is the following fact: Let $x_1, x_2, \cdots, x_n$ be any n entities, and X any combination of them constructed by means of application. Then there exists a unique Y, which is a combination of S and K and independent of $x_1, x_2, \cdots, x_n$ such that

$$X = Yx_1 x_2 \cdots x_n.$$

When we recall the interpretation to be given to application we have the following result: given any logical formula built up from functions $f_1, f_2 \cdots f$ and variables $x_1, x_2 \cdots x$ by substitution and rearrangement in any manner; then the formula is expressible in the form

$$Fx_1 x_2 \cdots x_m$$

where

$$F = Yf_1\ f_2 \cdots f_m.$$

Now as already remarked, in the formulas expressing propositions of logic, the variables are only apparent; which means that they are only a device by means of which rather complicated relations among the logical constants may be expressed; these relations, as the preceding argument shows, may also be expressed by means of the operators Y, so that when the Schönfinkel theory is used it is not necessary that variables should appear at all.

The theory of transformation of functions is included in the above as a special case; viz., when there is a single function f and F is a transform.

The theory is, however, open to objection from our point of view because of the complexity of Rule 4. This rule is not mentioned by Schönfinkel; but it is necessary in order that the Y mentioned be unique. In fact, the combinations SK and $K(SKK)$ determine the same X; yet it is evidently not possible to establish their identity by means of the first four rules.

This situation suggests a problem; viz., to find a set of postulates which, when adjoined to the Schönfinkel frame, enable us to dispense with Rule 4. In what follows we shall obtain the solution of a special case under this problem; specifically, we shall find a set of postulates such that, within a certain subclass of combinations of S and K, all the Y's which correspond to the same X may be proved equal by means of these postulates and Rules 0-3. The subclass is one which has particular reference to logical substitution.

To begin with, we make the following definitions (the first three were made by Schönfinkel):[*]

$$I = SKK$$
$$B = S(KS)K$$
$$C_1 = S(BBS)(KK)$$
$$C_2 = BC_1;\ C_3 = BC_2; \cdots \text{ etc.}$$
$$W = SS(SK)$$

then the reader may verify that whenever $x_0, x_1, x_2 \cdots$ are entities

$$Ix_0 = x_0$$
$$Bx_0\,x_1\,x_2 = x_0(x_1\,x_2)$$
$$C_1\,x_0\,x_1\,x_2 = x_0\,x_2\,x_1$$
$$C_2\,x_0\,x_1\,x_2\,x_3 = x_0\,x_1\,x_3\,x_2$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$Wx_0\,x_1 = x_0\,x_1\,x_1$$

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[*] We use B and C respectively in place of Schönfinkel's Z and T. Nothing corresponding to W or C_2, C_3, $\cdots$ is defined by him.

[illegible] we define multiplication thus

$$X \cdot Y = BXY$$

This multiplication is associative, and furthermore with respect to it, I is an identical element. These properties follow from Rule 4; they may also be derived from the postulates:

$$C_1(BB(BBB))B = BBBBB$$
$$BI = I$$
$$C_1 BI = I.$$

The subclass in question shall now consist of all the combinations obtainable by multiplication from the C's, W, and I.

Before discussing the significance of this subclass, we turn aside to some conventions in regard to transformations. Let us redefine a transformation as an operation converting the sequence 1, 2, 3, ··· into the sequence $a_1, a_2, \cdots$ The latter sequence will, in the cases we consider, be such that there exists and index m such that: 1) for $i \leqq m$ the a_i are a permutation, with repetitions allowed, but not omissions, of the integers from 1 to $m - p$, $p \geqq 0$, inclusive, 2) for $i > m$, $a_i = i - p$. Let m be the least integer having this property; then we shall call m the order of the transformation. We shall denote a transformation by writing in brackets the sequence into which it transforms the sequence 1, 2, 3, ···: furthermore there is no ambiguity if we indicate only the first n terms of the new sequence, if $m \leqq n$.

We shall further agree that a transformation of order m may operate on a function of any number of variables $\geqq m$; and that the effect of the transformation on the function $\phi(1, 2, \cdots, n)$, where $n \geqq m$, shall be the transform $\phi(a_1, a_2, \cdots, a_n)$. We shall regard as undefined the effect of a transformation on a function of multiplicity less than the order of the transformation. Then there is one and only one transformation which carries a given function into a given transform.

Multiplication of transformations we now define as follows:

$$(1) \quad [a_1, a_2, a_3 \cdots] \cdot [b_1, b_2, b_3 \cdots] = [a_{b_1}, a_{b_2}, a_{b_3} \cdots]$$

(The product transformation has a finite order, provided that the factors do.)

Suppose, now, that we have a combination, X, of S and K, having the property that there exists a transformation $\alpha = [a_1, a_2 \cdots a_m]$ such that for arbitrary $x_0, x_1, x_2, \cdots x_{n-p}$

$$X x_0 x_1 x_2 \cdots x_{m-p} = x_0 x_{a_1} x_{a_2} \cdots x_{a_m}$$

where m and p are defined as above. Then we shall say that X corresponds to α. It follows from Rules 0-3 that if X and Y are entities which correspond to transformations α and β, respectively, then $X \cdot Y$ corresponds to $\alpha\beta$.

I now assert that every entity of our subclass corresponds to a transformation; and conversely that every X corresponding to a transformation, is an entity of the subclass. For since the generating entities, the C's, W, and I, correspond to transformations, the first part of the statement follows from the closing sentence of the last paragraph. The last part follows similarly from the fact that every transformation can be obtained by multiplication from permutations of adjacent integers and the transformation [1, 1.] Of course the one-to-oneness of the correspondence depends essentially on Rule 4.

The problem now is to find a set of postulates for the C's and W. from which we may conclude, without the use of Rule 4, that the correspondence between our subclass and the set of all transformations is a simple isomorphism. To this we now turn.

III.

Theorem

1) *Let $\mathfrak{A}$ be a system of operators, in which there exists an associative multiplication and an identical element, I.*

2) *Let this system be generated by operators $W, C_1, C_2, \cdots$ Subject to the postulates*

I. $C_i \cdot C_i = I \qquad i = 1, 2, 3, \cdots$

II. $C_i \cdot C_{i+1} \cdot C_i = C_{i+1} \cdot C_i \cdot C_{i+1} \qquad i = 1, 2, 3, \cdots$

III. $C_i \cdot C_j = C_j \cdot C_i \qquad i = 1, 2, 3, \cdots \qquad j > i + 1$

IV. $C_i \cdot W = W \cdot C_{i+1} \qquad i = 2, 3, 4 \cdots$

V. $W \cdot C_1 = W$

VI. $W \cdot W \cdot C_2 = W \cdot W$ *

VII. $W \cdot D_j \cdot W = D_{j-1} \cdot W \cdot D_3 \cdot W \cdot C_2 \cdot C_3 \cdot C_1 \cdot C_2$ * $\qquad j = 3, 4, 5 \cdots$

where

$$D_1 = I$$

$$D_{j+1} = C_j \cdot D_j = C_j \cdot C_{j-1} \cdot C_{j-2} \cdot \cdots \cdot C_1$$

3) *Let a correspondence be set up between the system $\mathfrak{A}$ and the system of transformations of finite order as follows:*

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* VI and VII are equivalent respectively to (6) and (7) (see below). The latter may if desired replace VI and VII.

$$\begin{aligned} W &\sim [1,1] \\ C_1 &\sim [2,1] \\ C_2 &\sim [1,3,2] \\ C_i &\sim [1,2,3,\cdots i-1, i+1, i] \end{aligned} \tag{2}$$

and if $A \sim \alpha$, $B \sim \beta$[*] then $A \cdot B \sim \alpha \cdot \beta$ *where* $\alpha \cdot \beta$ *is defined by* (1).

Then, the correspondence so defined is a one-to-one isomorphism.

Proof.

There are four things to prove:

1) That to every product expression in 𝔄 there corresponds a unique transformation.

2) That this correspondence is an isomorphism.

3) That if two transformations correspond to two expressions in 𝔄 which are equal by Virtue of I-VII (inclusive), then these transformations are equal.

4) That conversely there corresponds an operator of 𝔄 to each transformation, and any two operators corresponding to the same transformation may be proved equal by I-VII.

Of these four things the first three follow immediately: the first two by definition; the third because propositions analogous to I-VII are true of transformations.[†]

It remains, therefore, only to prove the fourth.

By a theorem due to E. H. Moore[‡] postulates I-III insure that the subgroup of 𝔄 which is obtained from the first $n-1$ C's only is isomorph

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[*] Throughout this discussion we use Roman capitals to denote operators of 𝔄, Greek l. c. letters to denote transformations.

[†] The only question here is about VII. It follows, however, by definition that

$$D_i \sim [i, 1, 2, 3, \cdots, i-1]$$

and therefore the proposition analogous to VII is the identity

$$\begin{aligned} &[1, 1] \cdot [j, 1, 2, \cdots, j-1] \cdot [1, 1] \\ &\qquad = [j-1, 1, 2, \cdots, j-2] \cdot [1. 1] \cdot [3, 1, 2] \cdot [1, 1] \cdot [3, 4, 1, 2]. \end{aligned}$$

Both sides here are equal to $[j-1, j-1, 1, 1, 2, 3, \cdots, j-2]$.

[‡] *Proceedings of the London Mathematical Society*, Vol. 28 (1897), p. 357-366. Moore's proof involves some rather complicated theorems in group theory. It is, however, possible to prove the theorem by directly establishing the correspondence.

with the symmetric group on n letters; the correspondence is established in the same way as here. It follows readily that the subset generated by all the C's is isomorphic with the totality if all those transformations which do not allow repetition. In fact, suppose two such operators of $\mathfrak{A}$ correspond to the same transformation; let n be the largest index appertaining to a C in either one; then the two operators correspond to the same permutation on $n+1$ letters, and therefore, by Moore's result, are equal in virtue of I-III. In the following we shall assume this result as known; and therefore shall often designate operators composed of C's only by means of the transformations corresponding to them.

In order to prove the fourth statement above when W is considered, we shall show that every operator in $\mathfrak{A}$ can be expressed in a certain normal form, and then that one and only one such expression in normal form corresponds to a given transformation.

To obtain such a normal form we introduce the following definitions:

$$\begin{aligned} W_1 &= W \\ W_2 &= C_1 \cdot W_1 \cdot C_2 \cdot C_1 \\ &\cdots\cdots\cdots\cdots \\ W_{k+1} &= C_k \cdot W_k \cdot C_{k+1} \cdot C_k \end{aligned}$$

it then follows by induction * that

$$\begin{aligned} W_k &= D_k \cdot W \cdot C_2 \cdot C_3 \cdot C_4 \cdots C_k \cdot C_1 \cdot C_2 \cdot C_3 \cdots C_{k-1} \\ &= [k, 1, 2, \cdots k-1] \cdot W \cdot [3, 4, \cdots k+1, 1, 2] \end{aligned}$$

whence,

$$\begin{aligned} W_2 &\sim [1, 2, 2] \\ W_3 &\sim [1, 2, 3, 3] \\ &\cdots\cdots\cdots\cdots \\ W_k &\sim [1, 2, 3, \cdots k-1, k, k] \end{aligned}$$

The result we wish to prove now follows from a series of lemmas.

LEMMA 1. *Every product expression in $\mathfrak{A}$ can be reduced to the form*

$$W_{k_q} \cdot W_{k_{q-1}} \cdots W_{k_2} \cdot W_{k_1} \cdot B. \tag{3}$$

where B involves the C's only.

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* This and other proofs by mathematical deduction are such that by repeating the argument a sufficient number of times the formula may be established in any particular case with the use of I-VII.

Proof.

First we show that any expression of the form $A \cdot W$ can be reduced to the form $W_k \cdot B$, where A and B involve the C's only. If A is the identical element we are through. If not, consider the C nearest the W; if this is any other than C_1 it can be passed across the W by IV, otherwise we have an expression of the form $A' \cdot C_1 \cdot W$ where A' involves one less C than A.

Now suppose we have reduced $A \cdot W$ to the form $A' \cdot D_j \cdot W \cdot B'$ where A' and B' involve the C's only and A' is not I. Let C_i be that C in A' which is nearest D_j. Four cases can arise: 1) if $i < j - 1$, $C_i \cdot D_j = D_j \cdot C_{i+1}$, and since $i + 1 > 1$, C_{i+1} can be passed across the W by *IV;* 2) if $i = j - 1$, $C_i \cdot D_j = D_{j-1}$; 3) if $i = j$, $C_i \cdot D_j = D_{j+1}$; 4) if $i > j$, $C_i \cdot D_j = D_j \cdot C_i$, and since $i > j \geqq 1$, C_i can be passed across the W by IV. In all four cases we have reduced the expression to another one of the same form where the new A' involves one less C than the old.

Continuing in this way we must eventually reach a stage where $A' = I$. Then we have

$$\begin{aligned} A \cdot W &= D_k \cdot W \cdot B' \\ &= (D_k \cdot W \cdot C_2 \cdot C_3 \cdots C_k \cdot C_1 \cdot C_2 \cdots C_{k-1}) \\ &\qquad \cdot C_{k-1} \cdots C_2 \cdot C_1 \cdot C_k \cdots C_3 \cdot C_2 \cdot B' \text{ by } I \\ &= W_k \cdot C_{k-1} \cdots C_2 \cdot C_1 \cdot C_k \cdots C_3 \cdot C_2 \cdot B' \text{ by def.} \end{aligned}$$

which is of the form $W_k \cdot B$.

The proof of the lemma now follows. Let the given expression be of the form

$$A_q \cdot W \cdot A_{q-1} \cdot W \cdots A_2 \cdot W \cdot A_1 \cdot W \cdot A_0 \tag{4}$$

where $A_0, A_1, A_2 \cdots A_q$ involve the C's only. By what we have just proved,

$$\begin{aligned} A_q \cdot W &= W_{k_q} \cdot B_q \cdot \\ B_q \cdot A_{q-1} \cdot W &= W_{k_{q-1}} \cdot B_{q-1} \\ B_{q-1} \cdot A_{q-2} \cdot W &= W_{k_{q-2}} \cdot B_{q-2} \\ &\cdots\cdots\cdots \\ B_2 \cdot A_1 \cdot W &= W_{k_1} \cdot B_1 \\ B_1 \cdot A_0 &= B. \end{aligned}$$

Taking all these into consideration, we can reduce the form (4) to the form (3).

LEMMA 2. *Every expression in* $\mathfrak{A}$ *can be reduced further to a form* (3), in which

$$k_q \geqq k_{q-1} \geqq \cdots\cdots \geqq k_2 \geqq k_1$$

Proof. It is sufficient to prove that

$$W_j \cdot W_{k+1} = W_k \cdot W_j \qquad \text{for } k \geqq j. \tag{5}$$

The proof of (5) is as follows.

First,

$$W_1 \cdot W_2 = W_1 \cdot W_1. \tag{6}$$

For
$$\begin{aligned} W_1 \cdot W_2 &= W_1 \cdot C_1 \cdot W_1 \cdot C_2 \cdot C_1 && \text{by def.} \\ &= W_1 \cdot W_1 && \text{by V and VI.} \end{aligned}$$

Second,

$$W_1 \cdot W_{k+1} = W_k \cdot W_1 \qquad \text{for } k > 1. \tag{7}$$

For
$$\begin{aligned} W_1 \cdot W_{k+1} &= W_1 \cdot D_{k+1}\, W_1 \cdot [3,\, 4, \cdots k+2,\, 1,\, 2] && \text{by def.} \\ &= D_k \cdot W_1 \cdot [3,\, 1,\, 2] \cdot W \cdot [3,\, 4,\, 1,\, 2] \\ &\qquad \cdot [3,\, 4, \cdots k+2,\, 1,\, 2] && \text{by VII.} \end{aligned}$$

But since
$$\begin{aligned} &[3,\, 4,\, 1,\, 2] \cdot [3,\, 4, \cdots k+2,\, 1,\, 2] \\ &\qquad = [1,\, 2,\, 5,\, 6, \cdots k+2,\, 3,\, 4] \end{aligned}$$

and
$$\begin{aligned} &W_1 \cdot [1,\, 2,\, 5,\, 6, \cdots k+2,\, 3,\, 4] \\ &\qquad = [1,\, 4,\, 5, \cdots k+1,\, 2,\, 3] \cdot W_1 && \text{by IV.} \\ &[3,\, 1,\, 2] \cdot [1,\, 4,\, 5, \cdots k+1,\, 2,\, 3] \\ &\qquad = [3,\, 4, \cdots k+1,\, 1,\, 2] \end{aligned}$$

we have
$$\begin{aligned} W_1 \cdot W_{k+1} &= D_k \cdot W_1 \cdot [3,\, 4, \cdots k+1,\, 1,\, 2] \cdot W \\ &= W_k \cdot W_1 && \text{q. e. d.} \end{aligned}$$

Third,

$$C_k \cdot W_j = W_j \cdot C_k \qquad \text{for } k < j-1. \tag{8}$$

For by definition

$$C_k \cdot W_j = C_k \cdot D_j \cdot W \cdot [3,\, 4, \cdots j+1,\, 1,\, 2].$$

But we have
$$\begin{aligned} &C_k \cdot D_j = D_j \cdot C_{k+1} \\ &C_{k+1} \cdot W = W \cdot C_{k+2} \\ &C_{k+2} \cdot [3,\, 4, \cdots j+1,\, 1,\, 2] = [3,\, 4, \cdots j+1,\, 1,\, 2] \cdot C_k \end{aligned}$$

(the first and third relations follow from the known properties of permutations, the second from IV.) Combining these three we have (8).

Fourth,

$$W_j \cdot C_j = W_j. \tag{9}$$

For $j = 1$, this is true by V.

Suppose, now, (9) is true for $j-1$ (i. e. suppose $W_{j-1} \cdot C_{j-1} = W_{j-1}$) then

$$\begin{aligned} W_j \cdot C_j &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} \cdot C_j && \text{by def.} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_{j-1} \cdot C_j \cdot C_{j-1} && \text{by II.} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by hyp.} \\ &= W_j && \text{by def.} \end{aligned}$$

Fifth,

(10) $$C_{j+h} \cdot W_j = W_j \cdot C_{j+h+1} \qquad \text{for } h > 0$$

For $j = 1$, this is true by IV. Suppose it true for $j-1$, then

$$\begin{aligned} C_{j+h} \cdot W_j &= C_{j+h} \cdot C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by def.} \\ &= C_{j-1} \cdot C_{j+h} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by III} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_{j+h+1} \cdot C_j \cdot C_{j-1} && \text{by hyp.} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} \cdot C_{j+h+1} && \text{by III} \\ &= W_j \cdot C_{j+h+1} \qquad \text{q. e. d.} && \text{by def.} \end{aligned}$$

Sixth, $W_j \cdot W_{j+1} = W_j \cdot W_j$

For $j = 1$, this is true by (6). Suppose it true for $j-1$, then

$$\begin{aligned} W_j \cdot W_j &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by def. and I.} \\ &= C_{j-1} \cdot W_{j-1} \cdot W_{j-1} \cdot C_{j+1} \cdot C_j \cdot C_{j-1} && \text{by (10)} \\ &= C_{j-1} \cdot W_{j-1} \cdot W_j \cdot C_{j+1} \cdot C_j \cdot C_{j-1} && \text{by hyp.} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} \cdot C_{j-1} \cdot C_j \cdot W_j \cdot C_{j+1} \cdot C_j . C_{j-1} && \text{by I} \\ &= W_j \cdot C_{j-1} \cdot W_{j+1} \cdot C_{j-1} && \text{by def.} \\ &= W_j \cdot W_{j+1} \qquad \text{q. e. d.} && \text{by (8) and I.} \end{aligned}$$

Seventh,

$$W_j \cdot W_{k+1} = W_k \cdot W_j \qquad \text{for } k > j$$

For $j = 1$, this is true by (7). Suppose it true for $j-1$, then,

$$\begin{aligned} W_k \cdot W_j &= W_k \cdot C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by def.} \\ &= C_{j-1} \cdot W_k \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by (8), since } j-1 < k-1 \\ &= C_{j-1} \cdot W_{j-1} \cdot W_{k+1} \cdot C_j \cdot C_{j-1} && \text{by hyp.} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} \cdot W_{k+1} && \text{by (8)} \\ &= W_j \cdot W_{k+1} \qquad \text{q. e. d.} && \text{by def.} \end{aligned}$$

Definition: $W_k^r = W_k \cdot W_k \cdots W_k$ r times.

COROLLARY. *Every expression in* $\mathfrak{A}$ *can be reduced to the form*

(11) $$W^{r_q}_{k_q} \cdot W^{r_{q-1}}_{k_{q-1}} \cdots W^{r_2}_{k_2} \cdot W^{r_1}_{k_1} \cdot B$$

where $k_q > k_{q-1} > \cdots\cdots > k_2 > k_1, \qquad r_i > 0$

Proof. Simply collect the W's with equal indices in (3).

LEMMA 3. *If two expressions of the form* (11) *correspond to the same transformation, then they both have the same constants* $q, k_1, k_2, \cdots k_q, r_1\ r_2, \cdots r_q$.

Proof.

Let $$\Delta_q = W^{r_q}_{k_q} \cdot W^{r_{q-1}}_{k_{q-1}} \cdots W^{r_2}_{k_2} \cdot W^{r_1}_{k_1}$$

also let $$\Delta_q \sim \alpha = [a_1, a_2, a_3 \cdots\cdots]$$

Then for the explicit determination of the a_x we have

$$\begin{array}{lccc} \text{If} & 0 < x < k_1 & \text{then} & a_x = x \\ & k_1 \leqq x \leqq k_1 + r_1 & & a_x = k_1 \\ & k_1 + r_1 < x < k_2 + r_1 & & a_x = x - r_1 \end{array}$$

etc.

In general, if we define

$$s_i = r_1 + r_2 + \cdots + r_i$$
$$r_0 = 0, \quad s_0 = 0, \quad k_0 = 0$$

then, for $i = 1, 2, \cdots q$.

$$\begin{array}{lccc} \text{If} & k_{i-1} + s_{i-1} < x < k_i + s_{i-1} & \text{then} & a_x = x - s_{i-1} \\ & k_i + s_{i-1} \leqq x \leqq k_i + s_i & & a_x = k_i \\ & k_q + s_q < x & & a_x = x - s_q. \end{array}$$

These formulas follow directly from (1) and (2); their proofs are left to the reader.

The transformation has thus the following character: 1) the integers in the symbol $[a_1, a_2, a_3 \cdots]$ are arranged in their natural order with certain repetitions, 2) the only integers which appear more than once are $k_1, k_2 \cdots k_q$, and these appear respectively $r_1 + 1, r_2 + 1, \cdots\ r_q + 1$, times.

Now if $\beta = [b_1, b_2 \cdots]$ corresponds to B, then the transformation corresponding to $\Delta_q \cdot B$ is $\alpha \cdot \beta = [a_{b_1}, a_{b_2}, \cdots]$. By the restrictions on B the sequence $[a_{b_1}, a_{b_2}, \cdots]$ is merely a rearrangement of the sequence $[a_1, a_2 \cdots]$. Hence the second of the above properties applies just as much to $\alpha \cdot \beta$ as to α. But then the constants $q, k_1 \cdots k_q, r_1 \cdots r_q$, are uniquely determined by $\alpha \cdot \beta$, q. e. d.

LEMMA 4. *To each transformation of finite order there corresponds at least one expression of form* (11).

Proof.

Let the given transformation be $\gamma = [c_1, c_2 \cdots]$.

Let Δ_q be determined from γ as indicated in the preceding lemma, and suppose $\Delta_q \sim [a_1, a_2 \cdots] = \alpha$.

Then we can construct a permutation $\beta = [b_1, b_2 \cdots]$ such that $\alpha \cdot \beta = \gamma$, as follows. If c_i is distinct from any of the k's there is one and only one j such that $a_j = c_i$; in that case we must let $b_i = j$. On the other hand let the C's which are equal to k_j be $c_{i_1}, c_{i_2} \cdots c_{i_\rho}$ (where $\rho = r_j + 1$); then set $b_{i_1} = k_j + s_{j-1}$, $b_{i_2} = b_{i_1} + 1$, $b_{i_3} = b_{i_2} + 1$, etc. For definiteness we may suppose that $i_1 < i_2 < \cdots < i_\rho$. Then $\alpha \cdot \beta = \gamma$; for the i'th integer in the symbol for $\alpha \cdot \beta$ is a_{b_i}, and b_i has been so chosen that in all cases $a_{b_i} = c_i$.

To the β so constructed there corresponds a unique B in $\mathfrak{A}$, by Moore's result. For this B, $\Delta_q \cdot B \sim \gamma$, q. e. d.

LEMMA 5. *The operator in $\mathfrak{A}$ corresponding to a given transformation is unique.*

Proof.

Let A_1 and A_2 be two operators in $\mathfrak{A}$ which correspond to a given transformation γ. By Lemmas 2 and 3 there is a uniquely determined Δ_q such that

$$A_1 = \Delta_q \cdot B_1, \qquad A_2 = \Delta_q \cdot B_2$$

If $B_1 \sim \beta_1$, $B_2 \sim \beta_2$, β_1 and β_2 must be subject to all the restrictions to which β was subject in Lemma 4, except that it is not necessary to suppose $i_1 < i_2 < \cdots < i_\rho$. The only way in which β_1 and β_2 can differ is in the arrangement of the $b_{i_1}, b_{i_2} \cdots$ corresponding to each of the k_j. Hence

$$\beta_2 = \beta_3 \cdot \beta_1$$

where β_3 is a product of permutations, each of which permutes among themselves the integers composing one of the sets $k_j + s_{j-1}$, $k_j + s_{j-1} + 1$, $\cdots k_j + s_j$.

Let us now agree to denote by E_i^j, $i < j$ a combination of $C_i, C_{i+1} \cdots C_{j-1}$

(corresponding to a permutation of $i, i+1, \cdots j$). Then, in view of the isomorphism already established by Moore, we have the following result; there exists a B_3 such that

$$B_2 = B_3 \cdot B_1$$

$$B_3 = E^{k_1+r_1}_{k_1} \cdot E^{k_2+s_2}_{k_2+s_1} \cdot E^{k_3+s_3}_{k_3+s_2} \cdots E^{k_q+s_q}_{k_q+s_{q-1}}$$

To prove the lemma it is sufficient to show that $\Delta_q \cdot B_3 = \Delta_q$. This, in turn, follows from the above form for B_3, if we demonstrate that

$$\Delta_h \cdot E^{k_h+s_h}_{k_h+s_{h-1}} = \Delta_k \qquad (h = 1, 2, \cdots, q). \tag{12}$$

We turn to this last question forthwith.

In the first place, if

$$i > k+1,\ W_k \cdot E^j_i = E^{j-1}_{i-1} \cdot W_k.$$

This follows from (10), under Lemma 2.

Next, if $i > k+r$

$$W_k{}^r \cdot E_i{}^j = E^{j-r}_{i-r} \cdot W_k{}^r. \tag{13}$$

This is derived from the preceding by induction on r.

Third, if $i < h$

$$\Delta_i \cdot E^{k_h+s_h}_{k_h+s_{h-1}} = E^{k_h+s_h-s_i}_{k_h+s_{h-1}-s_i} \cdot \Delta_i. \tag{14}$$

For $i = 1$, this follows from the preceding, for the conditions $k_h + s_{h-1} > k_1 + r_1$ are satisfied since, if $h > 1$, $k_h > k_1$, $s_{h-1} \geqq s_1 = r_1$. For $i > 1$ we prove (14) by induction. Suppose it true for $i - 1$, then,

$$\begin{aligned} \Delta_i \cdot E^{k_h+s_{h-1}}_{k_h+s_h} &= W_{k_i}{}^{r_i} \cdot \Delta_{i-1} \cdot E^{k_h+s_h}_{k_h+s_{h-1}} \\ &= W_{k_i}{}^{r_i} \cdot E^{k_h+s_h-s_{i-1}}_{k_h+s_{h-1}-s_{i-1}} \cdot \Delta_{i-1}. \end{aligned}$$

In order, now, to apply (13), we need to know simply that

$$k_h + s_{h-1} - s_{i-1} > k_i + r_i$$

and this is fulfilled since for $h > i$,

$$s_{h-1} - s_{i-1} = r_i + r_{i+1} + \cdots + r_{h-1} \geqq r_i.$$

Hence

$$W_{k_i}{}^{r_i} \cdot E^{k_h+s_h-s_{i-1}}_{k_h+s_{h-1}-s_{i-1}} \cdot \Delta_{i-1} = E^{k_h+s_h-s_i}_{k_h+s_{h-1}-s_i} \cdot W_{k_i}{}^{r_i} \cdot \Delta_{i-1}$$

so that (14) is proved.

The following is the special case of (14) where $i = h - 1$

$$\Delta_{h-1} \cdot E^{k_h+s_h}_{k_h+s_{h-1}} = E^{k_h+r_h}_{k_h} \cdot \Delta_{h-1}.$$

In order to complete the proof of (12), it remains (since $\Delta_h = W^{r_h}_{k_h} \cdot \Delta_{h-1}$) simply to show

$$W^{r_h}_{k_h} \cdot E^{k_h+r_h}_{k_h} = W^{r_h}_{k_h} \tag{15}$$

In view of the composition of $E^{k_h+r_h}_{k_h}$, (15) follows from

$$W_k{}^r \cdot C_{k+j} = W_k{}^r \qquad \text{for } 0 \leqq j < r.$$

For $j = 0$, this is true by (9), under Lemma 2.

For $j = 1$

$$\begin{aligned} W_k \cdot W_k \cdot C_{k+1} &= W_k \cdot W_{k+1} \cdot C_{k+1} && \text{by (5)} \\ &= W_k \cdot W_{k+1} && \text{by (9)} \\ &= W_k \cdot W_k && \text{by (5)} \end{aligned}$$

$$\therefore \quad W_k{}^r \cdot C_{k+1} = W_k{}^r \qquad \text{for } r \geqq 2.$$

For $j > 1$, let $r = s + j - 1$, then $s \geqq 2$ and

$$\begin{aligned} W_k{}^r \cdot C_{k+j} &= W_k{}^s \cdot W_k{}^{j-1} \cdot C_{k+j} \\ &= W_k{}^s \cdot C_{k+1} \cdot W_k{}^{j-1} && \text{by (10)} \\ &= W_k{}^s \cdot W_k{}^{j-1} && \text{by the previous case} \\ &= W_k{}^r. \end{aligned}$$

These three cases together establish (15), which completes the proof of both the lemma and the theorem.

We have now achieved one of our objectives—viz. the elimination of Rule 4 so far as the selected subclass is concerned. For Rule 4 merely enables us to conclude that two entities of the subclass, whose application to a series $x_0 x_1 x_2 \cdots$ yields the same transformation, are equal. These entities correspond to the same transformation in the sense of our theorem, and therefore by that theorem are equal. Moreover the proof of the theorem is such that in any particular case a proof of the equality can be constructed in terms of I-VII, together with the rules for identity and no more. This proof will in general be quite long, but it may nevertheless be found.[*]

Finally we have an analysis of substitution in so far as that process is equivalent to transformation.

There remain open two questions: 1) What is the relation between the transforms of a function of $n-1$ variables, obtained by substituting a constant in a function of n variables, and the transforms of the original; 2) How can our infinite set of postulates be reduced to a finite set. Both these questions can be answered by introducing the entity B defined in Part II. But when we do that, it is expedient to consider a more extensive subclass. This takes us out of the domain of simple substitution. The topic is left for a later paper.

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[*] This statement is subject to reservation only in that the proof of Moore's theorem is indirect, but a direct proof may easily be found.

On Extending a Correspondence in the Sense of Antoine.[*]

By Harry Merrill Gehman.

Let M and M' denote point sets which are subsets of the point sets S and S' respectively and let T be a continuous $(1-1)$ correspondence[†] such that $T(M) = M'$; then if there exists a continuous $(1-1)$ correspondence U, such that $U(S) = S'$ and $U(M) = M'$, we shall say that the correspondence T can be *extended in the sense of Antoine*[‡] to a correspondence between S and S', or for brevity, T can be A-extended to a correspondence between S and S'.

Note than in order to be able to A-extend the correspondence T, it is not necessary that for each point P of M, $U(P)$ be the same as $T(P)$. That is, if the correspondence T can be *extended* in the sense in which we have used the word in a previous paper,[§] it can also be *A-extended,* but not conversely.

If M and M' are plane continuous curves and S and S' are planes, it has been proved in the paper just cited that the correspondence T can be extended (and hence A-extended) to a correspondence between the planes S and S' provided that sides are preserved under T. If sides are not preserved, T cannot be extended, but can nevertheless be A-extended provided that there exists another continuous $(1-1)$ correspondence V, such that $V(M) = M'$ and such that sides are preserved under V. If the continuous curve M is sufficiently simple in its constitution, this correspondence V always exists and *every* correspondence T can be A-extended to a correspondence between the planes.

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[*] Presented to the American Mathematical Society at Nashville, December 28th, 1927.

[†] A correspondence T between M and $T(M)$ is said to be *continuous,* if in case the point P of M is a limit point of N, a subset of M, then $T(P)$ is a limit point of $T(N)$. Since all the correspondences considered in this paper are continuous and $(1-1)$, we shall frequently use "correspondence" in place of "continuous $(1-1)$ correspondence."

[‡] L. Antoine, "Sur l'homéomorphie de deux figures et de leurs voisinages," *Journal de Mathématiques,* Ser. 8, Vol. 4 (1921), p. 221.

[§] H. M. Gehman, "On Extending a Continuous $(1-1)$ Correspondence of Two Plane Continuous Curves to a Correspondence of Their Planes," *Transactions of the American Mathematical Society,* Vol. 28 (1926), pp. 252-265.

The first object of this paper is to determine for what classes of plane continuous curves (classified according to the number of end points,[*] branch points [†] and simple closed curves, and a certain interior property to be defined next) it is true that if a correspondence T exists between two curves of the same class, a side preserving correspondence V also exists between them, and hence we can A-extend the correspondence T to a correspondence between the planes.

A necessary, but not sufficient, condition that a correspondence preserve sides is that it preserve interiors of simple closed curves. Hence in order that V exist, it is necessary to impose some interior condition on the two continuous curves. The condition used here (that they be in the same interior class) is an extremely light one, as is shown by Example 6 below. This example shows that the condition that two continuous curves be in the same interior class is not sufficient for the correspondence between them to be A-extended. The condition is necessary, of course.

DEFINITION. We shall say that the two plane continuous curves M and M' are *in the same interior class* with respect to the planes S and S' in which they lie, if there exists (a) a continuous $(1-1)$ correspondence T such that $T(M)=M'$, and (b) a $(1-1)$ correspondence between the set of all simple closed curves in M and the set of all simple closed curves in M', which is such that if J is a simple closed curve in M and J' is the corresponding simple closed curve in M', and if N is the set of all points of M which are interior to J and N' is the set of all points of M' which are interior to J', then there exists a continuous $(1-1)$ correspondence W, such that $W(N)=N'$.

THEOREM 1. *If M is a continuous curve lying in a plane S, and T is a continuous $(1-1)$ correspondence such that $T(M)=M'$, where M' lies in a plane S', then T can be A-extended to a correspondence between the planes S and S', provided that* (1) *M and M' are in the same interior class with respect to S and S', and* (2) *c, b, and e satisfy one of the following condi-*

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[*] For a number of equivalent definitions see: H. M. Gehman, "Concerning End Points of Continuous Curves and Other Continua," *Transactions of the American Mathematical Society*, Vol. 30 (1928), p. 63. Probably the simplest definition is one due to G. T. Whyburn: A point P of a continuous curve M is said to be an *end point* of M in case P is not an interior point of any arc in M.

[†] A point P of a continuous curve M is said to be a *branch point* of M in case M contains three arcs having P in common, but no two of them having any other point in common.

tions, where c is the number of simple closed curves in M, b is the number of branch points of M, and e is the number of end points of M:

(a) $c = 0$, $b \leqq 2$, *e is finite.*
(b) $c = 0$, $b = 3$, $e = 5$.
(c) $c = 1$, $b \leqq 1$, *e is finite.*
(d) $c = 1$, $b = e$, $e \leqq 3$.
(e) $c = 2$, $e \leqq 1$.
(f) $c = 3$, $e = 0$.
(g) $c = 4$, $b = 1$, $e = 0$.

Proof. By virtue of the preceding remarks, it will be sufficient to show that in each case there exists a continuous (1—1) correspondence V, such that $V(M) = M'$ and such that sides are preserved under V.

For each set of values of c, b, e satisfying the conditions of the theorem, we shall describe a collection $[M_x]$ of point sets, such that any continuous curve having the given values of c, b and e can be put into correspondence with one and only one of the sets of this collection. Then corresponding to each set M_x, we shall describe a collection of sets $M_{x,1}, M_{x,2}, \cdots, M_{x,k}$ lying in a plane S'', and such that if M is a given continuous curve which is in continuous $(1 - 1)$ correspondence with M_x and lies in a plane S, then there is one and only one set in this collection, say $M_{x,i}$, which is such that M and $M_{x,i}$ are in the same interior class with respect to S and S''. It will be obvious in each case that if M and $M_{x,i}$ are in the same interior class, there exists a continuous $(1 - 1)$ correspondence W, such that $W(M) = M_{x,i}$ and such that sides are preserved under W. If M and M' are in the same interior class with respect to S and S', then M' and $M_{x,i}$ will be in the same interior class with respect to S' and S''. Hence there exists a similar continuous $(1 - 1)$ correspondence W' such that $W'(M_{x,i}) = M'$, and such that sides are preserved under W'. The correspondence $W'W$ is the required correspondence V mentioned above.

We shall have need of these relations in the following discussion: if $c = 0$, then $e \geqq b + 2$; if $c = 1$, then $e \geqq b$; if $c = 2$, then $e + 2 \geqq b \geqq 1$; if $c = 3$, then $e + 4 \geqq b \geqq 1$. Note also that if $c = 0$, condition (1) of the theorem is superfluous.

Case (*a*1). If $c = b = 0$, then $e = 2$, and the set $M_1 = M_{1,1}$ is an arc.

Case (*a*2). If $c = 0$, $b = 1$, $e = n$, the set $M_2 = M_{2,1}$ consists of n arcs, each having the point B for one end point, and no two of them having any other point in common.

Case (*a*3). If $c = 0$, $b = 2$, $e = n$, the set $M_3 = M_{3,1}$ consists of (1) an arc joining the points B_1 and B_2; (2) a collection of i arcs, where

$2 \leqq i \leqq n-2$, each arc having the point B_1 and no other point in common with the arc B_1B_2, and no two of them having any other point in common; and (3) a collection of $n-i$ arcs each having the point B_2 and no other point in common with B_1B_2, no two of them having any other point in common, and no one of them having a point in common with any arc of the other collection.

Case (b). If $c=0$, $b=3$, $e=5$, the set $M_4=M_{4,1}$ consists of four arcs E_1B_1, E_2B_2, E_3B_3, E_4E_5 which are such that B_1, B_2, B_3 are interior points of the arc E_4E_5, and except for these three points, no two of these arcs have any point in common.

Case ($c1$). If $c=1$, $b=0$, then $e=0$, and the set $M_5=M_{5,1}$ is a simple closed curve.

Case ($c2$). If $c=1$, $b=1$, $e=n$, the set M_6 consists of a simple closed curve C, and n arcs $E_1B, E_2B, \cdots, E_nB$, each having the point B and no other point in common with C, and no two of them having any other point in common. The sets $M_{6,1}, M_{6,2}, \cdots, M_{6,n+1}$ are defined thus: in the set $M_{6,i}$, there are $i-1$ end points interior to C, and $n-i+1$ end points exterior to C.

Case ($d1$). The case $c=1$, $b=e=0$ is the same as ($c1$).

Case ($d2$). The case $c=1$, $b=e=1$ is a special case of ($c2$).

Case ($d3$). For $c=1$, $b=e=2$, there are two sets: M_7 and M_8. The set M_7 consists of a simple closed curve C and two arcs E_1B_1, E_2B_2 having no point in common and having respectively the point B_1, B_2 and no other point in common with C. In the set $M_{7,1}$ both end points are exterior to C; in the set $M_{7,2}$ one is interior and one exterior to C; in the set $M_{7,3}$ both are interior to C.

The set M_8 consists of (1) three arcs E_1B_2, E_2B_2, B_1B_2, no two of which have any point in common except B_2, and (2) a simple closed curve C which contains B_1, but which has no other point in common with any one of the three arcs. In the set $M_{8,1}$ the point B_2 is exterior to C; in the set $M_{8,2}$ it is interior to C.

Case ($d4$). For $c=1$, $b=e=3$, there are three sets: M_9, M_{10}, M_{11}. The set M_9 consists of the set M_7 and an arc E_3B_3 having B_3 in common with $C-B_1-B_2$ and having no other point in common with M_7. The sets $M_{9,1}, \cdots, M_{9,4}$ are defined thus: in the set $M_{9,i}$ there are $i-1$ end points interior to C, and $4-i$ exterior to C.

The set M_{10} consists of the set M_9 and an arc E_3B_3 having B_3 in common with $C - B_1$ and having no other point in common with M_9. The sets $M_{10,1}, \cdots, M_{10,4}$ are defined in the same way as $M_{9,1}, \cdots, M_{9,4}$ were defined.

The set M_{11} consists of the set M_8 and an arc E_3B_3 having B_3 and no other point in common with M_8, where B_3 is an interior point of the arc B_1E_1. The sets $M_{11,1}$ and $M_{11,2}$ are defined in the same way as $M_{8,1}$ and $M_{8,2}$ were defined.

Case (*e*1). If $c = 2$, $b = 1$, $e = 0$, the set M_{12} consists of two simple closed curves C_1, C_2 having the point B_1 and no other point in common. In the set $M_{12,1}$, C_2 is exterior to C_1; in the set $M_{12,2}$, C_2 is interior to C_1.

Case (*e*2). If $c = 2$, $b = 2$, $e = 0$, the set M_{13} consists of two simple closed curves C_1, C_2 having no point in common, and an arc B_1B_2 having B_1 in common with C_1 and B_2 in common with C_2, and having no other point in common with either of the simple closed curves. The sets $M_{13,1}$ and $M_{13,2}$ are defined in the same way as $M_{12,1}$ and $M_{12,2}$ were defined.

Case (*e*3). If $c = 2$, $b = 1$, $e = 1$, the set M_{14} consists of the set M_{12} and an arc E_1B_1 having no point in common with M_{12} except B_1. In the set $M_{14,1}$, C_1 is exterior to C_2, and E_1 is exterior to both C_1 and C_2; in $M_{14,2}$, C_1 is exterior to C_2, and E_1 is interior to C_1; in $M_{14,3}$, C_2 is interior to C_1, and E_1 is exterior to C_1; in $M_{14,4}$, E_1 and C_1 are interior to C_2, and E_1 is exterior to C_1; in $M_{14,5}$, E_1 is interior to C_1, and C_1 is interior to C_2.

Case (*e*4). For $c = 2$, $b = 2$, $e = 1$, there are two sets: M_{15} and M_{16}. The set M_{15} consists of the set M_{12} and an arc E_1B_2 having B_2 in common with $C_1 - B_1$ and having no other point in common with M_{12}. The sets $M_{15,1}, \cdots, M_{15,5}$ are defined in the same way as $M_{14,1}, \cdots, M_{14,5}$ were defined; in the set $M_{15,6}$, C_2 and E_1 are interior to C_1.

The set M_{16} consists of the set M_{13} and an arc E_1B_1 having B_1 and no other point in common with M_{13}. The sets $M_{16,1}, \cdots, M_{16,6}$ are defined in the same way as $M_{15,1}, \cdots, M_{15,6}$ were defined.

Case (*e*5). For $c = 2$, $b = 3$, $e = 1$, there are two sets: M_{17} and M_{18}. The set M_{17} consists of the set M_{13} and an arc E_1B_3 having B_3 in common with $C_1 - B_1$ and having no other point in common with M_{13}. The sets $M_{17,1}, \cdots, M_{17,6}$ are defined in the same way as $M_{15,1}, \cdots, M_{15,6}$ were defined.

The set M_{18} consists of the set M_{13} and an arc E_1B_3 having B_3 in common with $B_1B_2 - B_1 - B_2$ and having no other point in common with M_{13}. The sets $M_{18,1}$ and $M_{18,2}$ are defined in the same way as $M_{12,1}$ and $M_{12,2}$ were defined.

Case (*f*1). If $c=3$, $b=1$, $e=0$, the set M_{19} consists of three simple closed curves C_1, C_2, C_3 each containing the point B_1 and no two of them having any other point in common. In the set $M_{19,1}$, each simple closed curve is exterior to each of the other two; in $M_{19,2}$, C_2 and C_3 are interior to C_1 and exterior to each other; in $M_{19,3}$, C_1 is interior to C_2, and C_2 is exterior to C_3; in $M_{19,4}$, C_1 is interior to C_2, and C_2 is interior to C_3.

Case (*f*2). For $c=3$, $b=2$, $e=0$, there are three sets: M_{20}, M_{21}, M_{22}. The set M_{20} consists of three arcs joining the same pair of points, no two of the arcs having any interior points in common. Any two plane point sets which are in continuous $(1-1)$ correspondence with M_{20} are in the same interior class with respect to the planes in which they lie; hence $M_{20,1}=M_{20}$.

The set M_{21} consists of the set M_{12} and a simple closed curve C_3 having the point B_2 in common with C_2-B_1 and having no other point in common with M_{12}. The sets $M_{21,1}, \cdots, M_{21,4}$ are defined in the same way as $M_{19,1}, \cdots, M_{19,4}$ were defined; in the set $M_{21,5}$, C_1 and C_3 are interior to C_2 and exterior to each other.

The set M_{22} consists of the set M_{13} and a simple closed curve C_3 having B_2 and no other point in common with M_{13}. The sets $M_{22,1}, \cdots, M_{22,5}$ are defined in the same way as $M_{21,1}, \cdots, M_{21,5}$ were defined; in the set $M_{22,6}$, C_3 is interior to C_2, and C_2 is exterior to C_1; in the set $M_{22,7}$, C_3 is interior to C_2, and C_2 is interior to C_1.

Case (*f*3). For $c=3$, $b=3$, $e=0$, there are two sets: M_{23} and M_{24}. The set M_{23} consists of the set M_{13} and a simple closed curve having the point B_3 in common with C_2-B_2 and having no other point in common with M_{13}. The sets $M_{23,1}, \cdots, M_{23,7}$ are defined in the same way as $M_{22,1}, \cdots, M_{22,7}$ were defined; in the set $M_{23,8}$, C_1 and C_2 are interior to C_3 and exterior to each other.

The set M_{24} consists of (1) the set M_{13}; (2) a simple closed curve C_3 having no point in common with M_{13}; and (3) an arc B_2B_3 having B_2 and no other point in common with M_{13}, and having B_3 and no other point in common with C_3. The sets $M_{24,1}, \cdots, M_{24,5}$ are defined in the same way as $M_{21,1}, \cdots, M_{21,5}$ were defined.

Case (*f*4). For $c=3$, $b=4$, $e=0$, there are two sets: M_{25} and M_{26}. The set M_{25} consists of (1) the set M_{13}; (2) a simple closed curve C_3 having no point in common with M_{13}; and (3) an arc B_3B_4 having B_3 and no other point in common with C_3, having B_4 in common with $B_1B_2 - B_1 - B_2$, and having no other point in common with M_{13}. The sets $M_{25,1}$ and $M_{25,2}$ are defined in the same way as $M_{19,1}$ and $M_{19,2}$ were defined.

The set M_{26} consists of (1) the set M_{13}; (2) a simple closed curve C_3 having no point in common with M_{13}; and (3) an arc B_3B_4 having B_3 and no other point in common with C_3, having B_4 in common with $C_2 - B_2$, and having no other point in common with M_{13}. The sets $M_{26,1}, \cdots, M_{26,5}$ are defined in the same way as $M_{21,1}, \cdots, M_{21,5}$ were defined.

Case (*g*). If $c = 4$, $b = 1$, $e = 0$, the set M_{27} consists of four simple closed curves C_1, C_2, C_3, C_4 each containing the point B_1, and no two of them having any other point in common. In the set $M_{27,1}$, each simple closed curve is exterior to each of the other three; in $M_{27,2}$, C_1 is interior to C_2, and each of the simple closed curves C_2, C_3, C_4 is exterior to each of the other two; in $M_{27,3}$, C_1 is interior to C_2, C_2 is exterior to C_4, and C_3 is interior to C_4; in $M_{27,4}$, C_1 and C_2 are interior to C_3 and exterior to each other, and C_4 is exterior to C_3; in $M_{27,5}$, C_1 is interior to C_2, C_2 is interior to C_3, C_3 is exterior to C_4; in $M_{27,6}$, C_1, C_2, C_3 are each interior to C_4, and each one is exterior to each one of the other two; in $M_{27,7}$, C_1 is interior to C_2, C_3 is exterior to C_2, C_2 and C_3 are interior to C_4; in $M_{27,8}$, C_1 and C_2 are interior to C_3 and exterior to each other, C_3 is interior to C_4; in $M_{27,9}$, C_1 is interior to C_2, C_2 is interior to C_3, C_3 is interior to C_4.

This completes the proof of Theorem 1. We shall now show that the hypotheses of this theorem cannot be further weakened and the conclusion remain true. We give below a collection of examples in each of which are given two plane continuous curves M and M', which are in the same interior class, and for which there exists a continuous $(1-1)$ correspondence T such that $T(M) = M'$. In each case c, b, e do not satisfy any one of the conditions (2) of the theorem and T cannot be A-extended.

Example 1. For $c = 0$, $b = 1$, e *infinite,* the set M consists of the straight line intervals from $(0, 0)$ to $(1, 0)$, and from $(0, 0)$ to $(1/n, 1/n^2)$, for $n = 1, 2, 3, \cdots$, and the set M' consists of the straight line intervals from $(0, 0)$ to $(1, 0)$, and from $(0, 0)$ to $[1/n, (-1)^n/n^2]$, for $n=1, 2, 3, \cdots$.

Example 2. For $c = 0$, $b = 2$, e *infinite,* M and M' are the same sets as in Example 1 with the addition to each of the straight line interval from $(1/2, 0)$ to $(1, 1/2)$.

Example 3. For $c = 0$, $b = 3$, $e = 6$, the set M consists of the straight line intervals from $(0, 0)$ to $(4, 0)$, from $(1, 0)$ to $(1, 1)$, from $(2, 0)$ to $(2, 1)$, from $(2, 0)$ to $(5/2, 1)$, from $(3, 0)$ to $(3, 1)$, and M' is the same set except that the interval from $(2, 0)$ to $(5/2, 1)$ is replaced by the interval from $(2, 0)$ to $(2, -1)$.

Example 4. For $c = 0$, $b = 4$, $e = 6$, the set M consists of the straight line intervals from $(0, 0)$ to $(5, 0)$, and from $(n, 0)$ to $(n, 1)$ for $n = 1, 2, 3, 4$, and M' is the same set except that the interval from $(3, 0)$ to $(3, 1)$ is replaced by the interval from $(3, 0)$ to $(3, -1)$.

Example 5. For $c = 1$, $b = 1$, e *infinite*, M and M' are the same sets as in Example 1, with the addition to each of the circle $(x + 1)^2 + y^2 = 1$.

Example 6. For $c = 1$, $b = 2$, $e = 3$, the set M consists of the circle $x^2 + y^2 = 4$, and the straight line intervals from $(0, 1)$ to $(0, 2)$, and from $(1, 0)$ to $(3, 0)$, and M' consists of the same circle and the intervals from $(0, 2)$ to $(0, 3)$, from $(1, 0)$ to $(2, 0)$, and from $(1, 1)$ to $(2, 0)$.

Example 7. For $c = 1$, $b = 3$, $e = 4$, the sets M and M' are the same sets as in Example 6 with the addition to each of the interval from $(3/2, 0)$ to $(3/2, 1/4)$.

Example 8. For $c = 1$, $b = 4$, $e = 4$, the sets M and M' are the same sets as in Example 4 with the addition to each of the interval from $(0, 0)$ to $(1, 1)$.

Example 9. For $c = 2$, $b = 1$, $e = 2$, the set M consists of the straight line intervals from $(-1, 0)$ to $(1, 0)$, from $(0, -1)$ to $(0, 1)$, from $(-1, -1)$ to $(1, 1)$, from $(-1, -1)$ to $(-1, 0)$, and from $(1, 0)$ to $(1, 1)$, and M' is the same set except that the interval from $(1, 0)$ to $(1, 1)$ is replaced by the interval from $(0, 1)$ to $(1, 1)$.

Example 10. For $c = 3$, $b = 1$, $e = 1$, the set M consists of the straight line intervals from $(0, 0)$ to $(0, 4)$, from $(0, 4)$ to $(7, 4)$, from $(7, 4)$ to $(7, 0)$, from $(7, 0)$ to $(0, 0)$, from $(0, 0)$ to $(n, 1)$, for $n = 1, 2, 3, 4, 5$, from $(1, 1)$ to $(2, 1)$, and from $(3, 1)$ to $(4, 1)$, and M' is the same set except that the interval from $(3, 1)$ to $(4, 1)$ is replaced by the interval from $(4, 1)$ to $(5, 1)$.

Example 11. For $c = 4$, $b = 1$, $e = 1$, the sets M and M' are the same sets as in Example 10 with the addition to each of the circle $(x+1)^2+y^2=1$.

Example 12. For $c=4$, $b=2$, $e=0$, the sets M and M' are the same sets as in Example 10 with the addition to M of the circle $(x-5)^2+(y-2)^2=1$, and with the addition to M' of the circle $(x-3)^2+(y-2)^2=1$.

Example 13. For $c=5$, $b=1$, $e=0$, the set M consists of the set M of Example 10 with the addition of the straight line intervals from $(0, 0)$ to $(6, 1)$, from $(5, 1)$ to $(6, 1)$, from $(0, 0)$ to $(2/3, 1/2)$, from $(2/3, 1/2)$ to $(5/6, 1/2)$, and from $(5/6, 1/2)$ to $(0, 0)$, and M' is the same set except that the last three intervals are replaced by the intervals from $(0, 0)$ to $(5/3, 1/2)$, from $(5/3, 1/2)$ to $(11/6, 1/2)$, and from $(11/6, 1/2)$ to $(0, 0)$.

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The second object of this paper is to prove a theorem which includes as special cases two of Antoine's theorems which occur in sections 22 and 23 of his paper.

DEFINITION. We shall say that the plane point set M is *reversible,* if given a simple closed curve J_0 enclosing M and a finite collection of simple closed curves $J_1, J_2, \cdots, J_n$ each one exterior to each of the others and each lying in a bounded domain complementary to M, there exists a continuous $(1-1)$ correspondence V, such that (1) $V(D)=D$, where D denotes that portion of the plane which is interior to J_0 and exterior to each of the simple closed curves $J_1, J_2, \cdots, J_n$; (2) $V(M)=M$; (3) $V(J_i)=J_i$, for $i=0, 1, 2, \cdots, n$; and (4) if A, B, C are three points on the simple closed curve J_0, the sense ABC on J_0 is opposite to the sense $V(A)V(B)V(C)$ on J_0.

LEMMA *A. If the plane point set M is reversible, then* (5) *if D, E, F are three points on the simple closed curve J_i $(i=0, 1, \cdots, n)$ the sense DEF on J_i is opposite to the sense $V(D)V(E)V(F)$ on J_i.*

Of the plane continuous curves described in the proof of Theorem 1, all are reversible with the exception of the sets with the following subscripts: 11, 1 and 11, 2; 14, 1 and 14, 4; 16, 1 and 16, 4 and 16, 6; 18, 1 and 18, 2; 19, 1 and 19, 2; 22, 1 and 22, 2 and 22, 5; 24, 1 and 24, 2 and 24, 5; 25, 1 and 25, 2; 27, 1; 27, 2; 27, 4; 27, 6; 27, 7 and 27, 8. We can therefore state the following lemma:

LEMMA *B. The plane continuous curve M is reversible if c, b, e satisfy one of the following conditions:*

(a) $c=0$, $b\leqq 2$, *e is finite.*
(b) $c=0$, $b=3$, $e=5$.
(c) $c=1$, $b\leqq 1$, *e is finite.*
(d′) $c=1$, $b=e$, $e\leqq 2$.
(e′) $c=2$, $e=0$.

THEOREM 2. *Given* (1) *a point set* M *lying in a plane* S, *and a continuous* $(1-1)$ *correspondence* T, *such that* $T(M)=M'$, *where* M' *lies in a plane* S'; (2) *each component** *of* M *is bounded and except for at most one component, each is reversible;* (3) *if* C *denotes a simple closed curve in either* S *or* S', C *encloses points of at most a finite number of components of either* M *or* M'; (4) *for each component* M_i *of* M, *the correspondence* T *between* M_i *and the component* $T(M_i)=M_i'$ *of* M' *can be extended to a correspondence between the planes* S *and* S'; *and* (5) *if* M_i *and* M_j *denote any two components of* M, *then* M_i *lies in the domain of* $S-M_j$ *bounded by the subset* B *of* M_j, *if and only if* $T(M_i)=M_i'$ *lies in the domain of* $S'-M_j'$ *bounded by* $T(B)$. *Under these conditions, the correspondence* T *between* M *and* M' *can be A-extended to a correspondence between the planes* S *and* S'.

Proof. The components of M are countable, by (3). Let us denote them by $M_1, M_2, M_3, \cdots$. Then the components of M' are $M_1', M_2', M_3', \cdots$, where $T(M_i)=M_i'$. Let us suppose them so numbered that for $i>1$, M_i and M_i' are reversible.

By (4), there exists corresponding to each component M_i of M, a continuous $(1-1)$ correspondence U_i, such that $U_i(S)=S'$, and for each point P of M_i, $U_i(P)=T(P)$. Hence $U_i(M_i)=T(M_i)=M_i'$.

By (3), no point of M or of M' is a limit point of the set consisting of all components which do not contain the given point. Hence, corresponding to each component M_i, we can construct a simple closed curve J_i in S, such that (a) J_i has no points in common with M or with $J_1+J_2+\cdots+J_{i-1}$; (b) J_i encloses M_i but encloses no other components of M save those which lie in a bounded domain of $S-M_i$; (c) the simple closed curve $U_i(J_i)=J_i'$ in S' has no points in common with M' or with $J_1'+J_2'+\cdots+J'_{i-1}$; and (d) J_i' encloses M_i' but encloses no other components of M' save those which lie in a bounded domain of $S'-M_i'$.

By (3), each simple closed curve J_i encloses at most a finite number of simple closed curves of the collection $J_1+J_2+\cdots$. Of all those which are enclosed by J_i, let $J_{i_1}, J_{i_2}, \cdots, J_{i_x}$ denote those which are not interior to any other simple closed curve enclosed by J_i. Then $J_i+J_{i_1}+J_{i_2}+\cdots+J_{i_x}$ form the boundary of a bounded domain D_i which is complementary to the set $J_1+J_2+\cdots$. The simple closed curve J_i is the outer boundary of D_i; each of the simple closed curves J_{i_n} $(n=1,2,\cdots,x)$ is a component of the inner boundary of D_i. The domain D_i contains M_i but contains no points of $M - M_i$. Similarly in S', J_i' is the outer bounda and $J'_{i_1} + J'_{i_2} + \cdots + J'_{i_x}$ the inner boundary of a domain D_i', which co tains M_i' but contains no points of $M' - M_i'$.

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* A *component* (= maximal connected subset) of a point set M is a connected subset of M which is not a proper subset of any other connected subset of M.

In order to prove the theorem we must define a continuous (1 — 1) correspondence U such that $U(S) = S'$ and $U(M) = M'$. We shall do t in such a way that for each component M_i of M, $U(M_i) = T(M_i) = M_i'$. We shall first define U for points of D_1 and its boundary.

Let $d_1, d_2, \cdots, d_{k_1}$ denote those domains of $S - M_1$ that contain co ponents of the inner boundary of D_1. By (5), $U_1(d_i) = d_i'$ is a domain $S' - M_1'$ which contains a component of the inner boundary of D_1'. N let V_1 be a continuous (1 — 1) correspondence such that (a) $V_1(P) = U_1($ for all points P of $S - (d_1 + d_2 + \cdots + d_{k_1})$, and hence $V_1(M_1) = M$ and $V_1(J_1) = J_1'$; (b) $V_1(d_i) = d_i'$, for $i = 1, 2, \cdots, k_1$, and he $V_1(S) = S'$; and (c) $V_1(J_{1_n}) = J'_{1_n}$, for $n = 1, 2, \cdots, x_1$, and he $V_1(D_1) = D_1'$. We shall let $U(P) = V_1(P)$ for all points of D_1 and boundary.

Let D_a be a domain such that U has been defined for points of Let A, B, C be three points of J_a. If the sense $U(A)U(B)U(C)$ is t same as the sense $U_a(A)U_a(B)U_a(C)$ on the simple closed curve J_a', th exists a continuous (1 — 1) correspondence V_a such that (a) $V_a(S) = S'$ (b) for each point P of J_a, $V_a(P) = U(P)$, and hence $V_a(J_a) = J'$ (c) for each point P of M_a, $V_a(P) = U_a(P)$, and hence $V_a(M_a) = T(M_a) = M'$ and (d) $V_a(J_{a_n}) = J'_{a_n}$, for $n = 1, 2, \cdots, x_a$, and hence $V_a(D_a) = D'$ Since M_a is reversible, even if the two senses are opposite, there exists a correspondence V_a such that (a), (b) and (d) are true, and (c) is replac (c') $V_a(M_a) = T(M_a) = M_a'$. In both cases, we shall let $U(P) = V_a($ for all points of D_a and its boundary.

Let D_b be a domain such that U has been defined for points of t component J_{b_1} of its inner boundary. Here again there exists a continuo (1 — 1) correspondence V_b such that (a) for each point P of J $V_b(P) = U(P)$; (b) $V_b(M_b) = T(M_b) = M_b'$; (c) $V_b(J_b) = J_b'$; ($V_b(J_{b_n}) = J'_{b_n}$, for $n = 2, 3, \cdots, x_b$; and (e) $V_b(D_b) = D_b'$. We shall $U(P) = V_b(P)$ for all points of D_b and its boundary.

Having defined U for points of D_1, we can by the process outlined the two preceding paragraphs, define U for all points of S interior to and for all points of S interior to a simple closed curve J_c which encloses If this includes all points of S, the theorem has been proved. In case th does not include all points of S, we shall construct in S an infinite collection of simple closed curves $C_1, C_2, C_3, \cdots$, such that (a) C_i is interior to C_{i+1},

(b) C_i has no points in common with M; (c) for each point P of S there is some integer k such that P is interior to C_k. In S', we shall construct an infinite collection of simple closed curves $C_1', C_2', C_3', \cdots$, such that (a) C_i' is interior to C'_{i+1}; (b) C_i' has no points in common with M'; (c) for each point P of S' there is some integer k such that P is interior to C_k'; (d) C_i' encloses a component M_j' of M', if and only if C_i encloses M_j. Let W_i $(i = 1, 2, 3, \cdots)$ be a continuous $(1-1)$ correspondence such that $W_i(C_i) = C_i'$. If we let $N = M + C_1 + C_2 + \cdots$ and $N' = M' + C_1' + C_2' + \cdots$ and let V be a continuous $(1-1)$ correspondence such that for each point P of M, $V(P) = T(P)$, and for each point P of C_i, $V(P) = W_i(P)$, then N, N', V satisfy the conditions of Theorem 2. If then we order the components of N, construct a new collection of simple closed curves $J_1, J_2, J_3, \cdots$, and proceed as outlined above, a correspondence U will be defined for all points of S in such a way that $U(S) = S'$ and $U(M) = M'$, which proves Theorem 2.

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Green's Functions for Differential Systems Containing a Parameter

BY W. W. ELLIOTT.

1. *Introduction.* This paper is a continuation of the paper entitled 'Generalized Green's Functions for Compatible Differential Systems,' which appeared in Vol. 50 of this Journal, pp. 243-258. The notations and equation numbers of the former paper are continued in this paper, and references are made accordingly. The ϕ_i's and the ψ_i's are respectively assumed to form biorthogonal sets with the u_i's and the v_i's as suggested at the close of the above named paper.

2. *Relations Between Differential and Integral Systems Containing a Parameter.* Let us now consider in connection with the compatible differential system (1), the system

$$L(u) = \lambda u, \qquad U_p(u) = 0, \qquad (p = 1, \cdots, n), \tag{29}$$

where λ is a parameter ranging over the complex plane.

The system adjoint to (29) will be

$$M(v) = \lambda v, \qquad V_p(v) = 0, \qquad (p = 1, \cdots, n). \tag{30}$$

In general systems (29) and (30) are compatible for some values of λ and incompatible for others. However, in special cases systems (29) and (30) are incompatible for all values of λ. Such systems will possess ordinary Green's functions for all parameter values. Cases also arise in which systems (29) and (30) are compatible for all* values of λ. Such systems possess generalized Green's functions for all parameter values. In ordinary cases, systems (29) and (30) will possess ordinary Green's functions for some parameter values and generalized Green's functions for other parameter values. If systems (29) and (30) are compatible of order h for some value of λ, their solutions will be respectively denoted by $u_1^\lambda, \cdots, u_h^\lambda$, and $v_1^\lambda, \cdots, v_h^\lambda$. The ϕ_i's and ψ_i's will also be given a superscript to denote the value of λ for which they are taken.

The generalized Green's functions will be given two superscripts. The first one will be the value of λ, and the second will denote the order of compatibility of the given system for this value of λ. In this notation G with a second superscript zero will mean an ordinary Green's function, while with the second superscript different from zero it will mean a generalized Green's function. The assumption as to what parameter value we are considering, the order of compatibility for this value, and the indication whether the Green's function considered is the ordinary or generalized one will then be implied from this notation without the necessity of explicit statement in each case.

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* M. Bôcher, *Les Méthodes de Sturm*, p. 42.

At this point several natural choices of the ϕ_i^λ's and ψ_i^λ's may be presented.

The ϕ_i^λ's and ψ_i^λ's may be taken respectively as sets of functions linearly equivalent to the conjugates of the sets of solutions u_i^λ's and v_i^λ's. The Green's function resulting from this choice of the ϕ_i^λ's and ψ_i^λ's will be given the subscript 1.

Some of the following work will simplify if a set of ϕ_i^λ's can be found which, in addition to forming a biorthogonal set with the u_i^λ's, are orthogonal to each u^μ, $(\mu \neq \lambda)$, and if a set of ψ_i^λ's can be found which, in addition to forming a biorthogonal set with the v_i^λ's, are orthogonal to each v^μ, $(\mu \neq \lambda)$. The Green's functions resulting from this choice of the ϕ_i^λ's and ψ_i^λ's, when they exist, will be given the subscript zero.

Provided each u^λ multiplied by v^λ and integrated from a to b does not give zero, the above condition will be satisfied by the ϕ_i^λ's and ψ_i^λ's, if the ϕ_i^λ's are chosen linearly equivalent to the v_i^λ's and the ψ_i^λ's to the u_i^λ's. The Green's function resulting from this further special choice, when it exists, will be given the double subscript 01.

It should be noted that the functions $G_1(xy)$ and $G_{01}(xy)$ depend solely upon the given differential system and are independent of any particular choice of the complete sets of solutions.

The function $G_{01}(xy)$ always exists for Hermitian differential systems. In fact for Hermitian differential systems $G_{01}(xy) = G_1(xy)$. In dealing with Hermitian differential systems in the future we shall always use the special function $G_1(xy)$, as it will serve to give all the facts. For other systems, so far as possible, the general Green's function without special choice of the ϕ_i^λ's and ψ_i^λ's will be used. The results for special choices of the ϕ_i^λ's and ψ_i^λ's will be pointed out specifically only when they are more simple than in the general case.

If zero is taken as one of the chosen parameter values some of the work simplifies. An assumption about the differential system for $\lambda = 0$ is essentially equivalent to an assumption about it for any fixed parameter value, since we can always effect a linear transformation on the parameter.

From Theorem V and Cor. II readily follows

THEOREM VI. *If the differential system is Hermitian and* $n > 1$, *then*

$$G_1^{\lambda k}(xy) \equiv G_1^{\lambda k}(yx).$$

THEOREM VII. *Solutions of the integral equation*

$$u(x) = \lambda \int_a^b \left[G^{0k}(xs) + (1/\lambda) \sum_{i=1}^{k} u_i{}^0(x)\phi_i{}^0(s)\right] u(s)\,ds, \tag{31}$$

are linearly equivalent to those of the system

$$\begin{aligned} &L(u) = \lambda u(x) - \lambda \sum_{i=1}^{k} \psi_i{}^0(x) \int_a^b v_i{}^0(s)u(s)\,ds, \\ &\Gamma_p(u) = 0, \qquad\qquad (p = 1, \cdots, n). \end{aligned} \tag{32}$$

That any solution of (31) satisfies (32) is seen at once by applying the differential operator L and the boundary conditions to equation (31). To show that any solution of (32) satisfies (31) apply Green's theorem, letting $u = u(x)$, any solution of (32) which has a continuous n^{th} derivative, $v = H^{0k}(xy)$, and simplify.

THEOREM VIII. *Solutions of the equation*

$$\theta(x) = \lambda \int_a^b \left[G^{0k}(xs) + (1/\lambda) \sum_{i=1}^{k} u_i{}^0(x)\,\phi_i{}^0(s)\right] \theta(s)\,ds, \tag{33}$$

are expressible linearly in terms of those of the system

$$\begin{aligned} &\theta_i(x) = u_i{}^0(x) + \lambda \int_a^b G^{0k}(xs)\theta_i(s)\,ds, \qquad (i = 1, \cdots, k), \\ &\theta_0(x) = \lambda \int_a^b G^{0k}(xs)\theta_0(s)\,ds. \end{aligned} \tag{34}$$

By way of proof, equation (33) may be written in the form

$$\theta(x) = \lambda \int_a^b G^{0k}(xs)\theta(s)\,ds + \sum_{i=1}^{k} c_i u_i{}^0(x), \tag{35}$$

where $c_i = \int_a^b \phi_i{}^0(s)\theta(s)\,ds$, $(i = 1, \cdots, k)$. Each c_i is given in terms of θ, but if we write equation (35) for arbitrary c_i it follows, on multiplying by $\theta_i{}^0(x)$, $(i = 1, \cdots, k)$, and integrating with respect to x from a to b, that

$$c_i = \int_a^b \phi_i{}^0(s)\theta(s)\,ds, \qquad (i = 1, \cdots, k),$$

and (35) is satisfied. Therefore it suffices to consider (35) for arbitrary c_i. Assign to c_i, $(i=1,\cdots,k)$, the k sets of values

$$\begin{array}{l} 1,\ 0,\ 0,\ \cdots,\ 0, \\ 0,\ 1,\ 0,\ \cdots,\ 0, \\ 0,\ 0,\ 1,\ \cdots,\ 0, \\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . \\ 0,\ 0,\ 0,\ \cdots,\ 1. \end{array}$$

Upon substituting these sets of values in (35) the following equations are obtained

$$(36) \qquad \theta_i(x) = u_i^0(x) + \lambda \int_a^b G^{0k}(xs)\theta_i(s)\,ds, \qquad (i=1,\cdots,k).$$

Now let $\theta(x)$ be any solution of (35) and form the difference

$$(37) \qquad \theta_0(s) = \theta(s) - \sum_{i=1}^{k} c_i\theta_i(s).$$

On multiplying (37) by $\lambda G^{0k}(xs)$ and integrating with respect to s from a to b, we get

$$\begin{aligned} \lambda \int_a^b G^{0k}(xs)\theta_0(s)\,ds &= \lambda \int_a^b G^{0k}(xs)\theta(s)\,ds - \lambda \sum_{i=1}^{k} c_i \int_a^b G^{0k}(xs)\theta_i(s) \\ &= \theta(x) - \sum_{i=1}^{k} c_i\theta_i(x) \\ &= \theta_0(x). \end{aligned}$$

This shows that $\theta_0(x)$ is a solution of the equation

$$(38) \qquad \theta_0(x) = \lambda \int_a^b G^{0k}(xs)\theta_0(s)\,ds.$$

Then a complete set of solutions of (35) may be obtained by adding solutions of equation (38) to those of system (36).

It is obvious that solutions of (36) and (38) satisfy (35) for some set of c_i's.

COROLLARY I. *The equation*

$$\theta(x) = \lambda \int_a^b \left[G^{0k}(xs) + (1/\lambda) \sum_{i=1}^{k} u_1^0(x)\phi_i^0(s)\right] \theta(s)\,ds,$$

always has a solution not identically zero.

For at values of λ which are roots of the determinant function* $D(\lambda)$

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* T. Lalesco, *Équations Intégrales*, p. 21.

formed for $G^{0k}(xs)$, (38) has solutions not identically zero, and at all other values of λ, equations (36) have solutions.

THEOREM IX. *Solutions of the system*

$$(39) \qquad \begin{aligned} &L(u) = \lambda u - \lambda \sum_{i=1}^{k} \psi_i^0(x) \int_a^b v_i^0(s) u(s)\, ds, \\ &U_p(u) = 0, \qquad\qquad (p = 1, \cdots, n), \end{aligned}$$

are expressible linearly in terms of those of the systems

$$(40) \qquad \begin{aligned} &L(u) = \lambda u(x) - \lambda \psi_i^0(x), \qquad && (i = 1, \cdots, k), \\ &U_p(u) = 0, && (p = 1, \cdots, n), \end{aligned}$$

and

$$(41) \qquad L(u_0) = \lambda u_0, \qquad U_p(u_0) = 0, \qquad (p = 1, \cdots, n).$$

The first equation of system (39) may be written

$$(42) \qquad L(u) = \lambda u - \lambda \sum_{i=1}^{k} c_i \psi_i^0(x),$$

where $c_i = \int_a^b v_i^0(s) u(s)\, ds$. Here again the c_i's are given in terms of the function u, but if equation (42) is written for arbitrary c_i's it follows by an application of Green's theorem that

$$(43) \qquad c_i = \int_a^b v_i^0(s) u(s)\, ds, \qquad (i = 1, \cdots, k),$$

so that equation (42) is satisfied. The remainder of the proof is similar to that of theorem VIII.

From theorems VII, VIII, and IX follows a rather symmetrical result, which may be stated as a theorem.

THEOREM X. *A complete set of solutions of the integral system*

$$\begin{aligned} &\theta_i(x) = u_i^0(x) + \lambda \int_a^b G^{0k}(xs) \theta_i(s)\, ds, \qquad (i = 1, \cdots, k), \\ &\theta_0(x) = \lambda \int_a^b G^{0k}(xs) \theta_0(s)\, ds, \end{aligned}$$

taken as a whole, is linearly equivalent to a complete set of solutions of the differential systems

$$L(u) = \lambda u - \lambda \psi_i^0, \qquad U_p(u) = 0, \quad (p = 1, \cdots, n; \quad i = 1, \cdots, k),$$

and

$$L(u_0) = \lambda u_0, \qquad U_p(u_0) = 0, \qquad (p = 1, \cdots, n),$$

taken as a whole.

THEOREM XI. *Solutions of the equation*

$$(44)\qquad \theta(x) = \lambda \int_a^b G^{0k}(xs)\theta(s)\,ds,$$

are linearly equivalent to those of the system

$$(45)\qquad \begin{aligned} &L(u) = \lambda u(x) - \lambda \sum_{i=1}^{k} \psi_i^0(x) \int_a^b v_i^0(s)u(s)\,ds, \\ &U_p(u) = 0, \qquad\qquad (p = 1, \cdots, n), \\ &\int_a^b \phi_i^0(x)u(x)\,dx = 0, \qquad (i = 1, \cdots, k). \end{aligned}$$

That any solution of (44) satisfies system (45) is seen at once by performing the indicated operations on the function defined by (44). To show that any solution of system (45) satisfies (44) apply Green's theorem, letting $u =$ any solution of (45) and $v = H^{0k}(xy)$.

THEOREM XII. *Solutions of the integral system*

$$(46)\qquad \begin{aligned} &\theta(x) = \lambda \int_a^b G^{0k}(xs)\theta(s)\,ds + \sum_{i=1}^{k} u_i^0(x) \int_a^b \phi_i^0(s)\theta(s)\,ds, \\ &\int_a^b v_i^0(s)\theta(s)\,ds = 0, \qquad (i = 1, \cdots, k), \end{aligned}$$

are linearly equivalent to those of the differential system

$$(47)\qquad L(u) = \lambda u, \quad U_p(u) = 0, \quad (p = 1, \cdots, n).$$

That every solution of (46) satisfies (47) is seen by performing the indicated operations on the function $\theta(x)$ and taking into account the condition $\int_a^b v_i^0(s)\theta(s)\,ds = 0, \quad (i = 1, \cdots, k)$.

To show that every solution of (47) satisfies system (46) apply Green's theorem, letting $u = u(x)$, any solution of (47), and $v = H^{0k}(xy)$. In the simplification use is made of the fact that every solution of (47) satisfies the condition

$$\int_a^b v_i^0(s)u(s)\,ds = 0 \qquad (i = 1, \cdots, k).$$

If the Green's function $G_{01}^{0k}(xy)$ exists for the given system, we have the simple equivalence theorem:

THEOREM XIII. *If the function* $G_{01}^{0k}(xy)$ *exists*[*] *for the given differential system, then solutions of the integral equation*

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[*] This hypothesis is equivalent to assuming that G_{01}^{0k} exists for some value of λ.

$$\theta(x) = \lambda \int_a^b G_{01}^{0k}(xs)\theta(s)\,ds, \qquad \lambda \neq 0,$$

are equivalent to those of the differential system

$$L(u) = \lambda u, \quad \lambda \neq 0, \quad U_p(u) = 0, \quad (p = 1, \cdots, n).$$

Since for a Hermitian system G_{01}^{0k} always exists and equals G_1^{0k}, we may state:

THEOREM XIV. *For a given Hermitian differential system the solutions of the integral equation*

$$\theta(x) = \lambda \int_a^b G_1^{0k}(xs)\theta(s)\,ds, \qquad \lambda \neq 0,$$

are equivalent to those of the differential system

$$L(u) = \lambda u, \quad U_p(u) = 0, \quad (p = 1, \cdots, n), \qquad \lambda \neq 0.$$

Now suppose the function $D(\lambda)$ is formed for the kernel $G_1^{0k}(xs)$ of a Hermitian system and the roots of $D(\lambda)$ are found to be $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \cdots$. If we choose a real number λ' not equal to $0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \cdots$, and define

$$\Gamma_1^{0k}(xs) = G_1^{0k}(xs) + (1/\lambda') \sum_{i=1}^{k} u_i^0(x)\bar{u}_i^0(s),$$

we may state

THEOREM XV. *Solutions of the equation*

$$\theta(x) = \lambda \int_a^b \Gamma_1^{0k}(xs)\theta(s)\,ds$$

are equivalent to those of the Hermitian system

$$L(u) = \lambda u, \quad U_p(u) = 0, \quad (p = 1, \cdots, n),$$

the value $\lambda = 0$ not being excluded.

Since

$$\begin{aligned}\theta(x) &= \lambda \int_a^b \Gamma_1^{0k}(xs)\theta(s)\,ds \\ &= \lambda \int_a^b \Big[G_1^{0k}(xs) + (1/\lambda') \sum_{i=1}^{k} u_i^0(x)\,\bar{u}_i^0(s)\Big]\,\theta(s)\,ds,\end{aligned}$$

and

$$\int_a^b \bar{u}_p^0(x)\theta(x)\,dx = (\lambda/\lambda') \int_a^b \bar{u}_p^0(s)\theta(s)\,ds, \qquad (p = 1, \cdots, k),$$

Hence $\lambda = \lambda'$, or $\displaystyle\int_a^b \bar{u}_p^0(s)\theta(s)\,ds = 0, \qquad (p = 1, \cdots, k).$

If $\int_a^b \bar{u}_p^0(s)\theta(s)\,ds = 0$, then $\theta(x) = \lambda \int_a^b G_1^{0k}(xs)\theta(s)\,ds$, and λ must be one of the values $\lambda_1, \lambda_2, \lambda_3, \cdots$; and solutions of equation (50) must be linearly equivalent to those of equation (48).

If $\lambda = \lambda'$

$$\theta(x) = \lambda' \int_a^b G_1^{0k}(xs)\theta(s)\,ds + \sum_{i=1}^{k} u_i^0(x) \int_a^b \bar{u}_i^0(s)\theta(s)\,ds.$$

Let

$$\alpha(s) = \theta(s) - \sum_{i=1}^{k} u_i^0(s) \int_a^b \bar{u}_i^0(t)\theta(t)\,dt.$$

Multiplying by $\lambda' G_1^{0k}(xs)$ and integrating from a to b, we get

$$\begin{aligned}\lambda' \int_a^b G_1^{0k}(xs)\alpha(s)\,ds = \lambda' \int_a^b G_1^{0k}(xs)\theta(s)\,ds \\ - \lambda' \sum_{i=1}^{k} \int_a^b \int_a^b G_1^{0k}(xs)u_i^0(s)\bar{u}_i^0(t)\,\theta(t)\,dt\,ds \\ = \lambda' \int_a^b G_1^{0k}(xs)\theta(s)\,ds \\ = \theta(x) - \sum_{i=1}^{k} u_i^0(x) \int_a^b \bar{u}_i^0(t)\theta(t)\,dt \\ = \alpha(x).\end{aligned}$$

Since $\lambda' \neq \lambda_1, \lambda_2, \lambda_3, \cdots$, $\alpha(x) = 0$, and $\theta(x) = \sum_{i=1}^{k} c_i u_i^0(x)$. Hence solutions of (50) are linearly equivalent to those of (48) plus linear combinations of $u_i^0(x)$, $(i = 1, \cdots, k)$, and theorem (XV) follows from theorem (XIV).

THEOREM XVI. *The kernel* $\Gamma_1^{0k}(xs)$ *is a closed** *kernel.*

Suppose $\int_a^b \Gamma_1^{0k}(xs)\beta(s)\,ds = 0$. This becomes, on substituting for $\Gamma_1^{0k}(xs)$, $\int_a^b G_1^{0k}(xs)\beta(s)\,ds + (1/\lambda') \sum_{j=1}^{k} u_j^0(x) \int_a^b \bar{u}_j^0(s)\beta(s)\,ds = 0$. Multiplying by $\bar{u}_i^0(x)$, $(i = 1, \cdots, k)$, and integrating, we get

$$\int_a^b \bar{u}_i^0(s)\beta(s)\,ds = 0, \qquad (i = 1, \cdots, k),$$

and then

$$\int_a^b G_1^{0k}(xs)\beta(s)\,ds = 0.$$

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* D. Hilbert, *Göttinger Nachrichten* (1904), p. 49.

Applying the differential operator L to this equation gives

$$\beta(x) \equiv \sum_{i=1}^{k} u_i^0(x) \int_a^b \bar{u}_i^0(s)\beta(s)\,ds \equiv 0, \tag{52}$$

which shows the kernel is closed.

From theorems XIV, XV, XVI, together with the theory of integral equations with Hermitian kernels, follows[*]

THEOREM XVII. *A Hermitian differential system cannot be compatible for all values of the parameter* λ. *The values of* λ *for which the system is compatible are infinite in number, are all real, and have no finite limit point.*

3. *Relations between Green's functions for different values of the parameter.*

THEOREM XVIII. *In systems for which* $G^{\lambda h}(xy)$ *and* $G^{00}(xy)$ *both exist,* $G^{\lambda h}(xy)$ *is a pseudoresolvent*[†] *function to the function* $G^{00}(xy)$.

Apply Green's theorem,[‡] letting

$$U = G^{\lambda h}(x\xi), \text{ and } v = H^{00}(x\eta).$$

This gives after a simplification and a change of variables

$$G^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G^{00}(xs)\,G^{\lambda h}(sy)\,ds - \int_a^b G^{00}(xs) \sum_{i=1}^{h} \psi_i^\lambda(s)\,v_i^\lambda(y)\,ds, \tag{53}$$

or

$$G^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G^{00}(xs)\,G^{\lambda h}(sy)\,ds - (1/\lambda) \sum_{i=1}^{h} \Psi_i^\lambda(x)\,v_i^\lambda(y), \tag{54}$$

where

$$\Psi_i^\lambda(x) = \lambda \int_a^b G^{00}(xs)\,\psi_i^\lambda(s)\,ds, \qquad (i = 1, \cdots, k). \tag{55}$$

Similarly, by letting $u = G^{00}(x\xi)$, $v = H^{\lambda h}(x\eta)$, applying Green's theorem, and simplifying, we get

$$G^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G^{\lambda h}(xs)\,G^{00}(sy)\,ds - (1/\lambda) \sum_{i=1}^{h} \Phi_i^\lambda(y)\,u_i^\lambda(x), \tag{56}$$

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[*] This result has been previously shown; see Bôcher, *Proceedings of the Fifth International Congress of Mathematicians*, Vol. 1, p. 185.

[†] W. A. Hurwitz, *Transactions of the American Mathematical Society*, Vol. 13 (1912), p. 405.

[‡] See theorem V.

where

$$(57) \qquad \Phi_i{}^\lambda(y) = \lambda \int_a^b \phi_i{}^\lambda(s)\,G^{00}(sy)\,ds, \qquad (i = 1, \cdots, k).$$

From the theory of the ordinary[*] Green's function it is known that the $u_i{}^\lambda$'s and $v_i{}^\lambda$'s form complete sets of linearly independent solutions of the equations

$$u^\lambda(x) = \lambda \int_a^b G^{00}(xs)\,u^\lambda(s)\,ds,$$

$$v^\lambda(x) = \lambda \int_a^b H^{00}(xs)\,v^\lambda(s)\,ds.$$

Then

$$\int_a^b u_i{}^\lambda(x)\,\Phi_j{}^\lambda(x)\,dx = \lambda \int_a^b\int_a^b u_i{}^\lambda(x)\,G^{00}(sx)\,\phi_i{}^\lambda(s)\,ds dx$$

$$= \int_a^b \phi_j{}^\lambda(s)\,u_i{}^\lambda(s)\,ds = \left\{ \begin{array}{l} 0,\ i \neq j, \\ 1,\ i = j, \end{array} \right\} (i,\ j = 1, \cdots, h).$$

Similarly

$$\int_a^b v_i{}^\lambda(x)\,\Psi_j{}^\lambda(x)\,dx = \left\{ \begin{array}{l} 0,\ i \neq j, \\ 1,\ i = j, \end{array} \right\} (i,\ j = 1, \cdots, h).$$

Thus the necessary and sufficient conditions[†] that $G^{\lambda h}(xy)$ be a pseudo resolvent to $G^{00}(xy)$ are satisfied.

The function $G^{\lambda h}(xy)$ is not uniquely determined by equations (54) and (56). However, it may be completely determined by demanding in addition one of the relations,

$$\int_a^b \Phi_j{}^\lambda(x)\,G^{\lambda h}(xy)\,dx = (1/\lambda)\ [\sum_{i=1}^{h} v_i{}^\lambda(y) \int_a^b \Phi_j{}^\lambda(s)\,\psi_i{}^\lambda(s)\,ds - \Phi_j{}^\lambda(y)],$$

$$\int_a^b \Psi_j{}^\lambda(x)\,G^{\lambda h}(xy)\,dx = (1/\lambda)\ [\sum_{i=1}^{h} u_i{}^\lambda(y) \int_a^b \Psi_j{}^\lambda(s)\,\phi_i{}^\lambda(s)\,ds - \Psi_j{}^\lambda(y)].$$

That these relations are satisfied by the function $G^{\lambda h}(xy)$ is obtained by applications of Green's theorem.

The last theorem simplifies somewhat if the Green's function $G_{01}^{\lambda h}\ (xy)$ is used. In this case the functions $\Psi_j{}^\lambda$ and $\Phi_j{}^\lambda$ are respectively equivalent to the functions $u_j{}^\lambda$ and $v_j{}^\lambda$, and we may state

THEOREM XIX. *In systems for which* $G_{01}^{\lambda h}\ (xy)$ *and* $G^{00}(xy)$ *both exist,* $G_{01}^{\lambda h}\ (xy)$ *is that pseudoresolvent function to* $G^{00}(xy)$ *which satisfies the conditions,*

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[*] M. Bôcher, *Les Méthodes de Sturm*, p. 109.

[†] W. A. Hurwitz, *loc. cit.*

$$G_{01}^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G^{00}(xs)\, G_{01}^{\lambda h}(sy)\, ds$$
$$- (1/\lambda) \sum_{i=1}^{h} u_i^{\lambda}(x)\, v_i^{\lambda}(y),$$

$$G_{01}^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G_{01}^{\lambda h}(xs)\, G^{00}(sy)\, ds$$
$$- (1/\lambda) \sum_{i=1}^{h} v_i^{\lambda}(y)\, u_i^{\lambda}(x),$$

$$\int_a^b v_j^{\lambda}(x)\, G_{01}^{\lambda h}(xy)\, dx = 0, \qquad (j = 1, \cdots, k),$$

$$\int_a^b u_j^{\lambda}(x)\, G_{01}^{\lambda h}(yx)\, dx = 0, \qquad (j = 1, \cdots, k).$$

COROLLARY I. *In a Hermitian differential system, $G_1^{\lambda h}(xy)$ is that pseudoresolvent function to the kernel $G^{00}(xy)$ which satisfies the conditions*

$$G_1^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G^{00}(xs)\, G_1^{\lambda h}(sy)\, ds$$
$$- (1/\lambda) \sum_{i=1}^{h} \bar{v}_i^{\lambda}(x)\, v_i^{\lambda}(y),$$

$$G_1^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G_1^{\lambda h}(xs)\, G^{00}(sy)\, ds$$
$$- (1/\lambda) \sum_{i=1}^{h} \bar{u}_i^{\lambda}(y)\, u_i^{\lambda}(x),$$

$$\int_a^b \bar{u}_i^{\lambda}(x)\, G_1^{\lambda h}(xy)\, dx = 0,$$

$$\int_a^b \bar{v}_i^{\lambda}(x)\, G_1^{\lambda h}(yx)\, dx = 0.$$

The functions $\Psi_i^{\lambda}(x)$ and $\Phi_i^{\lambda}(x)$ defined by equations (55) and (57) are respectively identical with the solutions of the differential systems,

$$L[\Psi_i^{\lambda}(x)] = \lambda \psi_i^{\lambda}(x), \quad U_p[\Psi_i^{\lambda}(x)] = 0, \qquad (p = 1, \cdots, n;\ i = 1, \cdots, k). \tag{58}$$

and

$$M[\Phi_i^{\lambda}(x)] = \lambda \phi_i^{\lambda}(x), \quad V_p[\Phi_i^{\lambda}(x)] = 0, \qquad (p = 1, \cdots, n;\ i = 1, \cdots, k). \tag{59}$$

For each i the solution of each of these systems is unique. The Φ_i^{λ}'s and the Ψ_i^{λ}'s are then completely determined by the differential systems (58) and (59).

Define the functions

$$Q^{\lambda h}(x\xi) = G^{\lambda h}(x\xi) - (1/\lambda) \sum_{i=1}^{h} \sum_{j=1}^{h} c^{\lambda}_{ij}\, u_i^{\lambda}(x)\, v_j^{\lambda}(\xi)$$

$$+ (1/\lambda) \sum_{i=1}^{h} u_i^{\lambda}(x)\, v_i^{\lambda}(\xi),$$

$$Q^{00}(x\eta) = G^{00}(x\eta) - (1/\lambda) \sum_{i=1}^{h} \Psi_i^{\lambda}(x)\, \Phi_i^{\lambda}(\eta),$$

$$R^{\lambda h}(x\xi) = H^{\lambda h}(x\xi) - (1/\lambda) \sum_{i=1}^{h} \sum_{j=1}^{h} b^{\lambda}_{ij}\, v_i^{\lambda}(x) u_j^{\lambda}(\xi)$$

$$+ (1/\lambda) \sum_{i=1}^{h} v_i^{\lambda}(x)\, u_i^{\lambda}(\xi),$$

$$R^{00}(x\eta) = H^{00}(x\eta) - (1/\lambda) \sum_{i=1}^{h} \Phi_i^{\lambda}(x)\, \Psi_i^{\lambda}(\eta),$$

where

$$c^{\lambda}_{ij} = \int_a^b \Phi_i^{\lambda}(v)\psi_j^{\lambda}(x)\,dx, \qquad b^{\lambda}_{ij} = \int_a^b \Psi_i^{\lambda}(x)\phi_j^{\lambda}(x)\,dx.$$

Then $Q^{00}(x\eta) = R^{00}(\eta x)$, and also, since $c^{\lambda}_{ij} = b^{\lambda}_{ji}$, $Q^{\lambda h}(x\xi) = R^{\lambda h}(\xi x)$.

THEOREM XX. *The function* $Q^{\lambda h}(xy)$ *is the resolvent function to* $Q^{00}(xy)$ *for the parameter value* λ.

Apply Green's Theorem, letting

$$u = Q^{\lambda h}(x\xi), \text{ and } v = R^{00}(x\eta).$$

This gives, after simplification involving somewhat lengthy calculations and a change of variables,

$$Q^{\lambda h}(xy) = Q^{00}(xy) + \lambda \int_a^b Q^{00}(xs)Q^{\lambda h}(sy)\,ds. \tag{60}$$

Similarly, by letting $u = Q^{00}(x\xi)$, $v = R^{\lambda h}(x\eta)$, and applying Green's theorem, we have

$$Q^{\lambda h}(xy) = Q^{00}(xy) + \lambda \int_a^b Q^{\lambda h}(xs)Q^{00}(sy)\,ds. \tag{61}$$

Equations (60) and (61) determine the resolvent relationship between the functions $Q^{\lambda h}(xy)$ and $Q^{00}(xy)$.

In systems for which the more specialized function $G_{01}^{\lambda h}$ exists, we may state

THEOREM XXI. *The function* $G_{01}^{\lambda h}(xy)$, *if it exists, is the resolvent function to* $Q^{00}(xy)$ *where*

$$Q^{00}(xy) = G^{00}(xy) - (1/\lambda) \sum_{i=1}^{h} u_i^{\lambda}(x)v_i^{\lambda}(y).$$

COROLLARY. *For a Hermitian differential system* $G_1^{\lambda h}(xy)$ *is the resolvent function to* $G^{00}(xy) - (1/\lambda)\sum_{i=1}^{h} u_i^{\lambda}(x)v_i^{\lambda}(y)$.

It remains to find the relations between the generalized Green's functions for two different values of the parameter λ, each of which renders the system compatible. Here again it would be desirable to find a resolvent relationship between the Green's functions and combinations of other functions, at the two parameter values, but this seems rather difficult in the general case and a relationship of a different kind has been found. The resolvent relationship has been found for the more special function $G_0(xy)$.

Define the functions,

$$\Psi_j^{\lambda}(x) = \lambda\int_a^b G^{0k}(xs)\psi_j^{\lambda}(s)\,ds, \quad (j=1,\cdots,h), \tag{62}$$

$$\Psi_j^{0}(x) = \lambda\int_a^b G^{\lambda h}(xs)\psi_j^{0}(s)\,ds, \quad (j=1,\cdots,k), \tag{63}$$

$$\Phi_j^{\lambda}(x) = \lambda\int_a^b H^{0k}(xs)\phi_j^{\lambda}(s)\,ds, \quad (j=1,\cdots,h), \tag{64}$$

$$\Phi_j^{0}(x) = \lambda\int_a^b H^{\lambda h}(xs)\phi_j^{0}(s)\,ds, \quad (j=1,\cdots,k). \tag{65}$$

For each j there is a unique function defined by each of these equations, and this function is identical with the unique solution given, for the same j, in the corresponding systems,

$$\begin{aligned} &L[\Psi_j^{\lambda}(x)] = \lambda\psi_j^{\lambda}(x) - \lambda\sum_{i=1}^{k}\psi_i^{0}(x)\int_a^b v_i^{0}(s)\psi_j^{\lambda}(s)\,ds,\\ &\qquad U_p[\Psi_j^{\lambda}(x)] = 0, \quad (p=1,\cdots,n),\\ &\int_a^b \phi_i^{0}(x)\Psi_j^{\lambda}(x)\,dx = 0, \quad (i=1,\cdots,k;\ j=1,\cdots,h); \end{aligned} \tag{62a}$$

$$\begin{aligned} &L[\Psi_j^{0}(x)] = \lambda\Psi_j^{0}(x) + \lambda\psi_j^{0}(x) - \lambda\sum_{i=1}^{h}\psi_i^{\lambda}(x)\int_a^b v_i^{\lambda}(s)\psi_j^{0}(s)\,ds,\\ &\qquad U_p[\Psi_j^{0}(x)] = 0, \quad (p=1,\cdots,n),\\ &\int_a^b \phi_i^{\lambda}(x)\Psi_j^{0}(x)\,dx = 0, \quad (i=1,\cdots,h;\ j=1,\cdots,k); \end{aligned} \tag{63a}$$

$$\begin{aligned} &M[\Phi_j^{\lambda}(x)] = \lambda\phi_j^{\lambda}(x) - \lambda\sum_{i=1}^{k}\phi_i^{0}(x)\int_a^b u_i^{0}(s)\phi_j^{\lambda}(s)\,ds,\\ &\qquad V_p[\Phi_j^{\lambda}(x)] = 0, \quad (p=1,\cdots,n),\\ &\int_a^b \psi_i^{0}(x)\Phi_j^{\lambda}(x)\,dx = 0, \quad (i=1,\cdots,k;\ j=1,\cdots,h); \end{aligned} \tag{64a}$$

$$(65\text{a})\quad \begin{aligned} &M[\Phi_j{}^0(x)] = \lambda\Phi_j{}^0(x) + \lambda\phi_j{}^0(x) - \lambda \sum_{i=1}^{h} \phi_i{}^\lambda(x) \int_a^b u_i{}^\lambda(s)\phi_i{}^0(s)\,ds,\\ &\qquad\qquad V_p[\Phi_j{}^0(x)] = 0, \quad (p = 1,\cdots,n),\\ &\int_a^b \psi_i{}^\lambda(x)\Phi_j{}^0(x)\,dx = 0, \quad (i = 1,\cdots,h;\ j = 1,\cdots,k). \end{aligned}$$

THEOREM XXII. *In systems for which $G^{0k}(xy)$ and $G^{\lambda h}(xy)$ both exist, and $G^{0k}(xy)$ is known, $G^{\lambda}(xy)$ may be determined either by a solution of the equation*

$$(66)\quad \begin{aligned} G^{\lambda h}(xy) = G^{0k}(xy) &- (1/\lambda) \sum_{i=1}^{h} v_i{}^\lambda(y)\Psi_i{}^\lambda(x)\\ &+ (1/\lambda) \sum_{i=1}^{k} u_i{}^0(x)\Phi_i{}^0(y) + \lambda \int_a^b G^{0k}(xs)G^{\lambda h}(sy)\,ds,^* \end{aligned}$$

which satisfies the conditions

$$(67)\quad \int_a^b v_j{}^0(x)G^{\lambda h}(xy)\,dx = (1/\lambda)\ [\sum_{i=1}^{h} v_i{}^\lambda(y)\int_a^b v^0{}_j(s)\psi_i{}^\lambda(s)\,ds - v_j{}^0(y)], \quad (j = 1,\cdots, k),$$

$$(68)\quad \int_a^b \phi_j{}^\lambda(x)\ G^{\lambda h}(xy)\,dx = 0, \qquad (j = 1,\cdots, h),$$

or by a solution of the equation

$$(66\text{a})\quad \begin{aligned} G^{\lambda h}(xy) = G^{0k}(xy) &- (1/\lambda) \sum_{i=1}^{h} u_i{}^\lambda(y)\Phi_i{}^\lambda(x)\\ &+ (1/\lambda) \sum_{i=1}^{k} v_i{}^0(x)\Psi_i{}^0(y) + \lambda \int_a^b G^{\lambda h}(xs)G^{0k}(sy)\,ds, \end{aligned}$$

which satisfies the conditions

$$(67\text{a})\quad \int_a^b u_j{}^0(x)G^{\lambda h}(yx)\,dx = (1/\lambda)\ [\sum_{i=1}^{h} u_i{}^\lambda(y) \int_a^b u_j{}^0(s)\phi_i{}^\lambda(s)\,ds - u_j{}^0(y)] \quad (j = 1,\cdots,k),$$

$$(68\text{a})\quad \int_a^b \psi_j{}^\lambda(x)G^{\lambda h}(yx)\,dx = 0.$$

That a solution of equations (66), (67) and (68) exists may be shown indirectly by showing they are satisfied by the Green's function $G^{\lambda h}(xy)$. It may be seen by application of Green's theorem that (66) and (67) are satisfied by $G^{\lambda h}(xy)$. Condition (68) follows from the definition of $G^{\lambda h}(xy)$.

But a solution of (66), (67) and (68) is unique; for suppose there are two solutions $G^{\lambda h}(xy)$ and $G_2{}^{\lambda h}(xy)$, and form the difference

$$R(xy) = G^{\lambda h}(xy) - G_2{}^{\lambda h}(xy).$$

The function $R(xy)$ will satisfy the system

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[*] The Φ_i's and Ψ_i's are defined by the systems (62a), (63a), (64a) and (65a).

$$R(xy) = \lambda \int_a^b G^{0k}(xs)R(sy)\,ds,$$

$$(69)\qquad \int_a^b v_j^0(x)R(xy)\,dx = 0, \qquad (j = 1, \cdots, k).$$

$$\int_a^b \phi_j^\lambda(x)R(xy)\,dx = 0, \qquad (j = 1, \cdots, h).$$

Every solution of system (69) satisfies the system

$$(70)\qquad \begin{aligned} &(a)\quad L_x[R(xy)] = \lambda R(xy), \\ &(b)\quad U_p[R(xy)] = 0, && (p = 1, \cdots, n) \\ &(c)\quad \int_a^b \phi_j^\lambda(x)R(xy)\,dx = 0, && (j = 1, \cdots, h). \end{aligned}$$

But there are no solutions of (70) which do not vanish identically, for every solution of (70a, b) is given by

$$R(xy) = \sum_{i=1}^{h} c_i(y)u_i^\lambda(x),$$

and from (70c), $c_i(y) \equiv 0$, $(i = 1, \cdots, h)$. Hence

$$R(xy) \equiv 0, \text{ and } G^{\lambda h}(xy) \equiv G_2^{\lambda h}(xy).$$

It may be shown similarly that the Green's function $G^{\lambda h}(xy)$ may be obtained from the alternative equations (66a), (67a) and (68a).

The statement of theorem XXII does not simplify to a great extent for the functions $G_0^{0k}(xy)$ and $G_0^{\lambda h}(xy)$, but for the functions $G_{01}^{0k}(xy)$ and $G_{01}^{\lambda h}(xy)$ it is more symmetric, and it will be restated.

THEOREM XXIII. *In systems for which $G_{01}^{0k}(xy)$ and $G_{01}^{\lambda h}(xy)$ both exist, and G_{01}^{0k} is known, $G_{01}^{\lambda h}(xy)$ may be obtained, either as the solution of the equations*

$$G_{01}^{\lambda h}(xy) = G_{01}^{0k}(xy) - (1/\lambda)\sum_{i=1}^{h} v_i^\lambda(y)u_i^\lambda(x)$$
$$- (1/\lambda)\sum_{i=1}^{k} u_i^0(x)v_i^0(y) + \lambda \int_a^b G_{01}^{0k}(xs)G_{01}^{\lambda h}(sy)\,ds,$$

$$\int_a^b v_j^0(x)G_{01}^{\lambda h}(xy)\,dx = -(1/\lambda)v_j^0(y), \qquad (j = 1, \cdots, k),$$

$$\int_a^b v_j^\lambda(x)G_{01}^{\lambda h}(xy)\,dx = 0, \qquad (j = 1, \cdots, h),$$

or the solution of the equations

$$G_{01}^{\lambda h}(xy) = G_{01}^{0k}(xy) - (1/\lambda)\sum_{i=1}^{h} u_i^\lambda(y)v_i^\lambda(x)$$
$$-(1/\lambda)\sum_{i=1}^{k} v_i^0(x)u_i^0(y) + \lambda \int_a^b G_{01}^{0k}(xs)G_{01}^{\lambda h}(sy)\,ds,$$

$$\int_a^b u_j^0(x)G_{01}^{\lambda h}(yx)\,dx = -(1/\lambda)u_j^0(y), \qquad (j = 1, \cdots, k$$

$$\int_a^b u_j^\lambda(x)G_{01}^{\lambda h}(yx)\,dx = 0, \qquad (j = 1, \cdots, h).$$

If the function $G_{01}(xy)$ is used, we should have, instead of the functions defined by (62a), (63a), (64a), and (65a), those defined by the following systems

$$\begin{aligned} &L[\Psi^{\lambda}{}_{0j}(x)] = \lambda\psi_j{}^{\lambda}(x), \qquad U_p[\Psi^{\lambda}{}_{0j}(x)] = 0, \\ &\int_a^b \phi_i{}^0(x)\Psi^{\lambda}{}_{0j}(x)\,dx = 0, \qquad (i = 1,\cdot\cdot\cdot,k;\ j = 1,\cdot\cdot\cdot,h; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p = 1,\cdot\cdot\cdot,n); \end{aligned} \tag{71}$$

$$\begin{aligned} &L[\Psi^{0}{}_{0j}(x)] = \lambda\Psi^{0}{}_{0j}(x) + \lambda\psi_j{}^{0}(x), \qquad U_p[\Psi^{0}{}_{0j}(x)] = 0, \\ &\int_a^b \phi_i{}^{\lambda}(x)\Psi^{0}{}_{0j}(x)\,dx = 0, \qquad (i = 1,\cdot\cdot\cdot,h;\ j = 1,\cdot\cdot\cdot,k; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p = 1,\cdot\cdot\cdot,n); \end{aligned} \tag{72}$$

$$\begin{aligned} &M[\Phi^{\lambda}{}_{0j}(x)] = \lambda\phi_j{}^{\lambda}(x), \qquad V_p[\Phi^{\lambda}{}_{0j}(x)] = 0, \\ &\int_a^b \psi_i{}^0(x)\Phi^{\lambda}{}_{0j}(x)\,dx = 0, \qquad (i = 1,\cdot\cdot\cdot,k;\ j = 1,\cdot\cdot\cdot,h; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p = 1,\cdot\cdot\cdot,n); \end{aligned} \tag{73}$$

$$\begin{aligned} &M[\Phi^{0}{}_{0j}(x)] = \lambda\Phi^{0}{}_{0j}(x) + \lambda\phi_j{}^{0}(x), \qquad V_p[\Phi^{0}{}_{0j}(x)] = 0, \\ &\int_a^b \psi_i{}^{\lambda}(x)\Phi^{0}{}_{0j}(x)\,dx = 0, \qquad (i = 1,\cdot\cdot\cdot,h;\ j = 1,\cdot\cdot\cdot,k; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p = 1,\cdot\cdot\cdot,n). \end{aligned} \tag{74}$$

Define the functions

$$\begin{aligned} Q_0{}^{\lambda h}(xy) &= G_0{}^{\lambda h}(xy) + (1/\lambda)\sum_{i=1}^{k} \Psi^{0}{}_{0i}(x)\ \Phi^{0}{}_{0i}(y) \\ &\quad + (1/\lambda)\sum_{i=1}^{h}\sum_{j=1}^{h} c_{ij}{}^{\lambda}u_i{}^{\lambda}(x)v_j{}^{\lambda}(y) - (1/\lambda)\sum_{i=1}^{h} u_i{}^{\lambda}(x)v_i{}^{\lambda}(y), \\ Q_0{}^{0k}(xy) &= G_0{}^{0k}(xy) - (1/\lambda)\sum_{i=1}^{h} \Psi^{0}{}_{0i}(x)\ \Phi^{0}{}_{0i}(y) \\ &\quad + (1/\lambda)\sum_{i=1}^{k}\sum_{j=1}^{k} c^{\lambda}{}_{ij}u_i{}^{\lambda}(x)\ v_j{}^{\lambda}(y) - (1/\lambda)\sum_{i=1}^{k} u_i{}^{0}(x)v_i{}^{0}(y), \\ R_0{}^{\lambda h}(xy) &= H_0{}^{\lambda h}(xy) + (1/\lambda)\sum_{i=1}^{k} \Phi^{0}{}_{0i}(x)\ \Psi^{0}{}_{0i}(y) \\ &\quad + (1/\lambda)\sum_{i=1}^{h}\sum_{j=1}^{h} b_{ij}{}^{\lambda}v_i{}^{\lambda}(x)u_j{}^{\lambda}(y) - (1/\lambda)\sum_{i=1}^{h} v_i{}^{\lambda}(x)u_i{}^{\lambda}(y), \\ R_0{}^{0k}(xy) &= H_0{}^{0k}(xy) - (1/\lambda)\sum_{i=1}^{h} \Phi^{\lambda}{}_{0i}(x)\ \Psi^{\lambda}{}_{0i}(y) \\ &\quad + (1/\lambda)\sum_{i=1}^{k}\sum_{j=1}^{k} b^{0}{}_{ij}v_i{}^{0}(x)u_j{}^{0}(y) - (1/\lambda)\sum_{i=1}^{k} v_i{}^{0}(x)u_i{}^{0}(y), \end{aligned}$$

where the Φ_{0j}'s and Ψ_{0j}'s are the functions defined above and where

$$c^{\lambda}{}_{ij} = \int_a^b \psi_j{}^{\lambda}(t)\ \Phi^{\lambda}{}_{0i}(t)\,dt, \qquad c^{0}{}_{ij} = \int_a^b \psi_j{}^{0}(t)\ \Phi^{0}{}_{0i}(t)\,dt,$$

$$b^{\lambda}{}_{ij} = \int_a^b \phi_j{}^{\lambda}(t)\ \Psi^{\lambda}{}_{0i}(t)\,dt, \qquad b^{0}{}_{ij} = \int_a^b \phi_j{}^{0}(t)\ \Psi^{0}{}_{0i}(t)\,dt.$$

It is readily seen that $Q_0^{\lambda k}(xy) = R_0^{\lambda k}(yx)$, and $Q_0^{0k}(xy) = R_0^{0k}(yx)$.

THEOREM XXIV. *In systems for which both functions exist, the function $Q_0^{\lambda l}(xy)$ is the resolvent function to $Q_0^{0k}(xy)$ for the parameter value λ.*

The proof follows from proper application of Green's theorem though the calculations are rather laborious.

For the more specialized Green's functions we may state

THEOREM XXV. *In systems for which both functions exist the function $Q^{\lambda}_{01}(xy)$ is the resolvent function to $Q^{0}_{01}(xy)$ for the parameter value λ where $Q^{\lambda}_{01}(xy)$ and $Q^{0}_{01}(xy)$ are defined by*

$$Q^{\lambda}_{01}(xy) = G_{01}^{\lambda h}(xy) - (1/\lambda)\sum_{p=1}^{h} u_p^0(x)v_p^0(y),$$

$$Q^{0}_{01}(xy) = G_{01}^{0k}(xy) - (1/\lambda)\sum_{p=1}^{h} u_p^{\lambda}(x)v_p^{\lambda}(y).$$

COROLLARY. *For Hermitian differential systems*

$$G_1^{\lambda h}(xy) - (1/\lambda)\sum_{p=1}^{k} \overline{v}_p^0(x)v_p^0(y)$$

is the resolvent function to

$$G_1^{0k}(xy) - (1/\lambda)\sum_{p=1}^{h} \overline{v}_p^{\lambda}(x)v_p^{\lambda}(y)$$

for the parameter value λ.

[illegible]. *Expansion Theorems.* In this section, when solutions for more than one value of the parameter are considered, we shall denote by λ_i ($i = 1, 2, 3, \cdots$), the values of λ which render the given differential system compatible, the subscripts being arranged in the order of magnitude of the absolute value of λ. The order of compatibility for each λ_i will be denoted by k_i, and the normalized solutions of the system for each λ_i will be given a superscript λ_i. In the notation, if the system is compatible for $\lambda = 0$, then $\lambda_0 = 0$.

If complex numbers are used, the Hilbert-Schmidt* expansion theorem for integral equations with Hermitian kernels states that every function of the form

$$g(x) = \int_a^b K(xs)h(s)ds,$$

where $h(s)$ is any continuous function and $K(xs)$ is a continuous Hermitian kernel, is developable in a series uniformly and absolutely convergent of the form

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* E. Schmidt, *Mathematische Annalen*, Vol. 63 (1907), p. 433; E. Goursat, *Cours d'Analyse*, Vol. 3 (1915), p. 446.

$$g(x) = \sum_{\nu} \phi(x) \int_a^b \bar{\phi}_\gamma(s) g(s) ds,$$

the ϕ_γ's being normalized solutions of the equation

$$\phi(x) = \lambda \int_a^b K(xs) \phi(s) ds.$$

From this follows readily,

THEOREM XXVI. *If the differential system* (29) *is Hermitian, any given function* $f(x)$ *having* n *continuous derivatives and satisfying* $U_p[f(x)] = 0$, $(p = 1, \cdots, n)$, *may be expanded in an absolutely and uniformly convergent series of the form*

$$f(x) = \sum_{i=0}^{\infty} \sum_{p=1}^{k_i} u_p^{\lambda_i}(x) \int_a^b \bar{u}_p^{\lambda_i}(s) f(s) ds$$

$$= \sum_{i=1}^{\infty} \sum_{p=1}^{k_i} \bar{v}_p^{\lambda_i}(x) \int_a^b v_p^{\lambda_i}(s) f(s) ds.$$

In order to obtain this from the Hilbert-Schmidt theorem choose[*] $K(xs) = \Gamma_1^{0k_0}(xs)$. The ϕ_γ's are then equivalent to the $u_p^{\lambda_i}$'s.

To show that any $f(x)$ which satisfies the conditions of the theorem may be written

$$f(x) = \int_a^b \Gamma_1^{0k_0}(xs) h(s) ds.$$

where $h(s)$ is a continuous function, apply Green's theorem letting $u = f(x)$ and $v = \Gamma_1^{0k_0}(sx)$. This gives, after simplification

$$f(x) = \int_a^b \Gamma_1^{0k_0}(xs) L[f(s)] ds + \sum_{j=1}^{k_0} u_j^0(x) \int_a^b \bar{u}_j^0(t) f(t) dt.$$

But

$$\lambda' \sum_{i=1}^{k_0} \int_a^b \int_a^b \Gamma_1^{0k_0}(xs) \bar{v}_i^0(s) \bar{u}_i^0(t)\, f(t)\, dt\, ds$$

$$= \sum_{i=1}^{k_0} \sum_{j=1}^{k_0} u_j^0(x) \int_a^b \int_a^b \bar{u}_j^0(s) \bar{v}_i^0(s) \bar{u}_i^0(t)\, f(t)\, dt\, ds$$

$$= \sum_{i=1}^{k_0} u_i^0(x) \int_a^b \bar{u}_i^0(t)\, f(t)\, dt.$$

Then we may write

$$f(x) = \int_a^b \Gamma_1^{0k_0}(xs) \{L[f(s)] + \lambda' \sum_{i=1}^{k_0} u_i^0(s) \int_a^b \bar{u}_i^0(t)\, f(t) dt\}\, ds.$$

Hence, if

$$h(s) = L\,[f(s)] + \lambda' \sum_{i=1}^{k_0} u_i^0(s) \int_a^b \bar{u}_i^0(t)\, f(t) dt,$$

then

$$f(x) = \int_a^b \Gamma_1^{0k_0}(xs) h(s) ds.$$

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[*] The function Γ_1 was defined on page 403.

From Theorem XVII there are an infinite number of λ_i which render the system compatible.

LEMMA. *Solutions of the equations*

$$\phi(x) = \lambda \int_a^b G_1^{0k}(xs)\, \bar{\psi}(s)\, ds, \tag{75}$$

$$\psi(x) = \lambda \int_a^b H_1^{0k}(xs)\, \bar{\phi}(s)\, ds, \tag{76}$$

are equivalent to those of the systems

$$L\,[\phi(x)] = \lambda\bar{\psi}(x), \qquad U_p\,[\phi(x)] = 0, \qquad (p = 1, \cdots, n), \tag{77}$$

and

$$M[\psi(x)] = \lambda\bar{\phi}(x), \quad V_p[\psi(x)] = 0, \qquad (p = 1, \cdots, n), \quad \lambda \neq 0. \tag{78}$$

In order to show that any solution of (75) satisfies (77) apply the indicated operation to (75). This gives

$$L[\phi(x)] = \lambda\bar{\psi}(x) - \sum_{i=1}^{k} \bar{v}_i^0(x) \int_a^b v_i^0(s)\, \bar{\psi}(s)\, ds,$$
$$U_p[\phi(x)] = 0, \qquad (p = 1, \cdots, n).$$

But from (76)

$$\int_a^b \bar{v}_i^0(s)\, \psi(s)\, ds = 0, \qquad (i = 1, \cdots, k).$$

Hence

$$L[\phi(x)] = \lambda\bar{\psi}(x), \qquad U_p[\phi(x)] = 0, \qquad (p = 1, \cdots, n).$$

Similarly it may be shown that $\psi(x)$ of (76) satisfies (78).

The converse is shown by an application of Green's theorem.

E. Schmidt[*] also showed how to expand a function in terms of the fundamental functions, defined by him, for an integral equation with non-symmetric kernel. This result, written out for complex functions, would state that every function $g(x)$, which may be expressed in the form

$$g(x) = \int_a^b K(xs)\, h(s)\, ds,$$

where $K(xs)$ is any continuous function of x and s, and $h(s)$ is a continuous function, is developable in a uniformly and absolutely convergent series of the form

$$g(x) = \sum_i \sum_{p=1}^{i} \phi_p^{\lambda_i}(x) \int_a^b \bar{\phi}_p^{\lambda_i}(s)\, g(s)\, ds,$$

---

[*] E. Schmidt, *loc. cit.*; E. Goursat, *Cours d'Analyse*, Vol. 3 (1915), p. 470.

the $\phi_p^{\lambda_i}$'s being solutions of the system

$$\phi(x)=\lambda\int_a^b K(xs)\,\bar{\psi}(s)\,ds, \tag{79}$$

$$\psi(x)=\lambda\int_a^b K(sx)\,\bar{\phi}(s)\,ds. \tag{80}$$

From this theorem and the preceding lemma follows

THEOREM XXVII. *Any given function $f(x)$ having n continuous derivatives and satisfying $U_p[f(x)]=0$, $(p=1,\cdots,n)$, may be expanded in an absolutely and uniformly convergent series of the form*

$$f(x)=\sum_i\sum_{p=1}^{k_i}\phi_p^{\lambda_i}(x)\int_a^b\bar{\phi}_p^{\lambda_i}(s)\,f(s)\,ds,$$

where the $\phi_p^{\lambda_i}$'s are solutions of system (77).

To obtain this ϕ choose

$$K(xs)=G^{0k_0}(xs).$$

Then solutions of (79) and (80) are, by the lemma above, equivalent to those of (77) and (78) for $\lambda\neq 0$. Suppose we are given any $f(x)$ satisfying the conditions of the theorem. Form from this $f(x)$ the function

$$f_1(x)=f(x)-\sum_{j=1}^{k_0}u_j^0(x)\int_a^b\bar{u}_j^0(s)\,f(s)\,ds.$$

The function $f_1(x)$ satisfies the conditions of the theorem and also the condition

$$\int_a^b\bar{u}_j^0(x)\,f_1(x)\,dx=0, \qquad (j=1,\cdots,k).$$

To show that the function $f_1(x)$ may be expressed in the form

$$f_1(x)=\int_a^b G^{0k_0}(xs)\,h(s)\,ds, \tag{81}$$

where $h(s)$ is a continuous function, apply Green's theorem letting $u=f_1(x)$ and $v=G_1^{0k_0}(sx)$. After simplification, this gives $f_1(x)$ in form (81) where, $h(s)=L[f(s)]$. Then from Schmidt's theorem

$$f_1(x)=\sum_i\sum_{p=1}^{k_i}\phi_p^{\lambda_i}(x)\int_a^b\bar{\phi}_p^{\lambda_i}(s)f_1(s)\,ds, \quad i\neq 0.$$

Whence

$$f(x)=\sum_{j=1}^{k_0}u_j^0(x)\int_a^b\bar{u}_j^0(s)\,f(s)\,ds$$
$$+\sum_i\sum_{p=1}^{k_i}\phi_p^{\lambda_i}(x)\int_a^b\bar{\phi}_p^{\lambda_i}(s)\,f(s)\,ds, \quad i\neq 0,$$

or

$$f(x)=\sum_i\sum_{p=1}^{k_i}\phi_p^{\lambda_i}(x)\int_a^b\bar{\phi}_p^{\lambda_i}(s)\,f(s)\,ds,$$

where $i=0$ is not excluded.

A Class of Invariant Functionals of Quadratic Functional Forms.*

By T. S. Peterson.

---

1. *Introduction.* Michal has shown † that the coefficient $g_{\alpha\beta}$ of the absolute quadratic functional form

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^\alpha)^2 d\alpha \qquad (g_{\alpha\beta} = g_{\beta\alpha}) \tag{1.1}$$

has the law of transformation

$$\begin{aligned}\bar g_{\alpha\beta} = g_{\alpha\beta} + g_{\lambda\beta}K_\alpha{}^\lambda + g_{\alpha\mu}K_\beta{}^\mu + g_{\lambda\mu}K_\alpha{}^\lambda K_\beta{}^\mu \\ + K_\beta{}^\alpha + K_\alpha{}^\beta + \int_a^b K_\alpha{}^\sigma K_\beta{}^\sigma d\sigma\end{aligned} \tag{1.2}$$

under the Fredholm integral transformation

$$y^i = \bar y^i + K_\alpha{}^i \bar y^\alpha. \tag{1.3}$$

In the above formulae we have employed the extension to integration † of the Einstein summation notation. *This convention of letting the repetition of a continuous variable, once as a subscript and once as a superscript, in a term denote a Riemann integration with respect to this subscript over the interval* (a, b) *will be used throughout our paper.*

The study of invariant functionals of functional forms (1.1) was begun by Michal.† It is the purpose of this paper to develop still further the invariantive properties of such functional forms.

2. *Functional transformations.* We shall first demonstrate the following theorem.

Theorem I. *The law of transformation,*

$$\begin{aligned}\bar g_{\alpha\beta} = g_{\alpha\beta} + g_{\lambda\beta}K_\alpha{}^\lambda + g_{\alpha\mu}K_\beta{}^\mu + g_{\lambda\mu}K_\alpha{}^\lambda K_\beta{}^\mu \\ + K_\beta{}^\alpha + K_\alpha{}^\beta + \int_a^b K_\alpha{}^\sigma K_\beta{}^\sigma d\sigma\end{aligned} \tag{2.1}$$

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* Presented to the Society, April 6, 1928.

† Cf. A. D. Michal, "Affinely Connected Function Space Manifolds," *American Journal of Mathematics*, Vol. 50 (1928), pp. 473-517.

of the coefficient of the quadratic functional form

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^\alpha)^2 d\alpha,$$

which remains invariant under the Fredholm group of transformations (1.3) *with non-vanishing Fredholm determinants, possesses the group property.*

In order to prove the above theorem, let us consider two distinct members of the class of transformations (2.1) with kernels A_α^i, B_α^i, and denote them by T_1, T_2 respectively. By taking the product T_1T_2 of the transformations T_1 and T_2, we obtain

$$(2.2) \qquad \bar{\bar{g}}_{\alpha\beta} = g_{\alpha\beta} + g_{\gamma\beta}Q_\alpha^\gamma + g_{\alpha\delta}Q_\beta^\delta + g_{\gamma\delta}Q_\alpha^\gamma Q_\beta^\delta + Q_\beta^\alpha + Q_\alpha^\beta + \int_a^b Q_\alpha^\sigma Q_\beta^\sigma d\sigma$$

where $Q_\alpha^i = A_\alpha^i + B_\alpha^i + A_\sigma^i B_\alpha^\sigma$. This verifies the closure property.

To prove the existence of an inverse, we shall use a method of proof similar to that used by Volterra[*] in his treatment of equation (1.3). Let us substitute new variables in (2.1), multiply by k_α^i, and integrate in order to obtain the equations for the following terms: $\bar{g}_{\alpha\beta}$, $\bar{g}_{\alpha\mu}k_\beta^\mu$, $\bar{g}_{\lambda\beta}k_\alpha^\lambda$, and $\bar{g}_{\lambda\mu}k_\alpha^\lambda k_\beta^\mu$. If, to the sum of these four expressions we add $k_\beta^\alpha + k_\alpha^\beta + \int_a^b k_\alpha^\sigma k_\beta^\sigma d\sigma$ to both sides, we shall obtain

$$\bar{g}_{\alpha\beta} + \bar{g}_{\lambda\beta}k_\alpha^\lambda + \bar{g}_{\alpha\mu}k_\beta^\mu + \bar{g}_{\lambda\mu}k_\alpha^\lambda k_\beta^\mu + k_\beta^\alpha + k_\alpha^\beta + \int_a^b k_\alpha^\sigma k_\beta^\sigma d\sigma = g_{\alpha\beta} + g_{\gamma\beta}P_\alpha^\gamma + g_{\alpha\delta}P_\beta^\delta + g_{\gamma\delta}P_\alpha^\gamma P_\beta^\delta + P_\beta^\alpha + P_\alpha^\beta + \int_a^b P_\alpha^\sigma P_\beta^\sigma d\sigma$$

where $P_\alpha^i = K_\alpha^i + k_\alpha^i + K_\sigma^i k_\alpha^\sigma$. This however is the fundamental relation between a kernel and its resolvent kernel and is always zero. Thus we have established a continuous inverse

$$(2.3) \qquad g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \bar{g}_{\lambda\beta}k_\alpha^\lambda + \bar{g}_{\alpha\mu}k_\beta^\mu + \bar{g}_{\lambda\mu}k_\alpha^\lambda k_\beta^\mu + k_\beta^\alpha + k_\alpha^\beta + \int_a^b k_\alpha^\sigma k_\beta^\sigma d\sigma.$$

We see then, that under the conditions we have imposed, (2.1) cannot have more than one continuous solution; and, if it has one, this will be given by formula (2.3). In order to prove that the continuous function $g_{\alpha\beta}$ defined by (2.3) is really a solution of (2.1), one needs only to note that by substituting (2.3) in (2.1), we obtain an identity.

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[*] Cf. Bôcher, "Introduction to the Study of Integral Equations," *Cambridge Tract No.* 10, Cambridge (1926), pp. 21-22.

Obviously we shall have the identical transformation when the kernel $K_a{}^i$ is equal to zero.

3. *Functional invariants.* A sequence of continuous functionals ${f_{(i)}}^{\alpha\beta\cdots\gamma}_{\lambda\mu\cdots\nu}[g_{\sigma\tau}]$, $(i=1, 2, \cdots, n)$ of the coefficients $g_{\sigma\tau}$ of the quadratic functional form (1.1) that depends in particular on the variables $\alpha, \beta, \cdots \gamma, \lambda, \mu, \cdots, \nu$ will be said to form an *invariant class of functionals*[*] of weight $w(=\text{const.})$, if, under the Fredholm transformations (1.3) with non-vanishing Fredholm determinants, we have

$$
\begin{aligned}
(3.1)\qquad & {f_{(i)}}^{\alpha\beta\cdots\gamma}_{\lambda\mu\cdots\nu}[\bar{g}_{\sigma\tau}] \\
&= (D[K_t{}^s])^w \phi^{\alpha\beta\cdots\gamma}_{\lambda\mu\cdots\nu}\left[{f_{(1)}}^{ab\cdots c}_{lm\cdots n}, {f_{(2)}}^{ab\cdots c}_{lm\cdots n}, \cdots, {f_{(i)}}^{ab\cdots c}_{lm\cdots n}, K_q{}^p, k_v{}^n\right] \\
&\qquad\qquad (i=1, 2, \cdots, n)
\end{aligned}
$$

where $\phi^{\alpha\beta\cdots\gamma}_{\lambda\mu\cdots\nu}$ is a fixed functional of its functional arguments ${f_{(1)}}^{ab\cdots c}_{lm\cdots n}$, ${f_{(2)}}^{ab\cdots c}_{lm\cdots n}, \cdots, {f_{(i)}}^{ab\cdots c}_{lm\cdots n}, K_q{}^p, k_v{}^u$ that eliminates the variables $a, b, \cdots, c, l, m, \cdots, n, p, q, u, v$, and, moreover, $k_v{}^u$ is the resolvent kernel of $K_v{}^u$.

A relative functional invariant of weight zero shall be termed an absolute functional invariant. In regard to absolute functional invariants we have the following theorem.[†]

If the Fredholm determinant $D[g_{\alpha\beta}]$ of a quadratic functional form

$$
(3.2)\qquad g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^\alpha)^2 d\alpha
$$

is not zero, the resolvent kernel $g^{\alpha\beta}[g_{ab}]$ of $g_{\alpha\beta}$,

$$
(3.3)\qquad g^{\alpha\beta}[g_{ab}] = -\{D_\beta{}^\alpha[g_{ab}]/D[g_{ab}]\},
$$

is an absolute functional invariant of the quadratic form (3.2). The law of transformation of this functional invariant is

$$
\begin{aligned}
(3.4)\qquad g^{\alpha\beta}[\bar{g}_{ab}] &= g^{\alpha\beta}[g_{ab}] + g^{\gamma\beta}[g_{ab}]\, k_\gamma{}^\alpha + g^{\alpha\delta}[g_{ab}]\, k_\delta{}^\beta \\
&\quad + g^{\gamma\delta}[g_{ab}]\, k_\gamma{}^\alpha k_\delta{}^\beta + k_\beta{}^\alpha + k_\alpha{}^\beta + \int_a^b k_\sigma{}^\alpha k_\sigma{}^\beta d\sigma.
\end{aligned}
$$

In order to establish a theorem which follows as a direct consequence of the invariantive property of the resolvent kernel $g^{\alpha\beta}[g_{ab}]$ of $g_{\alpha\beta}$, let us consider the identity [*]

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[*] This notion is due to A. D. Michal.

[†] Cf. A. D. Michal, *loc. cit.*

$$(3.5)\qquad D^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n}[g_{ab}]+g_{\alpha_1\sigma}D^{\sigma\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n}[g_{ab}]$$
$$=g_{\alpha_1\beta_1}D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_2\beta_3\cdots\beta_n}[g_{ab}]-g_{\alpha_1\beta_2}D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_1\beta_3\cdots\beta_n}[g_{ab}]+\cdots$$
$$+(-1)^{n-1}g_{\alpha_1\beta_n}D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_{n-1}}[g_{ab}],$$

satisfied by the Fredholm minors. If we consider all the quantities α and β to be constant, with the exception of α_1, we readily observe that (3.5) reduces to an integral equation of the form

$$f^{\alpha_1}+g_{\alpha_1\sigma}f^{\sigma}=\phi^{\alpha_1},$$

where

$$(3.6)\qquad \begin{cases} f^{x}=D^{x\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n}[g_{ab}], \\ \phi^{x}=g_{x\beta_1}D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_2\beta_3\cdots\beta_n}[g_{ab}]-g_{x\beta_2}D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_1\beta_3\cdots\beta_n}[g_{ab}] \\ \qquad +\cdots+(-1)^{n-1}g_{x\beta_n}D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_{n-1}}[g_{ab}]. \end{cases}$$

Hence, we have the unique continuous solution

$$f^{\alpha_1}=\phi^{\alpha_1}+\int_a^b g^{\alpha_1\sigma}[g_{ab}]\,\phi^{\sigma}d\sigma,$$

which may be written in the form

$$(3.7)\qquad D[g_{ab}]f^{\alpha_1}=D[g_{ab}]\phi^{\alpha_1}-D_{\sigma}{}^{\alpha_1}[g_{ab}]\phi^{\sigma}.$$

This follows from (3.3) and the fact that $D[g_{ab}]\neq 0$. Interpreting (3.7) according to (3.6) we have

$$D[g_{ab}]D^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n}[g_{ab}]=\{D[g_{ab}]g_{\alpha_1\beta_1}-\int_a^b D_{\sigma}{}^{\alpha_1}[g_{ab}]g_{\sigma\beta_1}d\sigma\}$$
$$\times D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_2\beta_3\cdots\beta_n}[g_{ab}]$$
$$-\{D[g_{ab}]g_{\alpha_1\beta_2}-\int_a^b D_{\sigma}{}^{\alpha_1}[g_{ab}]g_{\sigma\beta_2}d\sigma\}D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_1\beta_3\cdots\beta_n}[g_{ab}]+\cdots$$
$$+(-1)^{n-1}\{D[g_{ab}]g_{\alpha_1\beta_n}-\int_a^b D_{\sigma}{}^{\alpha_1}[g_{ab}]g_{\sigma\beta_n}d\sigma\}D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_{n-1}}[g_{ab}].$$

Simplifying each expression in brackets by the identity

$$D[g_{ab}]g_{\alpha\beta}-\int_a^b D_{\sigma}{}^{\alpha}[g_{ab}]g_{\sigma\beta}d\sigma=D_{\beta}{}^{\alpha}[g_{ab}],$$

---

[*] Cf. R. D. D'Adhemar, *Leçons sur les principes de l'analyse*, Paris (1912), p. 237.

we obtain

$$(3.8)\qquad D[g_{ab}]D^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n}[g_{ab}] = D_{\beta_1}{}^{\alpha_1}[g_{ab}]D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_2\beta_3\cdots\beta_n}[g_{ab}] - D_{\beta_2}{}^{\alpha_1}[g_{ab}]D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_1\beta_3\cdots\beta_n}[g_{ab}] + \cdots + (-1)^{n-1}D_{\beta_n}{}^{\alpha_1}[g_{ab}]D^{\alpha_2\alpha_3\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_{n-1}}[g_{ab}].$$

In order to obtain (3. 8) in the desired form, consider the case in which $n=2$

$$(3.9)\qquad D[g_{ab}]D^{\alpha_1\alpha_2}_{\beta_1\beta_2}[g_{ab}] = D_{\beta_1}{}^{\alpha_1}[g_{ab}]D_{\beta_2}{}^{\alpha_2}[g_{ab}] - D_{\beta_2}{}^{\alpha_1}[g_{ab}]D_{\beta_1}{}^{\alpha_2}[g_{ab}] = \begin{vmatrix} D_{\beta_1}{}^{\alpha_1}[g_{ab}], & D_{\beta_2}{}^{\alpha_1}[g_{ab}] \\ D_{\beta_1}{}^{\alpha_2}[g_{ab}], & D_{\beta_2}{}^{\alpha_2}[g_{ab}] \end{vmatrix}.$$

Let us assume, then, that this form holds in general, i. e.

$$(3.10)\qquad \{D[g_{ab}]\}^{n-1}D^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n}[g_{ab}] = \begin{vmatrix} D_{\beta_1}{}^{\alpha_1}[g_{ab}], & D_{\beta_2}{}^{\alpha_1}[g_{ab}], & \cdots, & D_{\beta_n}{}^{\alpha_1}[g_{ab}] \\ D_{\beta_1}{}^{\alpha_2}[g_{ab}], & D_{\beta_2}{}^{\alpha_2}[g_{ab}], & \cdots, & D_{\beta_n}{}^{\alpha_2}[g_{ab}] \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ D_{\beta_1}{}^{\alpha_n}[g_{ab}], & D_{\beta_2}{}^{\alpha_n}[g_{ab}], & \cdots, & D_{\beta_n}{}^{\alpha_n}[g_{ab}] \end{vmatrix}.$$

We shall proceed to prove by mathematical induction that the formula (3. 10) is true. From (3. 8) we have

$$D[g_{ab}]D^{\alpha_1\alpha_2\cdots\alpha_{n+1}}_{\beta_1\beta_2\cdots\beta_{n+1}}[g_{ab}] = D_{\beta_1}{}^{\alpha_1}[g_{ab}]D^{\alpha_2\alpha_3\cdots\alpha_{n+1}}_{\beta_2\beta_3\cdots\beta_{n+1}}[g_{ab}] - D_{\beta_2}{}^{\alpha_1}[g_{ab}]D^{\alpha_2\alpha_3\cdots\alpha_{n+1}}_{\beta_1\beta_3\cdots\beta_{n+1}}[g_{ab}] + \cdots + (-1)^nD^{\alpha_1}_{\beta_{n+1}}[g_{ab}]D^{\alpha_2\alpha_3\cdots\alpha_{n+1}}_{\beta_1\beta_2\cdots\beta_n}[g_{ab}].$$

Multiplying both sides of the above equation by $\{D[g_{ab}]\}^{n-1}$ we readily observe, with the aid of (3. 10) that we may obtain

$$\{D[g_{ab}]\}^{n}D^{\alpha_1\alpha_2\cdots\alpha_{n+1}}_{\beta_1\beta_2\cdots\beta_{n+1}}[g_{ab}] = \begin{vmatrix} D_{\beta_1}{}^{\alpha_1}[g_{ab}], & D_{\beta_2}{}^{\alpha_1}[g_{ab}], & \cdots, & D^{\alpha_1}_{\beta_{n+1}}[g_{ab}] \\ D_{\beta_1}{}^{\alpha_2}[g_{ab}], & D_{\beta_2}{}^{\alpha_2}[g_{ab}], & \cdots, & D^{\alpha_2}_{\beta_{n+1}}[g_{ab}] \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ D^{\alpha_{n+1}}_{\beta_1}[g_{ab}], & D^{\alpha_{n+1}}_{\beta_2}[g_{ab}], & \cdots, & D^{\alpha_{n+1}}_{\beta_{n+1}}[g_{ab}] \end{vmatrix}.$$

Hence we see that the formula (3. 10)[*] holds in general.

If we divide both sides of (3. 10) by $\{D[g_{ab}]\}^n$ and take into consideration (3. 3), we shall obtain

---

[*] This result is due to Hurwitz, *Bulletin of the American Mathematical Society*, Vol. 20 (1914), p. 408. For an indirect proof cf. Platrier, "Sur les mineurs de la fonction déterminante de Fredholm et sur les systèmes d'équationes intégrales linéaires," *Journal de Mathématiques* (6e serie), tome 9 (1913), pp. 248-249.

$$(3.11)\qquad g^{\alpha_1\beta_1,\,\alpha_2\beta_2,\,\dots,\,\alpha_n\beta_n}[g_{ab}] = \begin{vmatrix} g^{\alpha_1\beta_1}[g_{ab}], & g^{\alpha_1\beta_2}[g_{ab}], & \cdots, & g^{\alpha_1\beta_n}[g_{ab}] \\ g^{\alpha_2\beta_1}[g_{ab}], & g^{\alpha_2\beta_2}[g_{ab}], & \cdots, & g^{\alpha_2\beta_n}[g_{ab}] \\ \cdot & \cdot & \cdot & \cdot \\ g^{\alpha_n\beta_1}[g_{ab}], & g^{\alpha_n\beta_2}[g_{ab}], & \cdots, & g^{\alpha_n\beta_n}[g_{ab}] \end{vmatrix}$$

where

$$(3.12)\qquad g^{\alpha_1\beta_1,\,\alpha_2\beta_2,\,\dots,\,\alpha_n\beta_n}[g_{ab}] = (-1)^n\{D^{\alpha_1\alpha_2\dots\alpha_n}_{\beta_1\beta_2\dots\beta_n}[g_{ab}]/D[g_{ab}]\},$$

which we shall call the nth resolvent. The name resolvent is not to be mistaken for the resolvent kernel (3.3).

Let us now investigate the invariant character of the nth resolvent, when the identity (2.1) is satisfied. Before taking up the general case, let us consider the transformation induced in the second resolvent under the conditions as stated.

In order to facilitate our handling of this problem, let us adopt the following conventions, in reference to (3.4),

$$(3.13)\qquad g^{\alpha\beta}[\bar{g}_{ab}] = g^{\lambda\mu}[g_{ab}]\binom{\alpha}{\lambda}\binom{\beta}{\mu} + t_\beta{}^\alpha$$

where $\binom{\alpha}{\lambda} = (k_\lambda{}^\alpha + \theta_\lambda{}^\alpha)$ and $t_\beta{}^\alpha = k_\beta{}^\alpha + k_\alpha{}^\beta + \int_a^b k_\sigma{}^\alpha k_\sigma{}^\beta d\sigma,$

$\theta_k{}^i$ being a symbolic operator such that

$$f^{\alpha k}\theta_k{}^i = f^{\alpha i}, \qquad f^{k\beta}\theta_k{}^i = f^{i\beta}.$$

According to this convention, we may obtain from (3.13) four such expressions for the quantities $g^{\alpha_1\beta_1}[\bar{g}_{ab}]$, $g^{\alpha_1\beta_2}[\bar{g}_{ab}]$, $g^{\alpha_2\beta_1}[\bar{g}_{ab}]$, and $g^{\alpha_2\beta_2}[\bar{g}_{ab}]$. Substituting these expressions in the equation

$$(3.14)\qquad g^{\alpha_1\beta_1,\,\alpha_2\beta_2}[\bar{g}_{ab}] = \begin{vmatrix} g^{\alpha_1\beta_1}[\bar{g}_{ab}], & g^{\alpha_1\beta_2}[\bar{g}_{ab}] \\ g^{\alpha_2\beta_1}[\bar{g}_{ab}], & g^{\alpha_2\beta_2}[\bar{g}_{ab}] \end{vmatrix},$$

we obtain

$$\begin{aligned} g^{\alpha_1\beta_1,\,\alpha_2\beta_2}[\bar{g}_{ab}] = {} & g^{\lambda_1\mu_1,\,\lambda_2\mu_2}[g_{ab}]\binom{\alpha_1}{\lambda_1}\binom{\beta_1}{\mu_1}\binom{\alpha_2}{\lambda_2}\binom{\beta_2}{\mu_2} + g^{\lambda_1\mu_1}[g_{ab}]\binom{\alpha_1}{\lambda_1}\binom{\beta_1}{\mu_1}t_{\beta_2}{}^{\alpha_2} \\ & + g^{\lambda_2\mu_2}[g_{ab}]\binom{\alpha_2}{\lambda_2}\binom{\beta_2}{\mu_2}t_{\beta_1}{}^{\alpha_1} - g^{\lambda_1\mu_2}[g_{ab}]\binom{\alpha_1}{\lambda_1}\binom{\beta_2}{\mu_2}t_{\beta_1}{}^{\alpha_2} \\ & - g^{\lambda_2\mu_1}[g_{ab}]\binom{\alpha_2}{\lambda_2}\binom{\beta_1}{\mu_1}t_{\beta_2}{}^{\alpha_1} + t_{\beta_1}{}^{\alpha_1}t_{\beta_2}{}^{\alpha_2} - t_{\beta_2}{}^{\alpha_1}t_{\beta_1}{}^{\alpha_2}, \end{aligned}$$

which may be written in the convenient form

$$(3.15)\qquad g^{\alpha_1\beta_1,\ \alpha_2\beta_2}[\bar{g}_{ab}] = g^{\lambda_1\mu_1,\ \lambda_2\mu_2}[g_{ab}]\binom{\alpha_1}{\lambda_1}\binom{\beta_1}{\mu_1}\binom{\alpha_2}{\lambda_2}\binom{\beta_2}{\mu_2}$$

$$-\begin{vmatrix} 0 & \binom{\beta_1}{\mu_1} & \binom{\beta_2}{\mu_1} \\ \binom{\alpha_1}{\lambda_1} & t_{\beta_1}{}^{\alpha_1} & t_{\beta_2}{}^{\alpha_1} \\ \binom{\alpha_2}{\lambda_1} & t_{\beta_1}{}^{\alpha_2} & t_{\beta_2}{}^{\alpha_2} \end{vmatrix} g^{\lambda_1\mu_1}[g_{ab}] + \begin{vmatrix} t_{\beta_1}{}^{\alpha_1} & t_{\beta_2}{}^{\alpha_1} \\ t_{\beta_1}{}^{\alpha_2} & t_{\beta_2}{}^{\alpha_2} \end{vmatrix}.$$

Let us now consider the third resolvent, i. e.,

$$(3.16)\qquad g^{\alpha_1\beta_1,\ \alpha_2\beta_2,\ \alpha_3\beta_3}[\bar{g}_{ab}] = \begin{vmatrix} g^{\alpha_1\beta_1}[\bar{g}_{ab}], & g^{\alpha_1\beta_2}[\bar{g}_{ab}], & g^{\alpha_1\beta_3}[\bar{g}_{ab}] \\ g^{\alpha_2\beta_1}[\bar{g}_{ab}], & g^{\alpha_2\beta_2}[\bar{g}_{ab}], & g^{\alpha_2\beta_3}[\bar{g}_{ab}] \\ g^{\alpha_3\beta_1}[\bar{g}_{ab}], & g^{\alpha_3\beta_2}[\bar{g}_{ab}], & g^{\alpha_3\beta_3}[\bar{g}_{ab}] \end{vmatrix}.$$

In precisely the same manner as for (3. 14), we substitute (3. 13) for the resolvent kernels in (3. 16). Expanding the determinant and collecting like terms, we obtain

$$(3.17)\qquad g^{\alpha_1\beta_1,\ \alpha_2\beta_2,\ \alpha_3\beta_3}[\bar{g}_{ab}]$$

$$= g^{\lambda_1\mu_1,\ \lambda_2\mu_2,\ \lambda_3\mu_3}[g_{ab}]\binom{\alpha_1}{\lambda_1}\binom{\beta_1}{\mu_1}\binom{\alpha_2}{\lambda_2}\binom{\beta_2}{\mu_2}\binom{\alpha_3}{\lambda_3}\binom{\beta_3}{\mu_3}$$

$$+ g^{\lambda_1\mu_1,\ \lambda_2\mu_2}[g_{ab}]$$

$$\Bigg\{\binom{\alpha_1}{\lambda_1}\binom{\alpha_2}{\lambda_2}\Bigg[\binom{\beta_1}{\mu_1}\binom{\beta_2}{\mu_2}t^{\alpha_3}_{\beta_3} - \binom{\beta_1}{\mu_1}\binom{\beta_3}{\mu_2}t^{\alpha_3}_{\beta_2} + \binom{\beta_2}{\mu_1}\binom{\beta_3}{\mu_2}t^{\alpha_3}_{\beta_1}\Bigg]$$

$$+\binom{\alpha_2}{\lambda_1}\binom{\alpha_3}{\lambda_2}\Bigg[\binom{\beta_1}{\mu_1}\binom{\beta_2}{\mu_2}t^{\alpha_1}_{\beta_3} - \binom{\beta_1}{\mu_1}\binom{\beta_3}{\mu_2}t^{\alpha_1}_{\beta_2} + \binom{\beta_2}{\mu_1}\binom{\beta_3}{\mu_2}t^{\alpha_1}_{\beta_1}\Bigg]$$

$$+\binom{\alpha_3}{\lambda_1}\binom{\alpha_1}{\lambda_2}\Bigg[\binom{\beta_1}{\mu_1}\binom{\beta_2}{\mu_2}t^{\alpha_2}_{\beta_3} - \binom{\beta_1}{\mu_1}\binom{\beta_3}{\mu_2}t^{\alpha_2}_{\beta_2} + \binom{\beta_2}{\mu_1}\binom{\beta_3}{\mu_2}t^{\alpha_2}_{\beta_1}\Bigg]\Bigg\}$$

$$+ g^{\lambda_1\mu_1}[g_{ab}]$$

$$\Bigg(\binom{\alpha_1}{\lambda_1}\Bigg\{\binom{\beta_1}{\mu_1}\Big[t^{\alpha_2}_{\beta_2}t^{\alpha_3}_{\beta_3} - t^{\alpha_2}_{\beta_3}t^{\alpha_3}_{\beta_2}\Big] - \binom{\beta_2}{\mu_1}\Big[t^{\alpha_2}_{\beta_1}t^{\alpha_3}_{\beta_3} - t^{\alpha_2}_{\beta_3}t^{\alpha_3}_{\beta_1}\Big]$$

$$+\binom{\beta_3}{\mu_1}\Big[t^{\alpha_2}_{\beta_1}t^{\alpha_3}_{\beta_2} - t^{\alpha_2}_{\beta_2}t^{\alpha_3}_{\beta_1}\Big]\Bigg\} - \binom{\alpha_2}{\lambda_1}\Bigg\{\binom{\beta_1}{\mu_1}\Big[t^{\alpha_1}_{\beta_2}t^{\alpha_3}_{\beta_3} - t^{\alpha_1}_{\beta_3}t^{\alpha_3}_{\beta_2}\Big]$$

$$-\binom{\beta_2}{\mu_1}\Big[t^{\alpha_1}_{\beta_1}t^{\alpha_3}_{\beta_3} - t^{\alpha_1}_{\beta_3}t^{\alpha_3}_{\beta_1}\Big] + \binom{\beta_3}{\mu_1}\Big[t^{\alpha_1}_{\beta_1}t^{\alpha_3}_{\beta_2} - t^{\alpha_1}_{\beta_2}t^{\alpha_3}_{\beta_1}\Big]\Bigg\}$$

$$+\binom{\alpha_3}{\lambda_1}\Bigg\{\binom{\beta_1}{\mu_1}[t_{\beta_2}{}^{\alpha_1}t_{\beta_3}{}^{\alpha_2} - t_{\beta_3}{}^{\alpha_1}t_{\beta_2}{}^{\alpha_2}] - \binom{\beta_2}{\mu_1}[t_{\beta_1}{}^{\alpha_1}t_{\beta_3}{}^{\alpha_2} - t_{\beta_3}{}^{\alpha_1}t_{\beta_1}{}^{\alpha_2}]$$

$$+\binom{\beta_3}{\mu_1}\left[t_{\beta_1}{}^{\alpha_1}t_{\beta_2}{}^{\alpha_2}-t_{\beta_2}{}^{\alpha_1}t_{\beta_1}{}^{\alpha_2}\right]\Big\}\Big)$$

$$+\left[t_{\beta_1}{}^{\alpha_1}t_{\beta_2}{}^{\alpha_2}t_{\beta_3}{}^{\alpha_3}+t_{\beta_2}{}^{\alpha_1}t_{\beta_3}{}^{\alpha_2}t_{\beta_1}{}^{\alpha_3}+t_{\beta_3}{}^{\alpha_1}t_{\beta_1}{}^{\alpha_2}t_{\beta_2}{}^{\alpha_3}\right.$$

$$\left.-t_{\beta_3}{}^{\alpha_1}t_{\beta_2}{}^{\alpha_2}t_{\beta_1}{}^{\alpha_3}-t_{\beta_2}{}^{\alpha_1}t_{\beta_1}{}^{\alpha_2}t_{\beta_3}{}^{\alpha_3}-t_{\beta_1}{}^{\alpha_1}t_{\beta_3}{}^{\alpha_2}t_{\beta_2}{}^{\alpha_3}\right].$$

In the same manner as for the second resolvent, we may write (3. 17) in the following form,

$$g^{\alpha_1\beta_1,\ \alpha_2\beta_2,\ \alpha_3\beta_3}[\bar{g}_{ab}]=g^{\lambda_1\mu_1,\ \lambda_2\mu_2,\ \lambda_3\mu_3}[g_{ab}]\binom{\alpha_1}{\lambda_1}\binom{\beta_1}{\mu_1}\binom{\alpha_2}{\lambda_2}\binom{\beta_2}{\mu_2}\binom{\alpha_3}{\lambda_3}\binom{\beta_3}{\mu_3}\tag{3. 18}$$

$$+[1/(2!)^2]\begin{vmatrix} 0 & 0 & \binom{\beta_1}{\mu_1} & \binom{\beta_2}{\mu_1} & \binom{\beta_3}{\mu_1} \\ 0 & 0 & \binom{\beta_1}{\mu_2} & \binom{\beta_2}{\mu_2} & \binom{\beta_3}{\mu_2} \\ \binom{\alpha_1}{\lambda_1} & \binom{\alpha_1}{\lambda_2} & t_{\beta_1}{}^{\alpha_1} & t_{\beta_2}{}^{\alpha_1} & t_{\beta_3}{}^{\alpha_1} \\ \binom{\alpha_2}{\lambda_1} & \binom{\alpha_2}{\lambda_2} & t_{\beta_1}{}^{\alpha_2} & t_{\beta_2}{}^{\alpha_2} & t_{\beta_3}{}^{\alpha_2} \\ \binom{\alpha_3}{\lambda_1} & \binom{\alpha_3}{\lambda_2} & t_{\beta_1}{}^{\alpha_3} & t_{\beta_2}{}^{\alpha_3} & t_{\beta_3}{}^{\alpha_3} \end{vmatrix} g^{\lambda_1\mu_1,\ \lambda_2\mu_2}[g_{ab}]$$

$$-\begin{vmatrix} 0 & \binom{\beta_1}{\mu_1} & \binom{\beta_2}{\mu_1} & \binom{\beta_3}{\mu_1} \\ \binom{\alpha_1}{\lambda_1} & t_{\beta_1}{}^{\alpha_1} & t_{\beta_2}{}^{\alpha_1} & t_{\beta_3}{}^{\alpha_1} \\ \binom{\alpha_2}{\lambda_1} & t_{\beta_1}{}^{\alpha_2} & t_{\beta_2}{}^{\alpha_2} & t_{\beta_3}{}^{\alpha_2} \\ \binom{\alpha_3}{\lambda_1} & t_{\beta_1}{}^{\alpha_3} & t_{\beta_2}{}^{\alpha_3} & t_{\beta_3}{}^{\alpha_3} \end{vmatrix} g^{\lambda_1\mu_1}[g_{ab}]+\begin{vmatrix} t_{\beta_1}{}^{\alpha_1} & t_{\beta_2}{}^{\alpha_1} & t_{\beta_3}{}^{\alpha_1} \\ t_{\beta_1}{}^{\alpha_2} & t_{\beta_2}{}^{\alpha_2} & t_{\beta_3}{}^{\alpha_2} \\ t_{\beta_1}{}^{\alpha_3} & t_{\beta_2}{}^{\alpha_3} & t_{\beta_3}{}^{\alpha_3} \end{vmatrix}.$$

As for the two preceding resolvents, we can find a similar expression for the fourth resolvent. It is highly probable that a similar expression holds good for the nth resolvent. A proof of such a relation however appears to be difficult.

THEOREM II. *The first i resolvents $g^{\alpha_1\beta_1}[g_{ab}],\cdots,g^{\alpha_1\beta_1,\ \cdots,\ \alpha_i\beta_i}[g_{ab}]$, $(i=1,2,3,4)$ form an absolute invariant class of i functionals.*

If we inquire as to the meaning of

$$t_\beta{}^\alpha=k_\beta{}^\alpha+k_\alpha{}^\beta+\int_a^b k_\sigma{}^\alpha k_\sigma{}^\beta d\sigma=0\tag{3. 19}$$

we find that this condition gives us the orthogonal group of Fredholm transformations. The characteristic property of this group is ordinarily given as

$$K_\beta^\alpha + K_\alpha^\beta + \int_a^b K_\alpha^\sigma K_\beta^\sigma d\sigma = 0$$

which immediately yields (3. 19), by noting the fact that K_α^β is the resolvent kernel of K_β^α and by making use of a theorem of Evans.[*]

Under the orthogonal Fredholm group we note that (3. 13) reduces to

$$(3.\,20) \qquad g^{\alpha\beta}[\bar{g}_{ab}] = g^{\lambda\mu}[g_{ab}] \binom{\alpha}{\lambda}\binom{\beta}{\mu}.$$

Substituting (3. 20) in (3. 11), we obtain

$$(3.\,21) \qquad g^{\alpha_1\beta_1,\ \alpha_2\theta_2,\ \dots,\ \alpha_n\beta_n}[\bar{g}_{ab}]$$
$$= g^{\lambda_1\mu_1,\ \lambda_2\mu_2,\ \dots,\ \lambda_n\mu_n}[g_{ab}] \binom{\alpha_1}{\lambda_1}\binom{\beta_1}{\mu_1}\binom{\alpha_2}{\lambda_2}\binom{\beta_2}{\mu_2}\cdots\binom{\alpha_n}{\lambda_n}\binom{\beta_n}{\lambda_n}$$

which gives us the general law of transformation of the nth resolvent under the orthogonal Fredholm group of transformations.

While it may seem that the orthogonality property imposes a great restriction on the Fredholm group of transformations, we shall show that this group is exactly the same as the group of Fredholm transformations which leaves the general Gramian invariant.

Let us consider, then, that we have a Gramian which is left invariant under a Fredholm transformation. That is to say

$$(3.\,22) \qquad \begin{vmatrix} (\bar{y}_1\bar{y}_1), & (\bar{y}_1\bar{y}_2), & \cdots, & (\bar{y}_1\bar{y}_n) \\ (\bar{y}_2\bar{y}_1), & (\bar{y}_2\bar{y}_2), & \cdots, & (\bar{y}_2\bar{y}_n) \\ \cdot & \cdot & \cdot & \cdot \\ (\bar{y}_n\bar{y}_1), & (\bar{y}_n\bar{y}_2), & \cdots, & (\bar{y}_n\bar{y}_n) \end{vmatrix} = \begin{vmatrix} (y_1y_1), & (y_1y_2), & \cdots, & (y_1y_n) \\ (y_2y_1), & (y_2y_2), & \cdots, & (y_2y_n) \\ \cdot & \cdot & \cdot & \cdot \\ (y_ny_1), & (y_ny_2), & \cdots, & (y_ny_n) \end{vmatrix}$$

where

$$(y_iy_k) = \int_a^b y_i^\sigma y_k^\sigma d\sigma, \qquad (i,\ k = 1,\ 2, \cdots,\ n).$$

Let us expand (3. 22) in order to have it in the following form

$$(3.\,23) \qquad (\bar{y}_1\bar{y}_1) \begin{vmatrix} (\bar{y}_2\bar{y}_2), & \dots, & (\bar{y}_2\bar{y}_n) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (\bar{y}_n\bar{y}_2), & \cdots, & (\bar{y}_n\bar{y}_n) \end{vmatrix} + Q_{\alpha\beta}\,[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n]\,\bar{y}_1^\alpha\bar{y}_1^\beta$$

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[*] Cf. G. C. Evans, "Functionals and Their Applications," *Cambridge Colloquium Lectures*, New York (1918), pp. 122.

$$= (y_1y_1)\begin{vmatrix} (y_2y_2),\cdots,(y_2y_n) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (y_ny_2),\cdots,(y_ny_n) \end{vmatrix} + Q_{\alpha\beta}\,[y_2, y_3,\cdots, y_n]\, y_1^{\alpha}y_1^{\beta}$$

where $Q_{\alpha\beta}$ is the coefficient of a quadratic functional form of $\bar{y}_1$ and y_1 respectively. Let us consider $\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n$ and $y_2, y_3, \cdots, y_n$ to be constant and then calculate the first variation of (3.23). Doing this, we obtain

$$\begin{aligned} &2\int_a^b \bar{y}_1^{\alpha}\delta\bar{y}_1^{\alpha}d\alpha \begin{vmatrix} (\bar{y}_2\bar{y}_2),\cdots,(\bar{y}_2\bar{y}_n) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (\bar{y}_n\bar{y}_2),\cdots,(\bar{y}_n\bar{y}_n) \end{vmatrix} + \{Q_{\alpha\beta}[\bar{y}_2,\cdots,\bar{y}_n] + Q_{\beta\alpha}[\bar{y}_2,\cdots,\bar{y}_n]\}\,\bar{y}_1^{\alpha}\delta\bar{y}_1^{\beta} \\ (3.24)\qquad &= 2\int_a^b y_1^{\alpha}\delta y_1^{\alpha}d\alpha \begin{vmatrix} (y_2y_2),\cdots,(y_2y_n) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (y_ny_2),\cdots,(y_ny_n) \end{vmatrix} + \{Q_{\alpha\beta}[y_2,\cdots,y_n] + Q_{\beta\alpha}[y_2,\cdots,y_n]\}\,y_1^{\alpha}\delta y_1^{\beta}. \end{aligned}$$

Recalling that

$$y^i = \bar{y}^i + K_{\alpha}^{i}\bar{y}^{\alpha}$$

and hence that

$$\delta y^i = \delta\bar{y}^i + K_{\alpha}^{i}\delta\bar{y}^{\alpha},$$

we can write (3.24) in the form

$$(3.25)\qquad \begin{aligned} &\{\bar{y}_1^{\alpha}\Omega[\bar{y}_2,\cdots,\bar{y}_n, y_2,\cdots,y_n] \\ &\qquad + \Lambda_{\beta}^{\alpha}[\bar{y}_2,\cdots,\bar{y}_n, y_2,\cdots,y_n, K_{\mu}^{\lambda}]\,\bar{y}_1^{\beta}\}\,\delta\bar{y}_1^{\alpha} = 0 \end{aligned}$$

where

$$(3.26)\qquad \Omega = 2\begin{vmatrix} (\bar{y}_2\bar{y}_2),\cdots,(\bar{y}_2\bar{y}_n) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (\bar{y}_n\bar{y}_2),\cdots,(\bar{y}_n\bar{y}_n) \end{vmatrix} - 2\begin{vmatrix} (y_2y_2),\cdots,(y_2y_n) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (y_ny_2),\cdots,(y_ny_n) \end{vmatrix}$$

and

$$\begin{aligned} \Lambda = {} & \{Q_{\alpha\beta}[\bar{y}_2,\cdots,\bar{y}_n] + Q_{\beta\alpha}[\bar{y}_2,\cdots,\bar{y}_n]\} \\ & - \{Q_{\alpha\beta}[y_2,\cdots,y_n] - Q_{\beta\alpha}[y_2,\cdots,y_n]\} \\ & - \{Q_{\gamma\beta}[y_2,\cdots,y_n] + Q_{\beta\gamma}[y_2,\cdots,y_n]\}K_{\alpha}^{\gamma} \\ & - \{Q_{\alpha\delta}[y_2,\cdots,y_n] + Q_{\delta\alpha}[y_2,\cdots,y_n]\}K_{\beta}^{\delta} \\ & - \{Q_{\gamma\delta}[y_2,\cdots,y_n] + Q_{\delta\gamma}[y_2,\cdots,y_n]\}K_{\alpha}^{\gamma}K_{\beta}^{\delta} \\ & - 2\begin{vmatrix} (y_2y_2),\cdots,(y_2y_n) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (y_ny_2),\cdots,(y_ny_n) \end{vmatrix} \left(K_{\beta}^{\alpha} + K_{\alpha}^{\beta} + \int_a^b K_{\alpha}^{\sigma}K_{\beta}^{\sigma}d\sigma\right). \end{aligned}$$

By the fundamental lemma of the calculus of variations, we see that (3. 25) implies

$$(3.27)\quad \Omega[\bar{y}_2,\cdots,\bar{y}_n,y_2,\cdots,y_n]\bar{y}_1{}^{a}+\Lambda_{\beta}{}^{a}[\bar{y}_2,\cdots,\bar{y}_n,y_2,\cdots,y_n,K_{\mu}{}^{\lambda}]\bar{y}_1{}^{\beta}=0.$$

In order, however, that this relation be true, we must have [*]

$$\Omega[\bar{y}_2,\cdots,\bar{y}_n,y_2,\cdots,y_n]=0,\qquad \Lambda_{\beta}{}^{a}[\bar{y}_2,\cdots,\bar{y}_n,y_2,\cdots,y_n,K_{\mu}{}^{\lambda}]=0.$$

Considering (3. 26), we see that $\Omega=0$ implies the invariance of $(n-1)$ rowed minor of our Gramian. Hence, a necessary condition that a Gramian of the nth order be an absolute invariant under the Fredholm group of transformations, is that the Gramian of the $(n-1)$ degree be absolutely invariant.

Repeating the above operation $(n-1)$ times, we see that the proof of our problem depends on the invariance of a Gramian of the first order. Consider, then, the Gramian of order one

$$(3.28)\qquad \int_a^b y_i{}^{a}y_k{}^{a}d\alpha=\int_a^b \bar{y}_i{}^{a}\bar{y}_k{}^{a}d\alpha.$$

Under the Fredholm group of transformations, we obtain the condition

$$\int_a^b\int_a^b[K_{\beta}{}^{a}+K_{a}{}^{\beta}+\int_a^b K_{a}{}^{\sigma}K_{\beta}{}^{\sigma}d\sigma]\bar{y}_i{}^{a}\bar{y}_k{}^{\beta}d\alpha d\beta=0,$$

in order that (3. 28) be true. This, however, implies

$$(3.29)\qquad K_{\beta}{}^{a}+K_{a}{}^{\beta}+\int_a^b K_{a}{}^{\sigma}K_{\beta}{}^{\sigma}d\sigma=0,$$

as we have seen above. Equation (3. 29) is the characteristic equation of the orthogonal Fredholm group of transformations and hence a necessary condition that a Gramian of the nth degree be an absolute invariant is that the Fredholm group of transformations be orthogonal.

Interpreting (3. 28) as a general element of a Gramian of the nth degree, we see that the above necessary condition is also sufficient; since when each entry in a determinant is an absolute invariant, it follows that the determinant is an absolute invariant.

THEOREM III. *A necessary and sufficient condition that a Gramian be an absolute invariant under the Fredholm group of transformations is that the group of transformations be orthogonal.*

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[*] Cf. G. Kowalewski, "Über Funktionenraume," *Sitzb. d. mathem. Naturw. Kl.*, CXX Bd., Abt. IIa, Novem. 1911, pp. 1455.

4. *The Fredholm group with a parameter.* Let us consider a Fredholm transformation with an arbitrary parameter λ,

$$y^a = \bar{y}^a + \lambda K_\sigma{}^a \bar{y}^\sigma, \tag{4.1}$$

such that λ is not a characteristic value of the kernel $K_b{}^a$. The totality of such Fredholm transformations form a group. In fact, let us consider two arbitrary transformations of the type (4.1);

$$\begin{aligned} T_1 &: \bar{y}^a = y^a + \lambda A_\sigma{}^a y^\sigma \\ T_2 &: \bar{\bar{y}}^a = \bar{y}^a + \mu B_\tau{}^a \bar{y}^\tau \end{aligned}$$

where λ and μ are not characteristic values for their respective kernels.

By taking the product T_1T_2 of the transformations T_1 and T_2, we obtain

$$\bar{\bar{y}}^a = y^a + K_\sigma{}^a y^\sigma,$$

where

$$K_\sigma{}^a = \lambda A_\sigma{}^a + \mu B_\sigma{}^a + \lambda\mu B_\tau{}^a A_\sigma{}^\tau.$$

By a theorem due to Jonesco,[*] we see that the kernel $K_b{}^a$ has for a Fredholm determinant, corresponding to the value 1 of the parameter,

$$D_K(1) = D_A(\lambda) \cdot D_B(\mu).$$

Since by hypothesis $D_A(\lambda) \neq 0$ and $D_B(\mu) \neq 0$, we readily observe that $D_K(1)$ is different from zero and hence that the transformations (4.1) satisfy the closure property.

Since $D_A(\lambda) \neq 0$, we have the unique, continuous inverse [†]

$$y^a = \bar{y}^a + \lambda \mathfrak{A}_\sigma{}^a \bar{y}^\sigma,$$

where

$$\mathfrak{A}_\sigma{}^a = -\,[D_\sigma{}^a(\lambda)/D_A(\lambda)].$$

Obviously the kernel, zero, gives us the identical transformation. We may state then the

THEOREM IV. *The Fredholm transformations with a parameter*

$$y^a = \bar{y}^a + \lambda K_\sigma{}^a \bar{y}^\sigma,$$

where λ is not a characteristic value of the kernel $K_b{}^a$ constitute a group.

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[*] Cf. Jonesco, "Sur la fonçtion caractéristique et le noyau résolvent d'un noyau donné," *Bulletin des Sciences Mathématique*, 2e serie, tome 50 (Juillet, 1926).

[†] Cf. Goursat, *Cours d'Analyse*, Paris (1927), Vol. III, pp. 373.

5. *Cogredient and contragredient functional transformations.* We shall say that the functions $u^\alpha, v^\alpha, \cdots$ transform cogrediently to the function y^i if

$$\begin{aligned} \bar{u}^\alpha &= u^\alpha + K_\sigma{}^\alpha u^\sigma \\ \bar{v}^\alpha &= v^\alpha + K_\sigma{}^\alpha v^\sigma \\ &\text{etc.} \end{aligned} \tag{5.1}$$

when

$$\bar{y}^i = y^i + K_\alpha{}^i y^\alpha$$

where the kernel $K_b{}^a$ is the same in both transformations.

In addition, we shall say that the functions $\xi_\alpha, \eta_\alpha, \cdots$ transform contragrediently to the function y^i, if

$$\begin{aligned} \bar{\xi}_\alpha &= \xi_\alpha + \int_a^b k_\sigma{}^\alpha \xi_\sigma d\sigma, \\ \bar{\eta}_\alpha &= \eta_\alpha + \int_a^b k_\sigma{}^\alpha \eta_\sigma d\sigma \\ &\text{etc.} \end{aligned} \tag{5.2}$$

where $k_b{}^a$ is the resolvent kernel of the kernel $K_b{}^a$.

With these conceptions clear in mind then, let us investigate the induced transformation in the function $h^{\alpha\beta}$ when the quadratic functional form

$$h^{\alpha\beta}\xi_\alpha\xi_\beta + \int_a^b (\xi_\alpha)^2 d\alpha, \qquad (h^{\alpha\beta} = h^{\beta\alpha})$$

is an absolute invariant under the contragredient transformations (5. 2). This problem is analogous to that of equation (2. 9) and, as a result, we obtain

$$\begin{aligned} h^{\alpha\beta} = h^{\alpha\beta} &+ h^{\gamma\beta}k_\gamma{}^\alpha + h^{\alpha\delta}k_\delta{}^\beta + h^{\gamma\delta}k_\gamma{}^\alpha k_\delta{}^\beta \\ &+ k_\beta{}^\alpha + k_\alpha{}^\beta + \int_a^b k_\sigma{}^\alpha k_\sigma{}^\beta d\sigma, \end{aligned} \tag{5.3}$$

which is precisely of the form (3. 4), the law of transformation of the kernel $g^{\alpha\beta}[g_{ab}]$ when the quadratic functional form (1. 1) is absolutely invariant.

Thus we see that the resolvent kernel itself may be considered as a coefficient of a quadratic functional form

$$g^{\alpha\beta}[g_{ab}]\xi_\alpha\xi_\beta + \int_a^b (\xi_\alpha)^2 d\alpha.$$

Let the functional $Q[\xi, \eta]$ be defined by

$$Q[\xi, \eta] = g^{\alpha\beta}[g_{ab}]\xi_\alpha\eta_\beta + \int_a^b \xi_\alpha\eta_\alpha d\alpha.$$

On the basis of the above result it is clear that the determinant

$$\Delta = \begin{vmatrix} Q[\xi, \xi], & Q[\xi, \eta] \\ Q[\eta, \xi], & Q[\eta, \eta] \end{vmatrix} \tag{5.4}$$

is an absolute quartic functional form under the contragredient transformations (5.2). The determinant Δ when expanded can be written in the form

$$\begin{aligned} &g^{\alpha_1\beta_1,\, \alpha_2\beta_2}[g_{ab}]\ \xi_{\alpha_1}\xi_{\beta_1}\eta_{\alpha_2}\eta_{\beta_2} \\ &+ g^{\alpha_1\beta_1}[g_{ab}] \begin{vmatrix} \xi_{\alpha_1}\xi_{\sigma_1} \\ \eta_{\alpha_1}\eta_{\sigma_1} \end{vmatrix} \cdot \begin{vmatrix} \xi_{\beta_1}\xi_{\sigma_1} \\ \eta_{\beta_1}\eta_{\sigma_1} \end{vmatrix} + \xi_{\sigma_2}\eta_{\sigma_1} \begin{vmatrix} \xi_{\sigma_2}\xi_{\sigma_1} \\ \eta_{\sigma_2}\eta_{\sigma_1} \end{vmatrix} \cdot \end{aligned} \tag{5.5}$$

(integration on σ_1 and σ_2)

This can be verified most easily by substituting for $g^{\alpha_1\beta_1,\, \alpha_2\beta_2}[g_{ab}]$ its expression in terms of $g^{\alpha\beta}[g_{ab}]$ as given by formula (3.11).

These results then imply the following interesting theorem:

THEOREM V. *The second resolvent* $g^{\alpha_1\beta_1,\, \alpha_2\beta_2}[g_{ab}]$ *and the resolvent kernel* $g^{\alpha\beta}[g_{ab}]$ *of an absolute quadratic functional form* (1.1) *may be taken as coefficients of the absolute quartic functional form* (5.5).

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On the Contact of a Quartic Surface with an Analytic Surface.

By Ernest P. Lane.

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1. *Introduction.* An algebraic surface A is said to have contact of order k with an analytic surface S at a point P on S in case every curve on S through P has $k+1$ consecutive points on A at P. If A and S are in ordinary space, and if S is represented in the neighborhood of P by a power series expansion of one non-homogeneous projective coordinate z of a point on the surface in terms of the other two coördinates x, y, with respect to a local coordinate system, while A is represented by an algebraic equation of degree h in x, y, z, then this equation is satisfied identically in x, y as far as the terms of degrees k when the series for z is substituted therein.

It is $(k+1)(k+2)/2$ conditions for an algebraic surface to have contact of order k at a point of an analytic surface. Since an algebraic surface of order h depends on $h(h^2+6h+11)/6$ parameters there is a limitation on the order of contact possible for a given algebraic surface. For instance, a quadric surface, being of order two, depends on nine parameters. Since contact of the second order imposes six conditions, there are ∞^3 quadrics having contact of the second order with S at P. These quadrics play a prominent part in the projective differential geometry of a surface. But, since contact of the third order imposes ten conditions, there is no non-composite quadric having contact of the third order with S at P, if S is not itself a quadric.

A cubic surface depends on nineteen parameters, so that there are ∞^4 cubic surfaces having contact of the fourth order with S at P. There is no non-composite cubic surface having contact of the fifth order with S at P, if S is not itself a cubic. A quartic surface depends on thirty-four parameters, so that there are ∞^6 quartic surfaces having contact of the sixth order with S at P. There is no non-composite quartic surface having contact of the seventh order with S at P, if S is not itself a quartic.

It is the purpose of this paper to summarize briefly the most interesting results that have been obtained concerning cubic surfaces having contact of various orders at a point of an analytic surface, and then to initiate an investigation of the contact of a general quartic surface with an analytic surface. The cases in which the quartics have contact of the fourth, fifth, and sixth orders receive particular attention. In each case the possibilities as to composite quartic surfaces are considered, and emphasis is laid on the curve of intersection of a non-composite quartic and the tangent plane at the point of contact.

2. *Analytic Basis.* If the four projective homogeneous coordinates $x^{(1)}, \cdots, x^{(4)}$ of a point P on a non-degenerate non-ruled surface S in ordinary space are given as single valued analytic functions of two independent variables u, v, and if the parametric net on S is the asymptotic net, then the functions x when multiplied by a suitably chosen common factor, are solutions of a completely integrable system of differential equations in Fubini's canonical form,

$$x_{uu} = px + \theta_u x_u + \beta x_v, \qquad x_{vv} = qx + \gamma x_u + \theta_v x_v, \qquad (\theta = \log \beta\gamma). \tag{1}$$

The coordinates of a point Q on S near P can be represented by Taylor's formula as power series in the increments Δu, Δv corresponding to displacement on S from P to Q. Then by means of (1) and the equations obtained from (1) by differentiation it is possible to express each of these series in the form $x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}$, where $x_1, \cdots, x_4$ are power series in Δu, Δv which represent the local coordinates of Q referred to the covariant tetrahedron x, x_u, x_v, x_{uv} with suitably chosen unit point. It is easy to calculate power series in Δu, Δv for the non-homogeneous coordinates x, y, z of Q defined by placing

$$x = x_2/x_1, \qquad y = x_3/x_1, \qquad z = x_4/x_1. \tag{2}$$

From these series it is possible to compute an expansion for z as a power series in x, y. To terms of the sixth degree the series thus obtained can be written* in the form

$$z = T + U + V + W + \cdots, \tag{3}$$

where

$$\begin{aligned} T &= xy - (\beta x^3 + \gamma y^3)/3, \\ 12U &= \beta\phi x^4 - 4\beta\psi x^3 y - 6\theta_{uv} x^2 y^2 - 4\gamma\phi x y^3 + \gamma\psi y^4, \\ 60V &= \beta c_0 x^5 + 5\beta c_1 x^4 y + 10\beta c_2 x^3 y^2 + 10\gamma c_3 x^2 y^3 + 5\gamma c_4 x y^4 + \gamma c_5 y^5, \\ W &= \beta d_0 x^6 + \beta d_1 x^5 y + \beta d_2 x^4 y^2 + d_3 x^3 y^3 + \gamma d_4 x^2 y^4 + \gamma d_5 x y^5 + \gamma d_6 y_6, \end{aligned} \tag{4}$$

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[*] Lane, "Power Series Expansions in the Neighborhood of a Point on a Surface," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), p. 808.

[illegible]

$$
\begin{aligned}
\phi &= (\log\beta\gamma^2)_u, & \psi &= (\log\beta^2\gamma)_v,\\
c\ &= \phi_u - \theta_u\phi - \phi^2 + 8\beta\psi - 4p, & c_5 &= \psi_v - \theta_v\psi - \psi^2 + 8\gamma\phi \ldots,\\
c\ &= \phi\psi - \psi_u + 7\theta_{uv} + 4\beta\gamma, & c_4 &= \phi\psi - \phi_v + 7\theta_{uv} + 4\beta\gamma,\\
c\ &= 3\gamma\phi - \psi^2 + \theta_v\psi - \psi_v + 2q, & c_3 &= 3\beta\psi - \phi^2 + \theta_u\phi - \phi_u + 2\ldots
\end{aligned}
$$

the coefficients $d_0, \cdots, d_6$ having never been calculated in terms of the co[illegible] [illegible] (1). The series (3) is the basis of the investigations of the [illegible]

[illegible]

3. *Cubic Surfaces.* If we write the most general equation of th[illegible] d[illegible] in x, y, z and demand that this equation be satisfied by the exp[illegible] [illegible] in (3) identically in x, y as far as the terms of the fourth degre[illegible] [illegible] can be written in the form

$$
\begin{aligned}
&A[z - T + \theta_{uv}z^2/2 + (z - xy)(\phi x + \psi y)/4 + 5z(\beta\psi x^2 + \gamma\phi y^2)/12]\\
&+ Bz^2 + Cxz^2 + Dyz^2 + Ez(z - xy) = 0,
\end{aligned}
$$

[illegible] coefficients, A, B, C, D, E, being arbitrary. Every one of the cubi[illegible] represented by (6) has contact of the fourth order with S at P. T[illegible] [illegible] of these surfaces which are composite, one component of each [illegible] the tangent plane $z = 0$ and the other being a quadric surface having [illegible] [illegible] second order with S at P; for these surfaces, $A = 0$. *All the ot[illegible] [illegible] surfaces represented by* (6) *cut the tangent plane in the same [illegible] [illegible] whose equations are*

$$z - xy - (\beta x^3 + \gamma y^3)/3 + xy(\phi x + \psi y)/4 = 0.$$

[illegible] is a covariant cubic curve having a node at P, with the asymptotic [illegible] [illegible] for double-point tangents, and having three inflexions which [illegible] [illegible] line whose equations in homogeneous coordinates are found [illegible] [illegible] be

(8) $$x_4 = 4x_1 + \phi x_2 + \psi x_3 = 0.$$

T[illegible] is the second canonical edge of Green. One of the points of inflexion [illegible] each of the tangents of Darboux,*

(9) $$x_4 = \beta x_2^3 + \gamma x_3^3 = 0.$$

If we demand that (6) be satisfied by (3) identically in x, y as [illegible] the terms of the fifth degree without restriction on S, we obtain the tangent plane counted three times as the only cubic surface having contact of the fifth order with S at P. But *if we suppose* $A \neq 0$, *then* S *is restricted to be itself a cubic, and necessary and sufficient conditions therefor are found to be*

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* B. Segre, "La cubique indicatrice de l'elément linéare projective d'une su[illegible]face," *Comptes Rendus*, Vol. 184 (1927), p. 729.

$$\begin{aligned} &(\log \beta/\gamma)_{uv} = 0, \\ (10) \qquad &4p = \phi_u = \theta_u\phi + \phi^2/4 - \beta\psi/3, \\ &4q = \psi_v - \theta_v\psi + \psi^2/4 - \gamma\phi/3. \end{aligned}$$

By a suitable choice of parameters[*] it is possible to make $\beta = \gamma$ and consequently to simplify these conditions.

4. *Quartic surfaces with contact of the fourth order.* If we write the most general equation of the fourth degree in x, y, z and demand that this equation be satisfied by the expression for z in (3) identically in x, y as far as the terms of the fourth degree, the result can be written in the form

$$\begin{aligned} &(gx + fy)(z - T) + h(z - T - U) + c(z^2 - x^2y^2) + Cz^4 \\ (11) \qquad &+ Ex^3z + Gy^3z + Hxz^3 + Iyz^3 + Kx^2z^2 + Ly^2z^2 + Mxyz^2 \\ &+ Nxy^2z + Qx^2yz + (z - xy)(E'x^2 + J'xy + G'y^2) \\ &+ C'z^3 + H'xz^2 + I'yz^2 = 0, \end{aligned}$$

the twenty coefficients $g, f, h, \cdots, H', I'$ being arbitrary. Every one of the ∞^{19} quartic surfaces represented by (11) has contact of the fourth order with S at P. There are ∞^{13} of these quartics which are composite, one component of each being the tangent plane $z = 0$ and the other being a cubic surface having contact of the second order with S at P. Any one of the ∞^6 other quartic surfaces represented by (11) intersects the tangent plane in a quartic curve whose equations are

$$(12) \qquad z = (gx + fy)T + h(T + U) + xy[E'x^2 + (c + J')xy + G'y^2] = 0.$$

There are only ∞^5 of these curves, instead of the ∞^6 that might have been expected. Each of these curves for which $h = 0$ has a node at P, the double-point tangents being the asymptotic tangents $z = xy = 0$. A curve (12) for which $h = 0$, but not both of f, g are zero, has a triple-point at P, the triple-point tangents being the asymptotic tangents and the line $z = gx + fy = 0$. It is clear that if either of f and g is zero while the other is not, two of the point tangents coincide in an asymptotic tangent. And if $f=g=h=0$, the curve is composite, the components being the asymptotic tangents and two straight lines

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[*] Lane, "The Contact of a Cubic Surface with an Analytic Surface," *Transactions of the American Mathematical Society*, Vol. 29 (1927), p. 471.

$$z = E'x^2 + (c+J')xy + G'y^2 = 0.$$

There is another case of interest in which the curve (12) is composite, namely, the case in which this curve consists of two proper conics, one tangent to each of the asymptotic tangents at P. Writing the equations of two such conics, multiplying them together, and comparing the result with (12), we easily find that *conditions necessary and sufficient that* (12) *may represent two proper conics are*

$$\begin{aligned} 12E' &= 4\beta f + 3\phi g + 3\beta\psi - 3\phi^2/4, \\ 12G' &= 3\psi f + 4\gamma g + 3\gamma\phi - 3\psi^2/4, \\ 12(c+J') &= 12fg - 3\phi f - 3\psi g + 3\phi\psi/2 \\ &\quad + 6\theta_{uv} + 4\beta\gamma/3. \end{aligned}$$

If every tangent plane of a quartic surface cuts the surface in two conics, the surface is a Roman surface of Steiner. Since such a surface depends on fifteen parameters, and since contact of the fourth order imposes fifteen conditions, it is not surprising that there should be a unique Roman surface having contact of the fourth order with S at P. Wilczynski's determination of this surface is the only instance, known to the writer, of any previous study of the contact of a quartic surface with an analytic surface. The method used by Wilczynski gives the following *parametric equations of the osculating Roman surface of* S *at* P, referred to the tetrahedron, and in the notation, that we are now using:

$$\begin{aligned} x_1 &= 24 - 3(2\beta\psi + \phi^2/8)\xi^2 + (3\phi\psi/4 - 40\beta\gamma/3 - 12\theta_{uv})\xi\eta \\ &\quad - 3(2\gamma\phi + \psi^2/8)\eta^2, \\ x_2 &= 24\xi + 3\phi\xi^2 - 3\psi\xi\eta + 8\gamma\eta^2, \\ x_3 &= 24\eta + 8\beta\xi^2 - 3\phi\xi\eta + 3\psi\eta^2, \\ x_4 &= 24\xi\eta. \end{aligned}$$

Contact of the fifth order. Proceeding as before, we find that the equation of the most general quartic surface having contact of the fifth order with S at P is

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Wilczynski, "Fifth Memoir," *Transactions of the American Mathematical Society*, (1909), p. 279. Darboux also determined this surface, *Comptes Rendus*, 1880), p. 969.

$$
\begin{aligned}
& g\,[x(z-T)+z(4\beta\psi x^3+6\theta_{uv}x^2y+5\gamma\phi xy^2-\gamma\psi y^3)/12 \\
& \quad +\phi x^2(z-xy)/4]+f\,[y(z-T)+z(-\beta\phi x^3+5\beta\psi x^2y \\
& \quad +6\theta_{uv}xy^2+4\gamma\phi y^3)/12+\psi y^2(z-xy)/4] \\
& \quad +h\,[z-T-U+z\,\{\beta(c_5-10c_2)x^2y+\gamma(c_0-10c_3)xy^2 \\
& \quad -5\beta c_1x^3-5\gamma c_4y^3\}/60+(z-xy)(c_0x^2+c_5y^2)/20] \\
& \quad +c\,[z^2-x^2y^2+2z(\beta x^3+\gamma y^3)/3]+Cz^4+Hxz^3+Iyz^3 \\
& \quad +Kx^2z^2+Ly^2z^2+Mxyz^2+C'z^3+z(z-xy)(H'x+I'y) \\
& \quad +J'[xy(z-xy)+z(\beta x^3+\gamma y^3)/3]=0.
\end{aligned} \tag{16}
$$

Since this equation contains fourteen arbitrary homogeneous constants, it follows that there are ∞^{13} quartic surfaces having contact of the fifth order with S at P. Of these there are ∞^9 that are composite, one component of each being the tangent plane $z=0$ and the other being a cubic surface having contact of the third order with S at P. Any one of the ∞^4 other quartic surfaces represented by (14) intersects the tangent plane in a quartic curve whose equations are

$$
\begin{aligned}
z={}& g(xT+\phi x^3y/4)+f(yT+\psi xy^3/4) \\
& +h\,[T+U+xy(c_0x^2+c_5y^2)/20] \\
& +(c+J')x^2y^2=0.
\end{aligned} \tag{7}
$$

There are ∞^3 of these curves, which can be discussed as in § IV, the only difference being that *when* $f=g=h=0$, *the curve breaks up into the asymptotic tangents each counted twice.*

The fact that two quadrics each of which has contact of the second order constitute a composite quartic with contact of the fifth order suggests that under certain conditions the left member of (16) is factorable into the product of two quadratic expressions of the form

$$z-xy+pxz+qyz+rz^2,$$

since the vanishing of such an expression gives the equation of the most general non-singular quadric having contact of the second order with S at P. In fact, multiplying together two such expressions and comparing the coefficients of the result with the coefficients of (16) we easily show that (16) *represents two non-singular quadrics each of which has contact of the second order with S at P if the coefficients of* (16) *satisfy the conditions.*

$$
(18)\qquad
\begin{aligned}
&f = g = h = 0, \qquad c = 1, \qquad J' = -2,\\
&(2H - C'H')^2 = (C'^2 - 4C)(H'^2 - 4K),\\
&(2I - C'I')^2 = (C'^2 - 4C)(I'^2 - 4L),\\
&[2(C' + M) - H'I']^2 = (I'^2 - 4L)(H'^2 - 4K).
\end{aligned}
$$

For such a composite quartic, the curve of intersection with the tangent plane is the asymptotic tangents, each counted twice.

6. *Contact of the sixth order.* Since the equation of the most general quartic surface having contact of the sixth order with S at P is rather long, we shall not write it here explicitly, although there would be no difficulty in doing so.* We shall merely write conditions on the coefficients of (16) which are necessary and sufficient that the quartic surface represented by (16) shall have contact of the sixth order. These conditions are found by demanding that (16) be satisfied by the expression for z in (3) identically in x, y as far as the terms of the sixth degree, and can be written in the form

$$
(19)\qquad
\begin{aligned}
4\beta(c + J') &= \beta\phi f + (12c_0 - 80\beta\psi + 15\phi^2)g/20\\
&\quad + (36d_0 + \beta c_1 + 3\phi c_0/20)h,\\
4\gamma(c + J') &= (12c_5 - 80\gamma\phi + 15\psi^2)f/20 + \gamma\phi g\\
&\quad + (36d_6 + \gamma c_4 + 3\psi c_5/20)h,\\
4H' - \phi(2c + J') &= (3c_0 - 25\beta\psi)f/15 + (c_1 - 2\theta_{uv} - \phi\psi)g\\
&\quad + [12d_1 + (10\beta c_2 - \beta c_5 - 3\psi c_0)/15]\,h,\\
4I' - \psi(2c + J') &= (c_4 - 2\theta_{uv} - \phi\psi)f + (3c_5 - 25\gamma\phi)g/15\\
&\quad + [12d_5 + (10\gamma c_3 - \gamma c_0 - 3\phi c_5)/15]\,h,\\
12K - 5\beta\psi(2c + J') &= (c_4 - c_1 - 5\phi\psi/4)\beta f + [2\beta(c_5 - 10c_2)\\
&\quad + 15\phi\theta_{uv}]\,g/10 + [12\beta(d_5 - d_2) + 3c_0\theta_{uv}/10\\
&\quad - \beta\phi c_5/4]\,h,\\
12L - 5\gamma\phi(2c + J') &= [2\gamma(c_0 - 10c_3) + 15\psi\theta_{uv}]\,f/10\\
&\quad + (c_1 - c_4 - 5\phi\psi/4)\gamma g + [12\gamma(d_1 - d_4)\\
&\quad + 3c_5\theta_{uv}/10 - \gamma\psi c_0/4]\,h,\\
6\theta_{uv}(2c + J') &+ 8\beta\gamma(c + J')/3 - 12(C' + M)\\
&= (2c_2 - \gamma\phi - \psi^2)\beta f + (2c_3 - \beta\psi - \phi^2)\gamma g\\
&\quad + [12d_3 + (c_1 + c_4)\beta\gamma/3 - (\gamma\phi c_0 + \beta\psi c_5)/5]\,h.
\end{aligned}
$$

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* For details see R. E. McPherson's master's thesis, University of Chicago, 1929.

There are ∞^6 of these quartics, and there are ∞^4 of them that break up into the tangent plane and a cubic surface with the contact of the fourth order. Any one of the ∞^2 others intersects the tangent plane in a quartic curve whose equations are (17) with $c + J'$ satisfying either of the first two of equations (19) while f, g, h satisfy a linear homogeneous equation deduced from these two equations by eliminating $c + J'$. *There are ∞^1 of these curves forming a pencil, and in this pencil there is just one curve which has a triple-point at P; for it, $h = 0$. Its triple-point tangents are the asymptotic tangents and the covariant line*

$$(20) \qquad z = (12c_5 - 100\gamma\phi + 15\psi^2)\beta x + (12c_0 - 100\beta\psi + 15\phi^2)\gamma y = 0.$$

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Non-Monoidal Involutions having a Congruence of Invariant Conics.

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The present paper has for its purpose the derivation of all the birational involutorial point transformations of space which have the following properties:

a) each transforms every conic of a linear congruence into itself,
b) the transformations cannot be reduced birationally to the monoidal type.

Every conic of a linear congruence, that is, a ∞^2 system such that through each point of space passes just one conic belonging to it, cuts an arbitrary plane in space in a group of an involution of order two which is reducible to the Jonquières, Geiser, or Bertini type, the harmonic homology being included under the Jonquières. We shall study such a linear congruence of class 1[*] (hence of the Geiser form), whose generating net consists of cubic surfaces, the basis curve of which is a C_7 of genus 5. We shall find such a system and examine its properties.

Then we shall consider those involutorial birational transformations of space under which the conics of the congruence are invariant and derive their equations. Associated with the plane of each conic is a point lying in it and uniquely determined by it, which is the vertex of the pencil of lines joining pairs of conjugate points on the conic. Thus it is the center of a quadratic involution under which the conic in the plane is invariant. The locus of this point may be a point, a curve, or a surface. We shall show in the last two cases, unless the basis curve breaks up or other peculiarities exist, that the congruence cannot be reduced to a bundle of lines, and hence that the transformations are non-monoidal.

In doing this we shall establish the fact that a linear congruence of conics is not reducible to a linear congruence of lines if every directrix curve has an even number of points on a conic of the system.

Next we shall study the forms into which the C_7 may break up, and few cases where other peculiarities exist. In each of these cases we shall derive the equations for the generating surfaces.

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[*] By the class of a congruence of conics is understood the number of conics in the congruence which have a given line for bisecant.

Finally we shall mention without discussion other linear congruences which are reducible to the Geiser type. We shall not discuss in detail the Jonquières or Bertini cases, but we shall give some criteria for determining the type to which the involution given by a linear congruence of conics can be reduced.

Some articles by Montesano * suggested this problem. He has developed all the properties of the congruence we have studied, and in parts his treatment is fuller. For instance, he discusses the complex of tangents to the conics, and derives all the elementary characteristics of the system of conics. His treatment, however, is purely synthetic, while much of our problem has been to develop the analytic point of view. Only one paragraph in his first paper is devoted to the transformation upon which we have centered our attention.

I. *Equations of the conics of the congruence.*

1. Consider a bundle of planes and a projective net of quadrics R. Let $O \equiv (0, 0, 0, 1)$ be chosen as the vertex of the bundle of planes. Then any plane of the system has an equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 = 0. \tag{1}$$

If $H_i = 0$ is the equation of the quadric corresponding to $x_i = 0$, then the net of quadrics is given by

$$a_1H_1 + a_2H_2 + a_3H_3 = 0. \tag{2}$$

Each plane meets the corresponding quadric surface in a conic. Through an arbitrary point (y) passes a single conic of the system $\sum a_ix_i = 0$, $\sum a_iH_i = 0$ the equations of which are given by

$$|\, x\, y\, H(y) \,| = 0, \quad |\, H(x) y\, H(y) \,| = 0. \tag{3}$$

Since each is uniquely fixed when the point (y) is given, the system $x \sim H$

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* D. Montesano: "Su di un sistema lineare di coniche nello spazio," *Torino Atti,* Vol. 27 (1892), pp. 660-690; "Su le congruenze lineari di coniche nello spazio," *Istituto Lombardo Rendiconti,* Ser. 2, Vol. 26 (1893), pp. 589-604; "Su i varii tipi di congruenze lineari di coniche dello spazio; Note I, II," *Napoli Rendiconti,* Ser. 3, Vol. 1 (1895), pp. 93-110, 155-181. A number of later memoirs by him and by others are concerned with the classification of various systems of conics, but do not discuss the birational transformation.

defines a linear congruence of conics. Denote this system of conics by $\sum$ and a conic of the system by γ.

When the plane describes a pencil, the corresponding quadric describes a pencil projective with it, and the conic of intersection generates a cubic surface. Thus $x_1 = px_2$, $H_1 = pH_2$ generates $x_1H_2 - x_2H_1 = 0$. There are three such cubic surfaces forming a net Ξ,

$$(4) \qquad F_i = x_jH_k - x_kH_j \qquad (i,\ j,\ k = 1,\ 2,\ 3).$$

Any cubic surface that can be generated by conics of the system can be expressed linearly in terms of these three. Equation (3a) may now be written

$$(5) \qquad F_1(y)x_1 + F_2(y)x_2 + F_3(y)x_3 = 0.$$

2. From the general condition $x_i = kH_i$ and (4) the curve of intersection of any two cubics $F_i = 0$, $F_k = 0$, lies on the third unless $x_m = 0$, $H_m = 0$, which lies on both. That is, any two cubic surfaces of the system intersect in a variable conic, and the residual is a basis curve lying on all the $F_i = 0$ of the system. Since the complete curve of intersection is of order 9, the basis curve is therefore of order 7. Its genus is found to be 5. It will be denoted by C_7. Since the quadrics of R do not pass through O, the variable conic does not pass through O, yet each surface of the system Ξ does. Hence C_7 passes through O. Any plane through O meets it in 6 points not at O, and this plane also contains the conic associated with it. These six points then must lie on the conic.*

The curve C_7 does not admit a quadrisecant, but its ∞^1 trisecants form a ruled surface θ of order 15 on which C_7 is five-fold. It follows that the surfaces of Ξ passing through any point of a trisecant t of C_7 contain t. Then the variable base of the pencil which these surfaces form is composed of t and a second trisecant t' of C_7 situated in the plane Ot. The trisecants of C_7 then, are distributed in pairs in such a way that the trisecants of one pair compose a degenerate conic of the system $\sum$ and the plane which they determine belongs to O. Any line r of the bundle (O) meets both components of five degenerate conics of $\sum$. These degenerate conics lie in planes of the pencil (r) of $F : C_7r$ of Ξ and therefore their planes envelop at O a cone Δ of class 5 and genus 6.

Consider the locus of points common to the pairs of trisecants of C_7

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* Every C_7 of genus 5 has just one point O having the property mentioned. R. Sturm: *Synthetische Untersuchungen über Flächen dritter Ordnung,* Leipzig 1867. See p. 229.

which form the degenerate conics of the system. This is a double curve on θ of order 10 and genus 3. It will be denoted by H_{10}.

3. The surfaces $F: C_7$ of the net Ξ cut an arbitrary plane α of space in a net ζ of cubics having in common the points $P_1 \cdots P_7$ in which α cuts C_7. Therefore all the conics of the system $\sum$, variable base of a pencil of Ξ, have for their section with α a pair of points which with $P_1 \cdots P_7$ form the base of a pencil of ζ and which form an involution I_α of the eighth order and first class having for fundamental points the points $P_1 \cdots P_7$ as triple points. This is the Geiser involution. The curve of invariant points is a C_6 with the seven fundamental points as double points. This curve which is the locus of the points of contact of the conics of $\sum$ with the plane α evidently contains the points of section of the plane α with H_{10}.

Any space C_7 of genus 5 can be used to obtain a system like the one described in the preceding paragraphs.

It can be shown that through any conic of the system $\sum$ can be passed ∞^4 quadric surfaces, any one of which meets C_7 in 8 points not on the conic. Through these 8 points can be passed a net of quadrics, which being made projective with the given bundle of planes, generates the original system $\sum$ of ∞^2 conics. Thus the C_7 contains an involution I_8 with 4 degrees of freedom.

The Geiser involution in the plane can be generated by the lines of the plane and a projective net of conics.

4. The polar plane of $O \equiv (0, 0, 0, 1)$ as to the quadric belonging to a point (y), that is, $H \equiv | H(x) y H(y) | = 0$ is simply $\partial H/\partial x_4 = 0$. Thus the polar of O as to γ, the conic determined by (y) is the line o whose equations are

$$\sum F_i(y) x_i = 0, \ \sum F_i(y) \partial H_i(x)/\partial x_4 = 0.$$

If each H_i were replaced by $H_i + k\sigma x_i$ where $\sigma = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$, it would give the same line.

If (y) varies, all the polar planes of the system pass through the fixed point O' conjugate to O in the polarity as to the net of quadrics, where O' is given by

$$\partial H_1(x)/\partial x_4 = 0, \ \partial H_2(x)/\partial x_4 = 0, \ \partial H_3(x)/\partial x_4 = 0.$$

The projective systems

$$(6) \qquad [\partial H_1(x)/\partial x_4]/x_1 = [\partial H_2(x)/\partial x_4]/x_2 = [\partial H_3(x)/\partial x_4]/x_3 = k$$

define a cubic curve C_3 for which every line o is a bisecant.

Now consider the polar quadric of O as to any cubic surface $F_i(x) = 0$ of the system. Since $F_1(x) = x_2 H_3 - x_3 H_2$, $\partial F_1(x)/\partial x_4 = x_2\, \partial H_3(x)/\partial x_4 - x_3\, \partial H_2(x)/\partial x_4$, we may say: given any point (y), it uniquely determines a conic γ of the system. This conic lies on a cubic F of the system, and the residual section of its plane is a line r passing through O. The polar quadric of the F is therefore the quadric determined by the C_3 and this line r. Thus the net of first polars of the system is exactly the net determined by the cubic curve C_3. This is seen at once from the system of equations (6).

Let P be the point (y). There is an F of the net which contains the line OP. Its equation is given by

$$| F(x)\ \partial F/\partial y_4\ \partial_2 F/\partial y_4{}^2 | = 0.$$

If P is on C_3, the polar quadric is the cone projecting C_7 from O. The cubic surface has a double point at O, and the quadric cone is the tangent cone of the cubic surface at the node. The surface contains 6 lines through O, the tangent at O and the 5 trisecants of C_7 at O. C_3 and C_7 always have the same tangent line at O.

II. *The transformations.*

5. No birational correspondence exists between two systems in space in which the conics of the system $\sum$ are related to the lines of a bundle. This fact will be established later. There do exist, however, birational correspondences in space in which the correspondence is between two systems of conics $\sum$. Among these correspondences we shall consider the involutorial ones in which each conic is conjugate to itself.

A correspondence of this kind determines on an arbitrary conic γ of the system $\sum$ an ordinary quadratic involution. Three cases arise: either the center of the involution is the fixed point O, or as γ varies in $\sum$ it describes in a plane ω of the bundle (O), a curve $C_\mu : O^{\mu-1}$ in such a way that every point G of this curve is related to all the conics γ which have the line OG as a bisecant; or as the conics vary, the point G describes a surface S_m.

6. Case I. Take any point (y). It defines a conic γ of the congruence uniquely, and it lies in a plane through O. The line $O(y)$ meets γ in a second point (y'). When (y) takes all positions in space, (y') will also. It groups the points in pairs, (y), (y') in such a way that the line joining any point (y) to its conjugate (y') always passes through the fixed point O. The

involution is therefore monoidal. It will be denoted by I_2. Under I_2 every γ of the congruence is transformed into itself.

To obtain the equations of the transformation, let (y) be the point determining the conic and (y') a variable point on the line $O(y)$. The coördinates of (y') are then given by

$$(7) \qquad y_i' = sy_i, \ (i = 1, 2, 3), \ y_4' = r + sy_4.$$

If (y') is a point on the conic its coördinates must satisfy equations (3). The first of these is satisfied identically since the line lies in the plane.

Substitute (7) in (3*b*). Hereafter we shall use the following notation for the quadric surfaces.

$$\begin{aligned}
H_1(x) &\equiv u_0x_4^2 + u_1x_4 + u_2 = 0 \text{ where } u_0 \text{ is a constant,}\\
&u_1 \equiv \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3,\\
&u_2 \equiv a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + b_1x_1x_2 + b_2x_1x_3 + b_3x_2x_3.\\
H_2(x) &\equiv v_0x_4^2 + v_1x_4 + v_2 = 0 \text{ where } v_0 \text{ is a constant,}\\
&v_1 \equiv \beta_1x_1 + \beta_2x_2 + \beta_3x_3,\\
&v_2 \equiv c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + d_1x_1x_2 + d_2x_1x_3 + d_3x_2x_3.\\
H_3(x) &\equiv w_0x_4^2 + w_1x_4 + w_2 = 0 \text{ where } w_0 \text{ is a constant,}\\
&w_1 \equiv \gamma_1x_1 + \gamma_2x_2 + \gamma_3x_3,\\
&w_2 \equiv f_1x_1^2 + f_2x_2^2 + f_3x_3^2 + g_1x_1x_2 + g_2x_1x_3 + g_3x_2x_3.
\end{aligned}$$

Then

$$\begin{aligned}
H_1(y') &= s^2H_1(y) + rs(2u_0y_4 + u_1) + u_0r^2,\\
H_2(y') &= s^2H_2(y) + rs(2v_0y_4 + v_1) + v_0r^2,\\
H_3(y') &= s^2H_3(y) + rs(2w_0y_4 + w_1) + w_0r^2.
\end{aligned}$$

After some manipulation we get

$$\begin{aligned}
r &= -2y_4^2 \ |u_0 \ y_2 \ w_1| - 2y_4 \ |u_0 \ y_2 \ w_2| - |u_1 \ y_2 \ w_2|,\\
s &= y_4 \ |u_0 \ y_2 \ w_1| + |u_0 \ y_2 \ w_2|,
\end{aligned}$$

in which the determinants are of order 4, 3, or 2. If these are substituted in (7) the coördinates of (y') become

$$\begin{aligned}
\rho y_i' &= y_i \qquad (i = 1, 2, 3),\\
\rho y_4' &= -[y_4 \ |u_0y_2w_2| + 2 \ |u_1y_2w_2| \ ]/[y_4 \ |u_0y_2w_1| + |u_0y_2w_2| \ ].
\end{aligned}$$

The transformation is of order 4. The surface of invariant points is

$$y_4^2 \ |u_0y_2w_1| + 2y_4 \ |u_0y_2w_2| + 2 \ |u_1y_2w_2| = 0.$$

This surface can be used to define the transformation for the image of any point P lies on OP and is the harmonic conjugate of P as to the two residual points in which OP meets this surface. This surface has O for a double point, contains C_7 simply and the trisecants $t_1 \cdots t_5$.

The image of O is $y_4 \mid u_0\, y_2\, w_1 \mid + \mid u_0\, y_2\, w_2 \mid = 0$. The basis curve is defined by

$$y_4 \mid u_0 y_2 w_2 \mid + 2 \mid u_1 y_2 w_2 \mid = 0, \;\; y_4 \mid u_0 y_2 w_1 \mid + \mid u_0 y_2 w_2 \mid = 0$$

which projects from the cone

$$K_6 = \mid u_0 y_2 w_2 \mid^2 - 2 \mid u_0 y_2 w_1 \mid \; \mid u_1 y_2 w_2 \mid = 0.$$

The Jacobian is of order 12 and consists of the image of O, taken twice, and this K_6. The lines $t_1 \cdots t_5$ are fundamental and are those common to the three cones

$$\mid u_0 y_2 w_1 \mid = 0, \;\; \mid u_0 y_2 w_2 \mid = 0, \;\; \mid u_1 y_2 w_2 \mid = 0.$$

7. Case 2. Each conic of the system $\sum$ is conjugate to itself in an ordinary quadratic involution, the center of which belongs to a plane rational curve $C_\mu : O^{\mu-1}$. Any plane through O meets the curve in one point not at O. Let (ξ) be the point on C_μ in the plane of this conic. The line joining (y) to (ξ) meets the conic in a second point (y'). The relations between (y) and (y') define the involution desired.

The coördinates of any point on the line joining (y) to (ξ) are given by $r\xi + sy$. If this point is to be (y') its coördinates must satisfy equations (3). Substitute in (3b) and expand. Denote by $H(y, C)$ the polar of $H(y)$ as to the point (ξ) on C_μ and $H(C)$ the result of the substitution of the point on C_μ itself. Then the quadric becomes

$$s \mid H(y, C)\, y\, H(y) \mid + r \mid H(C)\, y\, H(y) \mid = 0.$$

From this we get

$$s = \sum H_k(C) F_k(y), \;\; r = - \sum H_k(y, C) F_k(y).^*$$

The coördinates of y' are therefore

$$(8) \quad \rho y_i' = y_i \sum H_k(C) F_k(y) - \xi_i \sum H_k(y, C) F_k(y) \quad (i = 1 \cdots 4),$$

where ξ_i is still to be determined in terms of (y).

Let the equations of C_μ in parametric form be

$$(9) \qquad \rho\xi_i = R(\lambda)(\lambda - a_i) \;\; (i = 1, 2, 3), \;\; \rho\xi_4 = f(\lambda)$$

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* Here and following the summation is taken from $k = 1$ to $k = 3$.

in which $R(\lambda)$ is a polynomial of order $\mu - 1$ in λ. The curve lies in the plane

$$(a_2 - a_3)x_1 + (a_3 - a_1)x_2 + (a_1 - a_2)x_3 = 0 \tag{10}$$

and contains $O \equiv (0, 0, 0, 1)$ as a point of multiplicity $\mu - 1$, the tangents to which are the $\mu - 1$ roots of $R(\lambda) = 0$. The plane associated with a point (y) meets the curve in the point having for its λ the value determined by

$$|(\lambda - a)y\,H(y)| = 0. \tag{11}$$

From this it follows that $\lambda = \sum a_i F_i(y) / \sum F_i(y)$ and

$$\begin{aligned} \rho\xi_i &= R[\textstyle\sum a_k F_k(y),\ \sum F_k(y)]\ [\sum a_k F_k(y) - a_i \sum F_k(y)] \\ &\qquad\qquad (i = 1, 2, 3), \\ \rho\xi_4 &= f[\textstyle\sum a_k F_k(y),\ \sum F_k(y)]. \end{aligned} \tag{12}$$

On substituting these values in (8), the equations of the transformation are

$$\begin{aligned} &\rho y_i' = y_i \textstyle\sum H_k(C) F_k(y)\ - R[\sum a_k F_k(y), \\ &\textstyle\sum F_k(y)]\ [\sum a_k F_k(y) - a_i \sum F_k(y)] \sum H_k(y, C) F_k(y) \quad (i = 1, 2, 3), \\ &\rho y_4 = y_4 \textstyle\sum H_k(C) F_k(y) - \sum H_k(y, C) F_k(y)\ \ f[\sum a_k F_k(y), \sum F_k(y)]. \end{aligned} \tag{13}$$

$R(\lambda)(\lambda - a_i)$ and $f(\lambda)$ are of order μ in F_k and F_k are of order 3 in (y). Therefore they are of order 3μ in (y). The expression $H_k(y, C)$ is of order $3\mu + 1$, since it is linear in C and in (y). $H_k(C)$ is of order 6μ in (y). The transformation (13) is therefore of order $6\mu + 4$.

All of the cubic surfaces $F_k(y) = 0$ pass through C_7. Since these appear to order μ in the parametic equations of the curve C_μ, it follows that the basis curve C_7 is of multiplicity $2\mu + 1$ on all the surfaces of the web (13).

Now suppose that the point (y) is chosen on C_μ. The plane and the quadric are determined as in the general case, and since the conic determined by their intersection passes through the given point (y), for this point every point of the conic is a point (y'). Thus C_μ is a fundamental curve, every point of which goes into a conic. Hence C_μ is a double curve on every surface of the web (13).

The plane of C_μ contains a conic of the system. Its equation is (10). This is identical with $(3a')$. Hence for a point (y) on this conic

$$a_2 - a_3 = F_1(y), \quad a_3 - a_1 = F_2(y), \quad a_1 - a_2 = F_3(y).$$

Then $\lambda = 0/0$, and $\lambda - a_i = 0$. Therefore the conic in the plane of C_μ is a 2μ-fold fundamental curve on the web (13). Every point of the conic has the entire conic for image.

Finally, the trisecants of C_7 which meet C_μ apart from O are all fundamental, since C_7 is of multiplicity $2\mu+1$ and C_μ is double. Since each trisecant meets C_7 in three points we thus have $3(2\mu+1)+2$ points on each. Hence these lines lie on every surface of the web and are fundamental. The image of any point is the entire line passing through it. The order of the ruled surface θ was found to be 15. The order of C_μ is μ. Therefore the surface meets the curve in 15μ points. But O is 5-fold on θ_{15} and of multiplicity $\mu-1$ on C_μ, hence the number of remaining intersections is $15\mu-5(\mu-1)=10\mu+5$. Therefore there are $5(2\mu+1)$ simple base lines, trisecants of C_7, which meet C_μ on every surface of the web (13).

The images of C_7 and of C_μ together constitute the Jacobian of the web. The order of the Jacobian is $4(6\mu+3)$; it contains C_7 to multiplicity $4(2\mu+1)-1$ and C_μ to multiplicity 7, the conic in the plane of C_μ to multiplicity $4\cdot 2\mu$ and each trisecant to multiplicity 4. Hence we have

$$S_1 \sim S_{6\mu+4}: C_7^{2\mu+1}\, C_\mu^2\, \gamma^{2\mu}c_\mu\, t_1 \cdots t_{5(2\mu+1)}.$$
$$C_7 \sim \Gamma_{9(2\mu+1)}: C_7^{6\mu+2}\, C_\mu^6\, \gamma^{6\mu}c_\mu\, t_1^3 \cdots t^3_{5(2\mu+1)}.$$
$$C_\mu \sim M_{3(2\mu+1)}: C_7^{2\mu+1}\, C_\mu\, \gamma^{2\mu}c_\mu\, t_1 \cdots t_{5(2\mu+1)}.$$

To obtain the surface of invariant points, let $y'=y$ in (13). This becomes

$\sum H_i(y,C)F_i(y)=0$ which has the signature
$K_{3\mu+4}: C_7^{\mu+1}\, C_\mu\, \gamma c_\mu\, t_1 \cdots t_{5(2\mu+1)}\, H_{10}.$

The lines joining pairs (y), (y') of conjugate points in the involution are ∞^3. They consist of all the lines which meet C_μ. Given any line l meeting C_μ at P. Pass a plane through l and O. This plane will meet its associated quadric in a conic containing an involution of conjugate points, the center of which is P. This line in the general position will not contain more than one pair of conjugate points.

An interesting special case occurs when $\mu=1$. In this case T belongs to a special linear complex.[*]

8. Case 3. Now suppose the center of the involution under which each conic of $\sum$ remains invariant describes a surface S_m. The lines joining pairs of conjugate points PP' now form a complex. Every point of S_m is the vertex of a pencil, and the plane of our system determined by that point is the plane of the pencil. This is a very particular form of a line complex: ∞

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[*] M. Pieri: "Sulle trasformazioni birationali dello spazio inerenti a un complesso lineare speciale." *Palermo Rendiconti*, Vol. 6 (1892), pp. 234-44.

lines arranged in ∞^2 pencils, the vertices describing a surface and the planes all passing through a fixed point O. The planes and the points are in (1, 1) correspondence and in united position, that is, a plane π and its associated point P are perspective, π passes through P.

Conversely, every such complex Γ and the system of planes of $\sum$ determine an involutorial birational transformation which leaves every conic of $\sum$ invariant.

Through an arbitrary point of space passes just one γ of $\sum$, but through a point of C_7 pass ∞^1 such conics. Let P be such a point of C_7. There is one F of the net which has P for a double point. All the planes through OP cut from this surface the line OP and a conic of $\sum$ always passing through P but not through O. In each such plane is a point of S_m. The line joining P to this point meets the γ of the plane in one other point P'. The locus of P' is the image of the fundamental point P on C_7. The point P' must always lie on the F: P^2 having the point P for a double point, and every line must lie on the complex cone belonging to P. Let μ be the order of the complex. Every line of the bundle (O) is of multiplicity $\mu - 1$, hence OP is a line of multiplicity $\mu - 1$ on the cone K_μ of P. It is simple line of F: P^2, hence the residual curve of intersection is of order $3\mu - (\mu - 1) = 2\mu + 1$. Since P is of multiplicity μ on K_μ and 2 on F, the complete curve has a point of multiplicity 2μ at P. But OP counts for $\mu - 1$, hence the residual curve of order $2\mu + 1$ has $\mu + 1$ branches through P.

9. In the involution I there is now a surface K of invariant or coincident points. It is the locus of points of contact of tangents from points of S_m to the associated conic and contains $C_7^{\mu+1}$. Any conic of $\sum$ has just 2 points on K, not on C_7, namely the points of contact. Let x be the order of K. A conic of $\sum$ meets C_7 in six points, and each counts for $\mu + 1$ intersections with K, hence the $2x$ intersections of γ and K are made up of $6(\mu + 1) + 2$. Therefore $x = 3\mu + 4$.

10. In an arbitrary plane α of space the pairs of distinct points of the involution I are conjugate in the Geiser involution determined by the points in which the plane α meets C_7, and the lines joining such pairs of conjugate points all belong to the complex Γ; that is, they envelop a curve of class μ. We have seen that when a point describes a straight line r, the lines from O to the conjugate describe a cone of class 3. Therefore if r is in the plane α, these lines form an envelop of class 3. Hence the points in α whose conjugates also lie in α form a curve of order 3μ. Moreover, α cuts K in a curve of order $3\mu + 4$. These two curves together form the complete intersection

of z and its conjugate surface in I. Hence the order of the conjugate surface the order of the involution, is $6\mu + 4$.

11 Let r be a line passing through O. A plane through r contains an associated point of S_m. As the plane turns about r, the point describes a curve such that each position of the plane contains just one point of it, apart from points on r. Let Q be any fixed point of r. In every position of the plane connect Q with (ξ), the corresponding point of S_m in this plane. This line $Q(\xi)$ meets a pair of conjugate points on the conic of its plane, hence it is a line of the complex. The curve must then lie on the complex cone of Q and meet every generator once, hence the locus of the points of S_m is of order μ, and it therefore meets r in $\mu - 1$ points.

Consider any plane section π of S_m. To each of its points P corresponds a plane of the bundle (O) and passing through P. The lines of the pencil having P for vertex and lying in the associated plane through O all belong to the complex Γ. One of these lines lies in the given plane π. Conversely all the lines of Γ in π are in the corresponding planes.

If each of these lines of Γ in π is connected with O, the planes envelop a cone of class μ. Some of the planes may contain fundamental curves, to which the plane is associated rather than to a point. Let $\sigma_1 \cdots \sigma_l$ be such planes belonging to K_μ, and let σ_i be counted to multiplicity δ_i. Moreover let s_i be the order of the fundamental curve in σ_i. Then every line of the plane σ_i is a multiple line of Γ to multiplicity s_i. The conic of $\sum$ is invariant under ∞^1 involutions, namely, all those having a point of C_{s_i} for center. This requires that at every point in which γ meets C_{s_i} the point be fundamental, corresponding to the whole conic. Each such conic is therefore a $2s_i$-fold fundamental curve of the second kind.

An arbitrary line r meets S_m in m points. These are associated with m planes of (O) which belong to all the cones, images of all the plane sections through r. Hence the two K_μ, K'_μ, having μ^2 common tangent planes, have also the fundamental planes σ_i, hence $\mu^2 - \sum\delta_i^2 = m$.

12. There are ∞^2 conics in $\sum$. Among them are ∞^1, each passing through the point of S_m associated with its plane. Such conics are images of the associated points of S_m lying on them. The locus of such points is a fundamental curve, double for the surfaces, images of the planes of space in I, since the image of each point is a curve of the second order. Let ν be the order of this curve C_ν. It is of the first kind. It meets every fundamental γ of the second kind in the $2s_i$ points in which γ is met by the curve C_{s_i}, locus of associated points of its plane. Therefore the planes of the conics containing

points of S_m on the conic envelop a cone containing the singular planes σ_i to multiplicity $2s_i$. The class of this cone is $2\mu + 1$.

From this cone and that of class 5, the planes of which contain composite conics, we obtain the result that there are $5(2\mu + 1)$ planes through O which contain composite conics—that is, a pair of lines, and the associated point of S_m lies on one of them. There are consequently $5(2\mu + 1)$ lines t_i such that in I any point of any one of them has the entire line through it for image.

13. This completes the list of fundamental lines in the involution, hence we may write

$$S_1 \sim S_{6\mu+4} : C_7^{2\mu+1}\, C_\nu^2 : \gamma_1^{2s_1} \cdots \gamma_h^{2s_h}\, t_1 \cdots t_{5(2\mu+1)}.$$

Since the image of an arbitrary straight line is a curve of order $6\mu + 4$, this means that the variable curve of any two $S_{6\mu+4}$ of the web has all its intersections in the basis or fundamental curves, that is,

$$(6\mu + 4)^2 - 7(2\mu + 1)^2 - 4\nu - 8\sum \delta^2 - 5(2\mu + 1) = 6\mu + 4, \text{ hence}$$
$$\nu = 2\mu^2 + \mu - 2\sum \delta^2 = 2m + \mu.$$
$$C_\nu \sim \Gamma_{3(2\mu+1)} : C_7^{2\mu+1} \quad C_\nu\, \gamma_1^{2s_1} \cdots \gamma_h^{2s_h} \quad t_1 \cdots t_{5(2\mu+1)},$$
$$C_7 \sim M_{9(2\mu+1)} : C_7^{6\mu+2} \quad C_\nu^6\, \gamma_1^{6s_1} \cdots \gamma_h^{6s_h} \quad t_1^3 \cdots t^3_{5(2\mu+1)}.$$

The other fundamental curves are all of the second kind, each point having for image the entire curve through it. The Jacobian of the web $S_{6\mu+4}$ is of order $24\mu + 12$. It consists of $\Gamma_{3(2\mu+1)}$ and $M_{9(2\mu+1)}$. The surface of invariant points is

$$K_{3\mu+4} : C_7^{\mu+1}\, C_\nu\, \gamma_1^{s_1} \cdots \gamma_h^{s_h}\, t_1 \cdots t_{5(2\mu+1)}.$$

All the lines t_i are trisecants of C_7. The surface $K_{3\mu+4}$ of invariant points also contain H_{10}, double on θ_{15} for the plane of a pair intersecting at any point of H_{10} passes through O, and every γ of $\sum$ remains invariant under I.

The numbers m, μ, s_i, h, δ_i are connected by the single relation

$$\mu^2 - \sum \delta_i^2 = m, \text{ hence } \mu \geqq m.$$

14. Analytically, proceeding as in Case 2 we get as analogous to (8)

$$(14) \qquad \rho y_i' = y_i \sum H_k(S_m) F_k(y) - \xi_i \sum H_k(y, S_m) F_k(y) \qquad (i = 1, 2, 3, 4).$$

where ξ_i denotes a point on the surface S_m still to be determined in terms of y; $H(y, S_m)$ the polar of $H(y)$ as to the point (ξ) on S_m; $H(S_m)$ the result of the substitution of (ξ) itself.

Let us consider the following cases:

(e) Let the surface be a plane not passing through O. Let the equation of S_m be $x_4 = 0$. The coordinates of any point on S_m are $\xi \equiv (\xi_1, \xi_2, \xi_3, 0)$. Transform (ξ) into (ξ') by means of

$$\xi'_i = a_i\xi_i, \qquad (i = 1, 2, 3), \qquad \xi'_4 = 0.$$

where $a_i \neq 0$ and all are distinct. This establishes a (1, 1) correspondence between the points of S_m and the planes passing through these points. Hence if the plane of the bundle is so taken as to contain one of these lines, the required conditions are satisfied. The plane through O, (ξ), (ξ') is

$$| x\,\xi\,a\xi | = 0,$$

which becomes

$$(a_3 - a_2)\xi_3\xi_2x_1 + (a_1 - a_3)\xi_1\xi_3x_2 + (a_2 - a_1)\xi_2\xi_1x_3 = 0.$$

Since this is a plane of the bundle it is identical with (5). From this it follows that

$$(15) \quad F_1(y) = \xi_3\xi_2(a_3 - a_2), \quad F_2(y) = \xi_1\xi_3(a_1 - a_3), \quad F_3(y) = \xi_2\xi_1(a_2$$

Upon solving, we get

$$(16) \quad \rho\xi_1 = F_3F_2(a_3 - a_2), \quad \rho\xi_2 = F_1F_3(a_1 - a_3), \quad \rho\xi_3 = F_2F_1(a_2 - a_1).$$

When these values are substituted in (14) the equations of the transformation are given by

$$(17) \quad \begin{aligned} \rho y_1' &= y_1 \sum H_k(S_m)F_k(y) - (a_3 - a_2)F_3F_2 \sum H_k(y, S_m)F_k(y), \\ \rho y_2' &= y_2 \sum H_k(S_m)F_k(y) - (a_1 - a_3)F_1F_3 \sum H_k(y, S_m)F_k(y), \\ \rho y_3' &= y_3 \sum H_k(S_m)F_k(y) - (a_2 - a_1)F_2F_1 \sum H_k(y, S_m)F_k(y), \\ \rho y_4' &= y_4 \sum H_k(S_m)F_k(y). \end{aligned}$$

Since (ξ) is of degree 6 in (y), $H(y, S_m)$ of degree 7 in (y), and $H(S_m)$ of degree 12, the transformation is of order 16. Therefore $6\mu + 4 = 16$ and $\mu = 2$.

The vertices of the triangle of reference in the plane $x_4 = 0$ are singular points. The image of each of these points is the whole pencil of lines having the point for vertex. Therefore we have three bundles of lines, apart from O that are singular lines of the complex.

All the cubic surfaces $F_k = 0$ pass through C_7. Since these appear to order 4 on $H(S_m)$ and to order 2 on $H_k(y, S_m)$, it follows that C_7 is of multiplicity 5 on the image.

5. Consider a pencil of planes through the line $O(\xi)$. In $x_4 = 0$

each line of the pencil has a point associated with it, and for one position of the line the associated point is (ξ) itself. Hence the locus is a conic. Now take any point P in space. The line OP meets $x_4 = 0$ in (ξ). The complex cone belonging to P contains the conic in $x_4 = 0$. This again proves that the complex is quadratic, and that the bundle (O) belongs to it simply.

The point (ξ) is associated with the line

$$(a_2 - a_3)\xi_2\xi_3 x_1 + (a_3 - a_1)\xi_3\xi_1 x_2 + (a_1 - a_2)\xi_1\xi_2 x_3 = 0$$

in $x_4 = 0$, and hence with the plane of (O) having the same equation. The quadric of the net corresponding to the latter has the equation

$$(a_2 - a_3)\xi_2\xi_3 H_1(x) + (a_3 - a_1)\xi_3\xi_1 H_2(x) + (a_1 - a_2)\xi_1\xi_2 H_3(x) = 0.$$

The point (ξ) lies on the associated conic of $\sum$ in its plane if it lies on the quadric. Therefore

$$(a_2 - a_3)\xi_2\xi_3 H_1(\xi) + (a_3 - a_1)\xi_3\xi_1 H_2(\xi) + (a_1 - a_2)\xi_1\xi_2 H_3(x) = 0$$

is the equation of the C_4 in $x_4 = 0$, every point of which has the whole conic of $\sum$ passing through it for conjugate in I. Hence C_4 is double on the conjugate of every plane in I. This C_4 passes through the fundamental points in $x_4 = 0$ simply.

The planes containing the sides of the triangle of reference in the x_4-plane have a fundamental conic double.

Finally, for certain values of (ξ), the point (ξ) lies on the associated conic of $\sum$, and the latter is composite. The first condition requires that (ξ) lie on C_4: the second that the plane containing (ξ) shall be tangent to the associated quadric. This gives rise to a determinant of order 5, each element of which is quadratic in (ξ), and therefore represents a C_{10} which meets C_4 in 40 points. Each fundamental point counts for 5 intersections, and therefore there are 25 fundamental trisecants of C_7.

We have then

$$S_1 \sim S_{16} : C_7^5\, C_4^2\, \gamma_1^2\, \gamma_2^2\, \gamma_3^2\, t_1 \cdot \cdot \cdot t_{25},$$
$$C_4 \sim \Gamma_{15} : C_7^5\, C_4\, \gamma_1^2\, \gamma_2^2\, \gamma_3^2\, t_1 \cdot \cdot \cdot t_{25},$$
$$C_7 \sim M_{45} : C_7^{14}\, C_4^6\, \gamma_1^6\, \gamma_2^6\, \gamma_3^6\, t_1^3 \cdot \cdot \cdot t^3_{25}.$$

The surface of invariant points is obtained by letting $y = \gamma'$ in (17). This gives

$$K_{10} = C_7^3\, C_4\, \gamma_1\, \gamma_2\, \gamma_3\, t_1 \cdot \cdot \cdot t_{25}\, H_{10}.$$

16. (b) Let the surface S_m be a plane passing through O. Let its equation be $x_1 = 0$. Then in the correspondence between the lines passing

through points on S, and the planes of the bundle, it is necessary that the lines themselves pass through O. This correspondence can be established as follows. Let $\sum a_i x_i = 0$ be any fixed plane through O. Then

$$\lambda(a_2x_2 + a_3x_3) + \mu x_1 = 0$$

is a pencil of planes, all passing through $x_1 = 0$, $\sum a_i x_i = 0$. Make these planes projective with points on the line, in such a way that $x_1 = 0$ is associated with O for every set of a_i, and the plane $a_2x_2 + a_3x_3 = 0$ is associated with the point $O' \equiv (0, a_3, -a_2, 0)$. We get then

$$\lambda(0, a_3, -a_2, 0) + \mu(0, 0, 0, 1) = 0.$$

If $\lambda = \mu$, $x_1 + a_2x_2 + a_3x_3 = 0$ is associated with $\xi \equiv (0, a_3, -a_2, 1)$. This is a plane of the bundle of (3*a'*). We get then

$$\rho\xi_1 = 0, \quad \rho\xi_2 = F_3(y), \quad \rho\xi_3 = -F_2(y), \quad \rho\xi_4 = F_1(y).$$

Substituting these values in (14) which applies to this case also, gives as the equations of the transformation

$$(15) \quad \begin{aligned} \rho y'_1 &= y_1 \sum H_k(S_m)F_k(y), \\ \rho y'_2 &= y_2 \sum H_k(S_m)F_k(y) - F_3(y) \sum H_k(y, S_m)F_k(y), \\ \rho y'_3 &= y_3 \sum H_k(S_m)F_k(y) + F_2(y) \sum H_k(y, S_m)F_k(y), \\ \rho y'_4 &= y_4 \sum H_k(S_m)F_k(y) - F_1(y) \sum H_k(y, S_m)F_k(y). \end{aligned}$$

This transformation is of order 10 and $\mu = 1$. The multiplicity of C_3 is 3 on the image. The point (ξ) is associated with the plane

$$\xi_4x_1 - \xi_3x_2 + \xi_2x_3 = 0.$$

The equation of C_3 is then,

$$\xi_4H_1(\xi) - \xi_3H_2(\xi) + \xi_2H_3(\xi) = 0.$$

We have then

$$S_1 \sim S_{10} : C_7^3 C_3^2 t_1 \cdots t_{15},$$
$$C_3 \sim \Gamma_9 : C_7^3 C_3 t_1 \cdots t_{15}.$$
$$C_7 \sim M_{27} : C_7^8 C_3^6 t_1^3 \cdots t^3_{15}.$$

The surface of invariant points is

$$\sum H_k(y, S_m)F_k(y) = 0 \text{ of form } C_7^2 C_3 t_1 \cdots t_{15} H_{10}.^*$$

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[*] This is included as a particular case of the transformation belonging to a linear complex. D. Montesano: "Su le trasformazioni involutorie dello spazio che determinano un complesso lineare di rette." *Academia dei Lincei Rendiconti*, Ser. 4 Vol 4 (1888), pp. 107-16; 277-86.

17. (c) Let S_m be a quadric passing through O. One method of relating the planes of the bundle with the points of S_m is as follows. Let $S_m \equiv x_1x_4 - x_2x_3 = 0$. Let (ξ) be a point on S_m. Project (ξ) from 0 on the plane $x_4 = 0$. The coordinates of (ξ') are $\xi_1, \xi_2, \xi_3, 0$. Make a transformation in the plane $x_4 = 0$ such that $\xi_i'' = a_i\xi_i'$, a_i distinct and $\neq 0$. Take as the plane of the bundle, the plane containing O, (ξ'), (ξ''). This has for its equation

$$(a_3 - a_2)\xi_3\xi_2x_1 + (a_1 - a_3)\xi_1\xi_3x_2 + (a_2 - a_1)\xi_2\xi_1x_3 = 0.$$

From this we get equations (15). These and the one obtained from the fact that (ξ) lies on the surface give

$$\rho\xi_1 = F_3F_2(a_3 - a_2), \quad \rho\xi_2 = F_1F_3(a_1 - a_3), \quad \rho\xi_3 = F_2F_1(a_2 - a_1),$$
$$\rho\xi_4 = [(a_1 - a_3)(a_2 - a_1)/(a_3 - a_2)]\, F_1^2.$$

18. Another method is as follows: take the surface as before. It contains the line l whose equations are $x_1 = x_2$, $x_3 = x_4$. It does not contain the line l', the equations of which are $x_1 = 0$, $x_4 = 0$. The coordinates of points on l' will be denoted by (z). Through (ξ) passes only one transversal of these two lines.

Relate (ξ) to the transversal passing through it, and take as the plane of the bundle, the plane through O containing this transversal. To set up this relation, take the plane through (ξ) and two points on l. Its equation is

$$(24) + (41) + (13) + (32) = 0, \qquad [(ij) = x_i\xi_j - x_j\xi_i].$$

This will cut l' in the point (z) whose coordinates are $z_1 = z_4 = 0$ and

$$z_2 = \xi_1 - \xi_2, \qquad z_3 = \xi_4 - \xi_3.$$

Let the plane of the bundle be the plane through O, (ξ), (z). Its equation reduces to

$$(\xi_2\xi_4 - \xi_1\xi_3)x_1 + (-\xi_1\xi_4 + \xi_1\xi_3)x_2 + (\xi_1^2 - \xi_1\xi_2)x_3 = 0.$$

This gives

$$F_1 = \xi_2\xi_4 - \xi_1\xi_3, \quad F_2 = -\xi_1\xi_4 + \xi_1\xi_3, \quad F_3 = \xi_1^2 - \xi_1\xi_2.$$

and since (ξ) is on the surface, $\xi_4 = \xi_2\xi_3/\xi_1$. By solving these four equations, we get

$$\rho\xi_1 = F_2F_3, \quad \rho\xi_2 = -F_3(F_1 + F_2), \quad \rho\xi_3 = F_2^2, \quad \rho\xi_4 = -F_2(F_1 + F_2).$$

This is the same result as for the previous method if

$$a_3 - a_2 = 1, \quad a_1 - a_3 = -(F_1 + F_2)/F_1, \quad a_2 - a_1 = F_2/F_1.$$

Substituting these results in (14), gives as the equations of the transformation

$$\begin{aligned}
\rho y_1' &= y_1 \sum H_k(S_m)F_k(y) - (a_3 - a_2)F_3F_2 \ \sum H_k(y, S_m)F_k(y),\\
\rho y_2' &= y_2 \sum H_k(S_m)F_k(y) - (a_1 - a_3)F_1F_3 \ \sum H_k(y, S_m)F_k(y),\\
\rho y_3' &= y_3 \sum H_k(S_m)F_k(y) - (a_2 - a_1)F_2F_1 \ \sum H_k(y, S_m)F_k(y),\\
\rho y_4' &= y_4 \sum H_k(S_m)F_k(y)\\
&\qquad - [a_1 - a_3)(a_2 - a_1)/(a_3 - a_2)] \ F_1^2(y) \sum H_k(y, S_m)F_k(y).
\end{aligned}$$

The transformation is of order 16 and $\mu = 2$. The image contains C_7 to multiplicity 5, 25 trisecants, C_6^2, and 2 fundamental conics double. In the second method these conics lie in the planes, one of which contains l, and the other l'. The lines of the bundle (O) are simple for through O there is one transversal of l and l' and through that transversal there is a pencil of planes to which O is therefore related.

19. (d) Let S_m be a quadric not passing through O. Using the second method, let the equation of S_m *be* $x_1x_3 - x_2^2 + x_4^2 = 0$. It contains the line l whose equations are $x_1 = x_2 - x_4$, $x_3 = x_2 + x_4$. It does not contain the line l' the equations of which are $x_1 = 0$, $x_4 = 0$. Proceeding exactly as in the previous case, we find that

$$z_2 = \xi_2 - \xi_4 - \xi_1, \qquad z_3 = \xi_3 - 2\xi_4 - \xi_1.$$

The plane of the bundle reduces to

$$(\xi_2 z_3 - \xi_3 z_2)x_1 - \xi_1 z_3 x_2 + \xi_1 z_2 x_3 = 0.$$

We get

$$\begin{gathered}
F_1 = \xi_1\xi_2 - \xi_1\xi_3 - \xi_3\xi_4 + 2\xi_2\xi_4, \quad F_2 = \xi_1\xi_3 - 2\xi_1\xi_4 - \xi_1^2,\\
F_3 = -\xi_1\xi_2 + \xi_1\xi_4 + \xi_1^2.
\end{gathered}$$

Solving these together with the relation obtained from the fact that (ξ) lies on the quadric, gives

$$\begin{aligned}
&\rho\xi_1 = F_3^2(F_2 + 2F_3), \quad \rho\xi_2 = -F_3(F_2^2 + 2F_2F_3 - F_1F_3 + F_3^2),\\
&\rho\xi_3 = (F_2 + F_3)(F_2^2 + F_2F_3 - 2F_1F_3), \quad \rho\xi_4 = -F_8^2(F_1 + F_2 + F_3).
\end{aligned}$$

Substituting these in (14) gives the equations of the transformation which are of degree 22 in y. Therefore $\mu = 3$.

20. (e) Humbert[*] discusses a remarkable complex of conics by considering the doubly infinite system of space cubic curves which pass through 5 fixed points in space, namely, the four vertices of the tetrahedron of reference and the unit point. The locus of the point of contact of those cubics of the system which touch a given plane gives a conic of the system. These conics satisfy the two equations

$$\lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda_4 t = 0, \quad \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \lambda_4 t^2 = 0.$$

The tangents to the space cubics form a complex of the sixth order. It follows that the locus of the points of contact of the tangents which can be drawn through any point P in space to the cubics is a C_7 of genus 5 which passes simply through P and the 5 fundamental points and has 6 points on every conic in any plane through P. The point P is called the pole of the C_7.

The C_7 is situated on 5 cones of the third order whose vertices are the 5 fundamental points, and all of whose generators cut C_7 in 2 variable points, that is, they are trisecants of C_7. The curves of the system $| C_7 |$ also pass through the ten points which are the intersections of the straight lines joining two fundamental points with the plane containing the other three fundamental points. These curves lie on a doubly infinite linear system of cubic surfaces. There are a triple infinity of them, one for each point in space.

By considering only one bundle of planes this system reduces to a special case of the one under discussion. The vertex of the bundle is the pole of C_7. Let the pole be the point $0 \equiv (a_1, a_2, a_3, a_4)$. On changing the notation the equations determining the conics become

$$\lambda_1(a_1x_4 - a_4x_1) + \lambda_2(a_2x_4 - a_4x_2) + \lambda_3(a_3x_4 - a_4x_3) = 0,$$
$$\lambda_1(a_1x_4^2 - a_4x_1^2) + \lambda_2(a_2x_4^2 - a_4x_2^2) + \lambda_3(a_3x_4^2 - a_4x_3^2) = 0.$$

By means of the transformation

$$x_i = a_i y_4 + y_i, \qquad (i = 1,\ 2,\ 3), \qquad x_4 = a_4 y_4$$

the equations of the conics become

$$\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = 0, \quad \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 = 0$$

where $$H_i = y_4^2(a_i a_4 - a_i^2) - 2a_i y_i y_4 - y_i^2 = 0.$$

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[*] G. Humbert: "Sur un complexe remarquable de coniques," *Journal de l'école polytechnique*, Vol. 64 (1893), pp. 123-149.

The fundamental points are transformed as follows:

$$(0, 0, 0, 1) \sim (a_1, a_2, a_3, -1),$$
$$(1, 0, 0, 0) \sim (1, 0, 0, 0),$$
$$(0, 1, 0, 0) \sim (0, 1, 0, 0),$$
$$(0, 0, 1, 0) \sim (0, 0, 1, 0),$$
$$(1, 1, 1, 1) \sim (a_4 - a_1, a_4 - a_2, a_4 - a_3, 1).$$

$F'_1, \ldots, F'_5$ are cubic cones with vertices at (0, 1, 0, 0), (0, 0, 1, 0), (1, 0, [illegible] respectively. The surface of trisecants breaks up into 5 cubic cones, and [illegible] the 10 lines joining the 5 fundamental points by two's.

[illegible]. *Reducibility of a linear congruence of conics to a linear congruence of lines.*

21. If a linear congruence of conics has a directrix which has [illegible] points on each conic of the system, this congruence is reducible to a linear congruence of lines by means of a birational transformation.

22. Case 1. If a directrix has one point on each conic of Σ, a correspondence can be established in the following manner. Relate the conics of Σ birationally to the lines of a linear congruence Σ', and a pencil of [illegible] $(A - \alpha)$ to a pencil of planes (a′). Any point P in space determines a conic γ in the plane ω. This conic passes through P, has one point D on the directrix, and corresponds to the line g' of the congruence Σ'. PD cuts one line of the pencil $(A - \alpha)$, namely the line r, determined by the point in which PD cuts the line of intersection of the planes ω and α. The line corresponds to the plane ρ' of (a′). This plane ρ' is cut by the line g' in a point which is P', the desired conjugate of P.

23. Case 2. If a directrix has three points on a conic of Σ, the correspondence is established in the following way. Relate the conics of Σ to the lines of a congruence Σ', and the pairs of an ordinary involution I of lines in a pencil $(A - \alpha)$ to the planes of a pencil (a′). For any point P in space consider the conic of Σ which corresponds to the line g' of Σ'. Through P and the three points in which the directrix cuts γ, passes a pencil of conics cutting α in two points. For only one conic γ', do these two points lie on conjugate lines r and r' of the involution I. These lines r and r' correspond to a plane ρ' of the pencil (a'). The plane ρ' is cut by g' in a point P'.

24. Case 3. If the congruence Σ has a directrix which has five poi

on each γ, since the total number of such points is six, then the congruence has a directrix with one point on γ and this reduces to Case 1. In all three cases the conics of $\sum$ are evidently related to the lines of $\sum'$ in a $(1, 1)$ correspondence.

25. Conversely, a linear congruence of conics is not reducible by means of a birational transformation in space to a linear congruence of lines if it has only one directrix, or if it has two which have four and two points respectively on each conic of $\sum$, or if it has three, each of which cuts γ in two points.

In all three cases it is not possible to relate to a conic of the congruence $\sum$ one of its points, for if it were possible, the surface which is the locus of such points would have in common with each conic of the system $\sum$, aside from the directrices, the one point related to γ. This is absurd for the points common to γ and the directrices consist of an even number (or zero) points of section. Therefore in these cases there is no birational transformation in space making the conics of $\sum$ correspond to the lines of a linear congruence $\sum'$. If this correspondence existed, to a plane of the system $\sum'$ would correspond in the other system a surface having apart from the directrices, one point on every conic of $\sum$.

26. It is true, however, that a linear congruence of conics which has an exceptional point is reducible by means of a birational transformation in space, to a linear congruence of lines. The proof of this is entirely analogous to that given for the congruence when the conics have only one point on a directrix. It is merely necessary in the argument to substitute the exceptional point of the congruence for the single point of section of the indicated directrix.

IV. *The cases for which the C_7 is composite.*[*]

27. Case 1. If each cubic surface of the net contains a line not passing through O, the C_7 consists of this line which is a trisecant, and of a C_6 of genus 3.

Let the line t be the line whose equations are $x_1 = 0$, $x_4 = 0$. The equations of the cubic surfaces are given by (5), where as before

$$H_1 = u_0 x_4^2 + u_1 x_4 + u_2 = 0, \text{ etc.}$$

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[*] Montesano enumerates these, but without discussion or equations. "Su i varii tipi di congruenze lineari di coniche dello spazio; Note II," *Napoli Rendiconti*, Ser. 3, Vol. 1 (1895).

It t is on the cubic surfaces the quadrics of the net R are of the form

$$H_1 \equiv u_0x_4^2 + u_1x_4 + (a_1x_1 + b_1x_2 + b_2x_3)x_1 = 0,$$
$$H_2 \equiv v_0x_4^2 + v_1x_4 + (c_1x_1 + d_1x_2 + d_2x_3)x_1 + (c_2x_2 + d_3x_3)x_2 = 0,$$
$$H_3 \equiv w_0x_4^2 + w_1x_4 + (f_1x_1 + g_1x_2 + g_2x_3)x_1 + (c_2x_2 + d_3x_3)x_3 = 0.$$

If $u_1 = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3$ as before, the conic $x_1 = 0$, $H_1 = 0$ passes through $X \equiv (0, -\alpha_3, \alpha_2, 0)$. Similarly the conics of the system in the planes $x_2 = 0$, $x_3 = 0$ pass through $Y \equiv (0, 0, 1, 0)$; $Y' \equiv (-\alpha_2, 0, \alpha_1, 0)$ and $Z \equiv (0, 1, 0, 0)$; $Z' \equiv (g_1, -f_1, 0, 0)$ respectively. The section of F_1 by the plane $x_4 = 0$ consists of t and a conic passing through Y' and Z'. The section of F_2 by the same plane consists of t and a conic passing through Z. These two conics have three points of intersection apart from C_7. It follows that t is a trisecant of C_6 which is therefore of genus 3.

28. Case 2. Let each cubic surface of the net contain a line passing through O. Then this line is a quadrisecant and the C_6^0 is of genus 2.

Let the equations of the line q be $x_1 = 0$, $x_2 = 0$. If F_1 and F_2 contain q, then H_1 and H_2 must contain q. The third quadric is perfectly arbitrary. These quadrics are then of the form

$$H_1 \equiv (\alpha_1x_1 + \alpha_2x_2)x_4 + (a_1x_1 + b_1x_2 + b_2x_3)x_1 + a_2x_2^2 = 0,$$
$$H_2 \equiv (\beta_1x_1 + \beta_2x_2)x_4 + (c_1x_1 + d_1x_2 + d_2x_3)x_1 + c_2x_2^2 = 0,$$
$$H_3 \equiv w_0x_4^2 + w_1x_4 + w_2 = 0.$$

The point O is simple on each cubic surface and lies on q. The residua intersection of any plane through q with any two cubic surfaces consists o two conics which have two points in common apart from q. The line q i therefore a quadrisecant and the genus of C_6 is 2.

29. Case 3. Let every cubic surface of the net contain a conic no passing through O. Let the conic C_2 have for its equations $x_1x_2 - x_3^2 = 0$, $x_4 = 0$. Then the quadrics are of the form

$$H_1 \equiv u_0x_4^2 + u_1x_4 + a_1x_1^2 + a_3x_3^2 + b_1x_1x_2 + b_2x_1x_3 = 0,$$
$$H_2 \equiv v_0x_4^2 + v_1x_4 + (a_3 + b_1)x_2^2 + c_3x_3^2 + (a_1 - c_3)x_1x_2 + b_2x_2x_3 = 0,$$
$$H_3 \equiv w_0x_4^2 + w_1x_4 + (b_2 - g_1)x_3^2 + g_1x_1x_2 + a_1x_1x_3 + (a_3 + b_1)x_2x_3 = 0.$$

The conic $H_1 = 0$, $x_1 = 0$ is tangent to the plane $x_4 = 0$ at the poin $(0, 1, 0, 0)$. The conic $H_2 = 0$, $x_2 = 0$ is tangent to the same plane at th point $(1, 0, 0, 0)$. The conic $H_3 = 0$, $x_3 = 0$ passes through the point

(0, 1, 0, 0) and (1, 0, 0, 0). Therefore C_2 has four points on C_5 which is then of genus 2.

30. Case 4. Let the cubic surfaces of the net contain a conic passing through O. Let the equations of this conic be $x_1x_4 - x_3^2 = 0$, $x_2 = 0$. Then the quadrics are

$$\begin{aligned} H_1 &\equiv u_1x_4 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + b_1x_1x_2 + b_2x_1x_3 + b_2x_2x_3 = 0, \\ H_2 &\equiv (\beta_1x_1 + \beta_2x_2)x_4 + c_2x_2^2 - b_1x_3^2 + d_1x_1x_2 + d_3x_2x_3 = 0, \\ H_3 &\equiv \alpha_3x_4^2 + [(b_2 - f_3)x_1 + \gamma_2x_2 + (\alpha_1 + a_3)x_3]x_4 + f_2x_2^2 + f_3x_3^2 \\ &\qquad + g_1x_1x_2 + g_3x_2x_3 + a_1x_1x_3 = 0. \end{aligned}$$

The residual basis curve is a C_5 which does not pass through O. This C_5 has 5 points in the plane $x_2 = 0$ which must be on C_2 since C_2 is the only curve in the plane common to all the cubic surfaces of the net. Therefore C_5 has genus 1.

31. Case 5. Let each cubic surface of the net contain a space cubic not passing through O. Let the given C_3 be the residual intersection of

$$\begin{aligned} H &\equiv x_1x_2 - x_3x_4 = 0 \text{ and} \\ H' &\equiv m_0x_4^2 + (m_1x_1 + m_2x_2 + m_3x_3)x_4 + (\mu_1x_1 + \mu_2x_2 + \mu_3x_3)x_1 = 0. \end{aligned}$$

Eliminate x_4 and remove the factor x_1 giving

$$m_0x_1x_2^2 + (m_1x_2x_1 + m_2x_2^2)x_3 + (m_3x_2 + \mu_1x_1 + \mu_2x_2)x_3^2 + \mu_3x_3^3 = 0$$

as the equation of the cone projecting C_3 from O. Eliminate x_4 between F_1 and H which gives the cone projecting the curve of intersection from O. Impose the necessary conditions that this shall contain the above cone. Do likewise for F_2 and F_3 and the quadrics become

$$\begin{aligned} H_1 &\equiv (\alpha_1x_1 + \alpha_3x_3)x_4 + a_1x_1^2 + b_2x_1x_3 = 0, \\ H_2 &\equiv [\alpha_1x_2 + (a_1 - d_1)x_3]x_4 + \alpha_3x_2^2 + d_1x_1x_2 + b_2x_2x_3 = 0, \\ H_3 &\equiv m_0x_4^2 + [m_1x_1 + m_2x_2 + (m_3 + \mu_2 + \alpha_1 - g_1)\, x_3]x_4 + \mu_1x_1^2 \\ &\qquad + b_2x_3^2 + g_1x_1x_2 + (\mu_3 + a_1)x_1x_3 + \alpha_3x_2x_3 = 0. \end{aligned}$$

The C_3 and C_4 compose the C_7. The C_3 has 5 points on C_4 which is therefore of genus 1.

32. Case 6. The C_7 may break up into a C_3 passing through O and a C_4 which have 6 points in common. The genus of C_4 is therefore 0.

Consider a C_4 which is the intersection of $Q \equiv x_1x_2 - x_3x_4 = 0$ and

$$F_1 \quad m_0x_4^2 + (m_1x_1 + m_2x_2 + m_3x_3)x_4 + n_1x_1^2 + n_2x_2^2 + n_3x_3^2 + p_1x_1x_2 + p_2x_1x_3 + p_3x_2x_3 = 0.$$

Eliminate x_4 giving as the projecting cone of C_4 from O,

$$K_1 \quad n_3x_3^4 + (p_3x_2 + p_2x_1)x_3^3 + (m_3x_1x_2 + n_1x_1^2 + n_2x_2^2 + p_1x_1x_2)x_3^2 + (m_1x_1^2x_2 + m_2x_1x_2^2)x_3 + m_0x_1^2x_2^2.$$

Now eliminate x_4 between F_1 and Q and impose the condition that this quartic shall contain K_1. Doing likewise for the other two cubic surfaces of the net we find that the quadrics of the net R have the form

$$R_1 \quad m_0x_4^2 + [\alpha_1x_1 - m_2x_2 + (n_2 + c_2 - b_1 - m_3 - p_1)x_3]x_4 + n_1x_1^2 - n_2x_2^2 - n_3x_3^2 + b_1x_1x_2 + (p_3 + d_3 - p_2)x_1x_3 - p_3x_2x_3 = 0,$$

$$R_2 \quad -m_0x_4^2 - [-m_1x_1 + (\alpha_1 + m_1 - m_2)x_2 + (a_1 + n_1 - d_1 - m_3 - p_1)x_3]x_4 - n_1x_1^2 + c_2x_2^2 - n_3x_3^2 + d_1x_1x_2 - p_2x_1x_3 + d_3x_2x_3 = 0,$$

$$R_3 \quad (\alpha_1 + m_1 - g_1)x_3x_4 + (p_3 + d_3)x_3^2 + g_1x_1x_2 + (a_1 + n_1)x_1x_3 + (n_2 + c_2)x_2x_3 = 0.$$

33. An interesting special case of (6) occurs when the C_7 breaks up into a C_4 of the second kind and three concurrent lines which are its bisecants. Let the three concurrent lines be

$$x_1 = 0, \quad x_4 = 0; \quad x_2 = 0, \quad x_4 = 0; \quad x_1 = 0, \quad x_2 = 0.$$

If we impose the condition that these lines lie on every cubic surface of the net, the quadrics have the form

$$H_1 \equiv (\alpha_1x_1 + \alpha_2x_2)x_4 + a_1x_1^2 + b_1x_1x_2 + b_2x_1x_3 = 0,$$
$$H_2 \equiv (\beta_1x_1 + \gamma_2x_2)x_4 + a_1x_1x_2 + b_2x_2x_3 + b_1x_2^2 = 0,$$
$$H_3 \equiv w_0x_4^2 + w_1x_4 + b_2x_3^2 + g_1x_1x_2 + a_1x_1x_3 + b_1x_2x_3 = 0.$$

This completes the cases, given by Montesano, where the C_7 may be composite by breaking up into two pieces, but two other cases are worthy of note. It may happen that the cubic surfaces of the net have a common point which is not exceptional for the congruence. If the C_7 does not break up and the net has no other peculiarities, this point is quadruple whose genius is therefore 1.

Let each cubic surface of the net have a double point at $P \equiv (1, 0, 0, 0)$. The quadrics are then of the form

$$H_1 \equiv u_0x_4^2 + u_1x_4 + a_2x_2^2 + a_3x_3^2 + b_1x_1x_2 + b_2x_1x_3 + b_3x_2x_3 = 0,$$
$$H_2 \equiv v_0x_4^2 + (\beta_2x_2 + \beta_3x_3)x_4 + c_2x_2^2 + c_3x_3^2 + d_3x_2x_3 = 0,$$
$$H_3 \equiv w_0x_4^2 + (\gamma_2x_2 + \gamma_3x_3)x_4 + f_2x_2^2 + g_3x_2x_3 = 0.$$

34. Now suppose each quadric of the net R passes simply through O. Then O is a double point on each cubic surface of the net Ξ and a triple point on C_7, the genus of which is reduced to 3. In this case the conics of Σ can be mapped on a linear congruence as follows:

Establish a $(1, 1)$ correspondence between the planes and the lines of the bundle (O) so that a plane and its related line are in united position. Let $\sum a_ix_i = 0$ be the equation of a plane of the bundle. Let $\sum a_i'x_i = 0$, $\sum a_ix_i = 0$ be the equations of the related line where a_i' is linear in a_i. This relation can be established in ∞^3 ways.

In the plane $\sum a_ix_1 = 0$ is a conic whose equation is $\sum a_iH_i = 0$. If the conic passes through O, the line related to the plane cuts the conic apart from O, in only one point G. To get the locus of G, solve the first and third equations for a_i giving $a_i = F_i(x)$, and substitute in the second. Therefore the locus of $F \equiv \sum a_iF_i(x)x_k = 0$, is a monoid of order 4, with vertex at O, and passing simply through C_7. These monoids form a homoloidal web and hence are suitable for a birational transformation. Two monoids have for variable intersection a conic γ. Therefore a conic of the system would go into the intersection of two planes, that is, would be mapped on a line.

By the linear congruence of conics we have been discussing there is determined on a plane in space an involution reducible to the Geiser type. The congruence had a net of generating surfaces of the third order with a C_7 of genus 5 as directrix.

35. *Other linear congruences.* There are other linear congruences which determine on an arbitrary plane in space an involution reducible to the Geiser type, but every one can be reduced to one or another of the types already discussed. Montesano[*] gives eleven of these with the transformations by which the reduction can be made.

A linear congruence of conics which determines in a plane an involution reducible to the Bertini type can be obtained by considering a pencil of cubic surfaces having in common 2 double points D, D'. The residual intersection of two surfaces of the pencil is a C_8 of genus 3 with two triple points. The plane which is tangent to a cubic surface of the pencil along DD' contains as residual a line c. Any plane through c cuts the cubic surface containing in a conic belonging to the congruence. The line c has 2 points on C_8, hence C_8 has 6 points on each conic. The class of this congruence is 4 and therefore it is of the Bertini type.

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[*] D. Montesano: "Su i varii tipi di congruenze lineari di coniche dello spazio; Note II," *Napoli Rendiconti*, Ser. 3, Vol. 1 (1895).

By a suitable non-involutorial birational transformation this congruence can be reduced to a congruence of conics of class 0. It follows that the only possible transformations for this case are reducible to the monoidal type.

VI. *Criteria for determining the various types of linear congruences of conics in space.*

36. In conclusion let us consider some criteria for determining the various types of linear congruences of conics in space. For such a congruence Σ, every surface formed by ∞^1 conics of the system cuts two arbitrary planes α and α' of space (which do not contain conics of the surface and which are not tangent to it) in two curves of the same order and same genus. These curves are respectively invariant in the two involutions which Σ determines on the planes α and α'.

Conversely, consider in an arbitrary plane α, a curve C which is conjugate to itself in an involution j_a which Σ determines on the plane α. The conics of the congruence which cut C are on a surface which cuts another arbitrary plane α' of space in a curve of the same order and genus as C, except for the cases when the two planes α and α' are tangent to the surface or contain a conic.

Now it is known that in a plane involution j_a there exists either a pencil of invariant rational curves, or a net of elliptic invariant curves, or only a pencil of elliptic curves. Then the involution is reducible to the Jonquières, Geiser or Bertini type. It is also known that in every case such an invariant curve of the pencil or of the indicated net, corresponds to itself in a rational involution.

Correspondingly, the type is known if a linear congruence of conics has a generating pencil of homoloidal surfaces with rational plane sections; or has a generating net of homoloidal surfaces with elliptic plane sections; or has a generating pencil of homoloidal surfaces with elliptic plane sections.

37. There are then three different families of linear congruences of conics. For the first kind, if the linear congruence of conics determines on an arbitrary plane in space an involution reducible to the Jonquières type,

it is generated by a pencil of Steiner surfaces or ruled surfaces. For this type the congruence can always be reduced to a bundle of lines, and our transformations to monoidal involutions.

For the congruence of conics of the other two families it is to be noted that a homoloidal surface with elliptic plane sections which contain ∞^1 conics can always be represented on a plane in such a way that the plane sections have for images either a $C_4 : P^2Q^2$ or a $C_3 : P$ having in common the other simple points (in number equal to or less than 5) so that for the congruence in question the order of the generating net or generating pencil formed by the surfaces having elliptic plane sections is equal to or less than 8.

Conversely, given a homoloidal surface S_m $(2 < m < 9)$ with elliptic plane sections, it is easy to recognize whether the S_m can be part of a net or of a pencil of surfaces generating a linear congruence of conics, formed by the surfaces having the same order and singularities as S_m.

38. Hence the only types that give rise to a birational involutorial transformation of space that cannot be reduced to the monoidal type are those for which the conics of $\sum$ meet an arbitrary plane in the groups of a Geiser involution. Even then the transformations are non-monoidal only when the conics cut every directrix curve in an even number of points.

CORNELL UNIVERSITY,
June, 1927.

Extensions of Clifford's Chain-Theorem.

By F. Morley.

propose to state more fully what is implied in the final section of my "Metric Geometry of the plane n-line," *Transactions of the American Mathematical Society*, Vol. 1 (1900), p. 115. I refer to this as M. G. let us recall Clifford's Theorem, *Works*, p. 51.*

a plane we take lines, say 1, 2, 3, $\cdots$, n. We complete the figure: We mark the intersections 12, $\cdots$. We mark the circles 12 23, 31, $\cdots$. There is a point 1234 on the 4 circles 123, 234, There is a circle 12345 on the 5 points 1234, $\cdots$. There is a point on the 6 circles 12345 $\cdots$. And so on. For an even number of the figure ends with a point—the Clifford point; for an odd number circle—the Clifford circle.

Regarding the lines as circles on the point ∞, we have a configuration Clifford configuration.

The fundamental curves C^n. An account of these curves and nt theory is given in my paper on Reflexive Geometry,† to which as R. G. I shall state a little differently what is needed here. The to obtain for any given number of lines a curve which plays the circumcircle for three lines.

We take a base circle $|t| = 1$. An equation $x = f(t)$ maps this some curve. The point x is stationary—that is a cusp of the $dx/dt = 0$.

especially simple class of curves with $n-2$ cusps will the by

$$dx/dt = \kappa(t-t_1)(t-t_2)\cdots(t-t_{n-2}).$$

such a curve by C^n.

Thus when $n = 2$, $dx/dt = \kappa$, $x = \kappa t + x_0$, so that C^2 is a circle. cardioid, and so on.

is convenient here to mean by C^1 a point, for which $dx/dt = 0$; and a line. If $dx/dt = \kappa/t$ and $t = e^{\iota\theta}$, then $x = x_0 + \iota\kappa\theta$, which defines

If the equation (1) which gives the cusp-parameters is written

$$dx/dt + (n-1)\{a_1 + (n-2)a_2 t + \cdots + \bar{a}_1 t^{n-2}\} = 0,$$

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or exceptional cases, see W. B. Carver, *American Journal of Mathematics*, Vol 20). pp. 137-167.

Transactions of the American Mathematical Society, Vol. 8 (1907), pp. 14-24.

then the lines of the curve C^n are given by

$$(2) \qquad x - x_0 + na_1 t + \binom{n}{2} a_2 t^2 + \cdots + n\bar{a}_1 t^{n-1} + (\bar{x} - \bar{x}_0)\, t^n = 0.$$

We call x_0 the center of the curve. If conversely we write a self-conjugate equation in t, and regard the end-coefficients as conjugate variables, we have a curve C^n with center 0. R. G. § 3.

The fully polarized form of (2) is

$$(3) \qquad x - x_0 + a_1 s_1 + a_2 s_2 + \cdots + (\bar{x} - \bar{x}_0) s_n = 0,$$

where $s_1, s_2 \cdots s_n$ are the product-sums of $t_1, t_2 \cdots t_n$. This is the osculant line of C^n for the parameters t_i, or points t_i of C^n. If we set $t_1 = t_2 = \cdots = t_m = t$ we have the osculant C^m for the points $t_{m+1} \cdots t_n$. Two osculant C^{n-1}'s have a common osculant C^{n-2} and conversely two C^{n-1}'s with a common osculant C^{n-2} are osculants of a C^n. R. G. § 6.

A curve C^n is defined save as to homologies $x = ay + b$ by its $n - 2$ cusp-parameters. Suppose $n = 2m + 1$. Then there are $2m - 1$ cusp-parameters. There is then a canonizant, m parameters apolar (or harmonic) to the cusp-parameters. These m points give an osculant C^{m+1} which is a repeated point. This point is the Clifford point of the C^{2m+1}. The reason for the existence of this point may be stated thus. An osculant C^m is defined by taking $n - m$ points t_i on the given C^n. These have a polar as to the cusp-parameters. This polar gives the cusp-parameters of C^m. Thus when the polar is arbitrary, the cusp-parameters are arbitrary. And this implies that in (1′) $a_1 = a_2 = \cdots = 0$. For the curve C^{2m+2} with $2m$ cusps the simplest set apolar to the cusp-parameters is a pencil of sets of $m + 1$ points. Such a set has an osculant C^{m+1} which again is a repeated point. The locus of such points is a circle.* This is the Clifford circle of the C^{2m+2}.

To prove this it will suffice to take one case. Suppose we have a C^4, with 2 cusps,

$$x - x_0 + 3a_1 t + 3\rho t^2 + \bar{a} t^3 = 0.$$

The osculant of t_1, t_2 is the circle

$$x - x_0 + a_1 s_1 + \rho s_2 + \bar{a} s_3 = 0.$$

If t_1 and t_2 are apolar or harmonic to

$$a_1 + 2\rho t + \bar{a} t^2 = 0$$

then

$$a_1 + \rho(t_1 + t_2) + \bar{a} t_1 t_2 = 0$$

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* Strictly, we should speak of the Clifford circle or arc, for the circle may or may not be complete.

and the circle became the point

$$x - x_0 + a_1(t_1 + t_2) + \rho t_1 t_2.$$

If we eliminate $t_1 + t_2$ from these two equations we have

$$\begin{vmatrix} x - x_0 & a_1 \\ a_1 & a\rho \end{vmatrix} = t_1 t_2 \begin{vmatrix} a_1 & \rho \\ \rho & \bar{a} \end{vmatrix}$$

which is a circle (or the arc of a circle). The argument is general—see M. G. § 4, p. 103. We are here restating that § 4 in geometrical language.

Given a C^{2m+2}, each osculant C^{2m+1} has a Clifford point. The locus of these is the Clifford circle. The reason is that if we have $m + 1$ points t_i apolar to $2m$ points τ_i the polar of t_i has as canonizant the points $t_2 \cdots t_{m+1}$. The algebraic argument is in M. G. § 4.

Given a C^{2m+1} each osculant C^{2m} has a Clifford circle. These are all on the Clifford point. The reason is that if we have m points t_i apolar to $2m - 1$ points, τ_i the polar of t_1 has an apolar set the m points. The algebraic argument is again in M. G. § 4. The gist is that we regard the Clifford chain as a property of a curve C^n and its osculants. The lines come in by taking $n + 1$ points t_i on C^n. Each n of these points has an osculant line, so that we have a set of $n + 1$ osculant lines. These are any lines and they determine the C^n uniquely (M. G. § 1). This C^n is the fundamental covariant, under the group $x = ay + b$ of homologies, of the $n + 1$ lines. The fundamental covariant C^m of any $m + 1$ of the lines is an osculant of C^n—in fact the osculant for the unused t_i. Thus for two lines the C^1 is the intersection. For three lines the 3 intersections are on the C^2 which is the circumcircle. For four lines, the 4 circumcircles are osculants of the C^3, and meet at the cusp (the Clifford point). For five lines the 5 C^3's are osculants of C^4. The five cusps are on its Clifford circle. And so on.

2. *The incenters of an* $(n + 1)$*-line.* A three-line has four inscribed circles. Their centers are the incenters of the three-line. We ask then for the curves C^n which touch $n + 1$ given lines, and more particularly for their centers. These centers we call the *incenters* of the $(n + 1)$-line.

A line is given most simply by the image in it of the base point 0. Let the images be $x_1, x_2 \cdots x_{n+1}$. We have then from (2) $n + 1$ equations

$$-x_0 + a_1 t_i + \cdots + \bar{a}_1 t_i^{n-1} + (\bar{x}_i - x_0) t_i^n = 0.$$

Eliminating $x_0, a_1 \cdots \bar{a}_1$ we have a determinant

$$| \, 1 \; t_i t_i^2 \cdots (\bar{x}_i - \bar{x}_0) t_i^n \, | = 0,$$

and therefore

$$\bar{x}_0 = \sum^{n+1} [\bar{x}_1 t_1^n / (t_1 - t_2) \cdots (t_1 - t_{n+1})]. \tag{4}$$

But the clinant of the line given by x_i is $-x_i/\bar{x}_i$ and is from (2) $-t_i^n$. Thus we have

$$(5) \qquad x_i = \bar{x}_i t_i^n,$$

so that (4) takes the form, the equation of incenters,

$$(4') \qquad \bar{x}_0 = \sum^{n+1} [x_1/(t\ - t_2)\cdots(t_1 - t_{n+1})].$$

These incenters can be constructed, if we can solve (5) for t_i, that is if we assume that we can divide an angle into n equal parts. This operation is all we need, in addition to Euclid's, in this paper. For any t_i we may substitute ϵt_i where $\epsilon^n = 1$. The set t_i and the set ϵt_i give the same value of x_0. Hence there are n^n incenters, and n^n inscribed C^n's.

Each can be named $1, \epsilon_1, \cdots, \epsilon_n$, where $\epsilon_i^n = 1$, and repetitions are allowed.

3. *The axes of an n-line.* The conjugate of (4) is

$$(5) \qquad x_0 = \sum [x_1 t_2 \cdots t_{n+1}/(t_2 - t_1)\cdots(t_{n+1} - t_1)].$$

Eliminating x_{n+1} from (4′) and (5) we have

$$(6) \qquad x_0 \pm \bar{x}_0 t_1 t_2 \cdots t_n = \sum^{n} [x_1 t_2 \cdots t_n/(t_2 - t_1)\cdots(t_n - t_1)]$$

the sign being + for n odd, — for n even. This is a self-conjugate equation, denoting a line. There are n^{n-1} such lines. We call them the *axes* of the n-lines. They constitute the locus of the centers of inscribed C^n's. The clinant of an axis is a geometric mean of the clinants of the n lines; in other words if a line makes with a base line an angle θ_i (to the modulus π) an axis makes an angle θ where

$$n\theta \doteq \sum \theta_i.$$

Therefore the n^{n-1} axes fall into n sets of parallel lines, inclined successively at angles π/n. In each set are n^{n-2} parallel lines.

Each axis can be named $1, \epsilon_1, \cdots, \epsilon_n$. If in two axes so named we have the same ϵ in the same place, the axes are parallel.

Two curves C^n touching n lines may or may not have their centers on an axis. Then the common lines of two curves C^n fall into sets. To verify this directly we compare with (2) a second equation

$$(2') \qquad x - y_0 + b_1\tau + \cdots + (\bar{x} - \bar{y}_0)\tau^n = 0$$

and we see at once that if these are to be the same equation we must have $\tau^n = t^n$, $\tau = \epsilon t$. For a selected root of unity ϵ, we have on subtraction an equation of degree n in t giving what we call a *tied* set of n common tangents. This equation is

$$y_0 - x_0 + \cdots + (\bar{y}_0 - \bar{x}_0)\, t^n$$

so that

$$(y - x_0)/(\bar{y}_0 - \bar{x}_0) = (-)^n t_1 \cdots t_n.$$

Accordingly, when the centers of two curves on n lines are on an axis, the n lines are a tied set.

The curves C^n which touch n lines fall then into n^{n-1} discrete systems. The transition from one system to another is when the center falls on two axes. One of the n lines is then a double line of C^n.

It follows that when n is not a prime number the double lines of C^4 will fall into sets. For example if $n = 4$, the center may be where axes meet at right angles, or where axes meet at $\pi/4$.

If we apply the theory of this section to a triangle abc, we obtain as the locus of centers of inscribed cardioids three sets of three parallel lines, forming equilateral triangles. The vertices of the triangles are the centers of the cardioids which touch a side (say bc) of the given triangle twice. If x_0 be such a center, then the angle x_0bc is a third of the angle abc. For x_0b is an axis of the 3 lines ab and bc twice. Thus if we take the interior trisectors of the angles of a triangle, the points where those adjacent to a side meet form an equilateral triangle.[*]

4. *The chain for an incenter.* Taking the equation of an incenter, (4′) we interpolate, as in M. G. § 2,

$$\bar{x} = \sum^{n+2} \frac{x_1(t_1 - \tau)}{(t_1 - t_2) \cdots (t_1 - t_{n+2})} \tag{11}$$

this becomes (4′) when $\tau = \epsilon_{n+2}$. It is a circle on selected incenters of any $n + 1$ lines out of $n + 2$.

There are then for $n + 2$ lines n^{n+2} circles, each on $n + 2$ incenters. For a second interpolation we write

$$\bar{x} = \sum^{n+3} \frac{x_1(t_1 - \tau_1)(t_1 - \tau_2)}{(t_1 - t_2) \cdots (t_1 - t_{n+3})} . \tag{12}$$

This becomes (11) when $\tau_2 = t_{n+3}$. It is an osculant of the curve

$$\bar{x} = \sum \frac{x_1(t_1 - \tau)^2}{(t_1 - t_2) \cdots (t_1 - t_{n+3})} .$$

And the point is that this curve is a C^3. For differentiating we get as cusp-condition

$$\sum \frac{x_1(t_1 - \tau)}{(t_1 - t_2) \cdots (t_1 - t_{n+3})} = 0, \tag{13}$$

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[*] Morley, *Mathematical Association of Japan for Secondary Mathematics*, Vol. 6, Dec. 1924. This theorem, which I obtained in this way long ago, has excited much interest.

whose conjugate is

$$\Sigma \frac{x_1(\tau - t)}{(t_2 - t_1)\cdot\cdot\cdot(t_{n+3} - t_1)} \times t_1 t$$

The equation (13) is then self-conjugate.

Thus the circles attached to $n+2$ out of $n+3$ line a C^3, and therefore meet at its cusp. There are then for cardioids. For $n+4$ lines, the cardioids are osculants of

$$(14) \qquad \bar{x} = \Sigma \frac{x_1(t-\tau)^3}{(t_1 - t_2)\cdot\cdot\cdot(t_1 - t_{n+4})}.$$

Writing $d\bar{x}/dt$, and forming its conjugate we see again cusps, and is a C^4. And so on.

We have then the table:

number of lines	1	2	3	4	5
axes C^0	1	2	3^2	4^3	5^4
incenters C^1		1	2^2	3^3	4^4
circles C^2			1	2^3	3^4
cardioids C^3				1	2^4
C^4					1
C^5					
:					

The table is read diagonally; each C^n is a first osculant of next column.

The first column says that a line is its own axis.

The second column says that a two-line has two axes an

The third column says that a three-line has 3^2 axe incenters, the intersection of the axes of the two-lines con that it has one circumcircle, on the intersection of the two-

For the nth column, the axes are new; the incente axes of the preceding column, that is the $n(n-1)^{n-2}$ axes $(n-1)$-lines meet in the $(n-1)^{n-1}$ incenters, there be point and $n-1$ points on an axis. The circles arise fr of the preceding column. That is the $n(n-2)^{n-2}$ ince $(n-2)^{n-1}$ circles, there being n points on each circle, each point. These $n-2$ circles cut at the angle $\pi/(n-2)$. cardioids arise from the $n(n-3)^{n-2}$ circles; the circles the cardioids, each C^3 having n osculant C^2's; and ea $n-3$ C^3's. And so on.

The leading diagonal indicates Clifford's chain. We n-line has a unique C^{n-1}.

The second diagonal indicates the chain discussed by F. H. Loud, *Transactions of the American Mathematical Society.*[*] The ambiguities which enter from (5) are there cleared up by regarding lines as directed. This makes the incenter of a 3-line unique. In general the ambiguities disappear if we recall that two osculants of C^n have a common osculant. Consider the 2^4 C^3's for 5 lines. These for 5/6 lines are osculants of C^4's. The 6×2^4 C^3's are osculants of the 2^5 C^4's; $6C^3$'s on each C^4; $2C^4$'s on each C^3. The C^3 taken from lines 12345 and that taken from 23456 must have a common osculant C^2 (taken from 2345).

The ambiguity is explained in another specific case involving axes by P. S. Wagner in the article following this one.

There is in fact no ambiguity where we name the C^n's by the roots of unity. The naming is carried on from one column to the next.

This completes the object of this paper, but it is convenient to add a canonical equation of the curve C^n.

4. *Canonical equation of a* C^n. The curve C^n is defined save as to homologies by the $n-2$ cusp-parameters. It is proper to give these intrinsically, so far as possible. When n is odd, say $n = 2m+1$, Sylvester pointed out the proper intrinsic or canonical form for an equation, here

$$a_1 + (n-2)\, a_1 t + \cdots + \bar{a}_1 t^{2m-1} = 0$$

namely the equation

$$\sum^{m} A_i (t - \tau_i)^{2m-1} = 0. \tag{15}$$

Here the m numbers τ_i give the canonizant, the unique equation of degree m apolar to the given equation. Accordingly the canonical form for a C^{2m+1} is

$$(1/2m)(dx/dt) = \sum^{m} A_i (t - \tau_i)^{2m-1}$$

or

$$x = \sum^{m} A_i (t - \tau_i)^{2m}. \tag{16}$$

The equation (15) must be self-conjugate, that is the same as

$$\sum^{m} \bar{A}_i (t - \tau_i)^{2m-1} / \tau_i^{2m-1} = 0.$$

This is secured by

$$\bar{A}_i = A_i T_i^{2m-1}. \tag{17}$$

The base-point in this canonical form (8) is the Clifford point of any $2m+2$ lines which are an osculant set of the C^{2m+1}. It is in fact the osculant of the canonizant. The simplest case, $m = 1$, is the cardioid, with the canonical equation

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[*] Vol. 1 (1900), pp. 323-338.

$$x = (1 - t)^2.$$

An osculant point is here

$$x = (1 - t_1)(1 - t_2)$$

and 3 such points are on the line

$$x = (1 - t_i)/(1 - t).$$

Four such lines are given by

$$x = \prod^{4} [(1 - t_i)/(1 - t)(1 - t')]$$

and are tangents of the parabola

$$x = [\prod^{4}/(1 - t)^2].$$

And so in general the C^{2m+1} can be immediately connected with

$$x = \sum^{m} [B_i/(t - \tau_i)^2]$$

which is Clifford's m-fold parabola (*loc. cit.*). The case of C^5, with 3 cusps, was analysed, with figures, by Father E. C. Phillips (*American Journal,* Vol. 31, 1909). It may be remarked that if the cusps c_i are given there are four curves, for the Clifford point is given by $\sum [1/(x - c_i)^{1/3}] = 0$, which rationalized is a quartic for which $g_2 = 0$.

When n is $2m$, we have for C^{2m} a cusp form of degree $2(m - 1)$. The lowest apolar forms are of degree m, and therefore as in Sylvester's theory we take

$$\frac{1}{2m - 1}\,\frac{dx}{dt} = \sum^{m} A_i(t - \tau_i)^{2(m-1)}$$

whence the canonical form is

$$x = \sum^{m} A_i(t - \tau_i)^{2m-1}.$$

This may be regarded as a first osculant of (8), namely

$$x = \sum^{m} A_i(t_0 - \tau_i)(t - \tau_i)^{2m-1}.$$

There is no advantage for small values of m. The circle C^2 is naturally best as $x = t$ and the C^4 as $x = 3t - 3\mu t^2 + t^3$.

The canonical form might have been used throughout, but it would not have been possible to use the references.

It is to be remarked that in the form used the coefficients $a_1, a_2 \cdots$ of the C^{n-1} of an n-line are invariants of the n-line. They form with the parameters t_i a complete set of rational invariants (under homologies). The parameters are by (3) inversely proportional to the clinants of the lines.

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An Extension to Clifford's Chain.[*]

By PAUL SMITH WAGNER.

1. *Tied Tangents.* Two non-intersecting circles which are exterior to each other have four real tangents which can be separated into two pairs, the one exterior and the other interior. This pairing may also be effected by noticing that the sum of the angles formed by each pair of these tangents with the line of centers is congruent to zero modulo π. We shall define as a *set* of tangents any pair which fulfils this condition.

Algebraically, if we express these circles as

$$x - 2t + \bar{x}t^2 = 0 \text{ and } x - c - 2\rho\tau + (\bar{x} - \bar{c})\tau^2 = 0,$$

then for a common tangent, we identify the equations so that $\tau^2 = t^2$. We have then $\tau = \pm t$, so that the common tangents fall into two sets. With $\tau = t$, we have

$$c - 2(1 - \rho)t + \bar{c}t^2 = 0,$$

giving one pair of tangents. With $\tau = -t$, we have

$$c + 2(1 - \rho)t + \bar{c}t^2 = 0,$$

giving the other pair.

From these equations it follows that, if the parameters of the tangents in either set are t_1 and t_2, $t_1t_2 = c/\bar{c}$, which is the clinant of the line of centers. Obviously,

$$t_1{}^2t_2{}^2 = (c/\bar{c})^2,$$

which means that if θ_1, θ_2 and ϕ, respectively, are the angles which the lines t_1, t_2 and the line of centers make with the axis of reals, $\theta_1 + \theta_2 \equiv 2\phi \pmod{\pi}$. In case we take the axis of reals as the line of centers,

$$\theta_1 + \theta_2 \equiv 0 \pmod{\pi}.$$

Such a set of tangents is a set of *tied* tangents.

This classification leads to an immediate generalization. If we take the two cardioids †

$$(1) \qquad x - 3t + 3t^2 - \bar{x}t^3 = 0$$

$$(2) \qquad x - c - 3a\tau + 3\bar{a}\tau^2 - (\bar{x} - \bar{c})\tau^3 = 0$$

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[*] W. K. Clifford, *Mathematical Papers*, pp. 51-54.

† F. Morley, *Journal of the Mathematical Society of Japan for Secondary Education*, Vol. 6 (1924).

and identify these equations, we have

$$\tau^3 = t^3.$$

Thus the common lines fall into three sets. If we take $\tau = t$, then

$$c - 3(1-a)t + 3(1-\bar{a})t^2 - \bar{c}t^3 = 0$$

gives the three common lines of one set. For then

$$t_1t_2t_3 = c/\bar{c} \text{ and } t_1{}^3t_2{}^3t_3{}^3 = (c/\bar{c})^3;$$

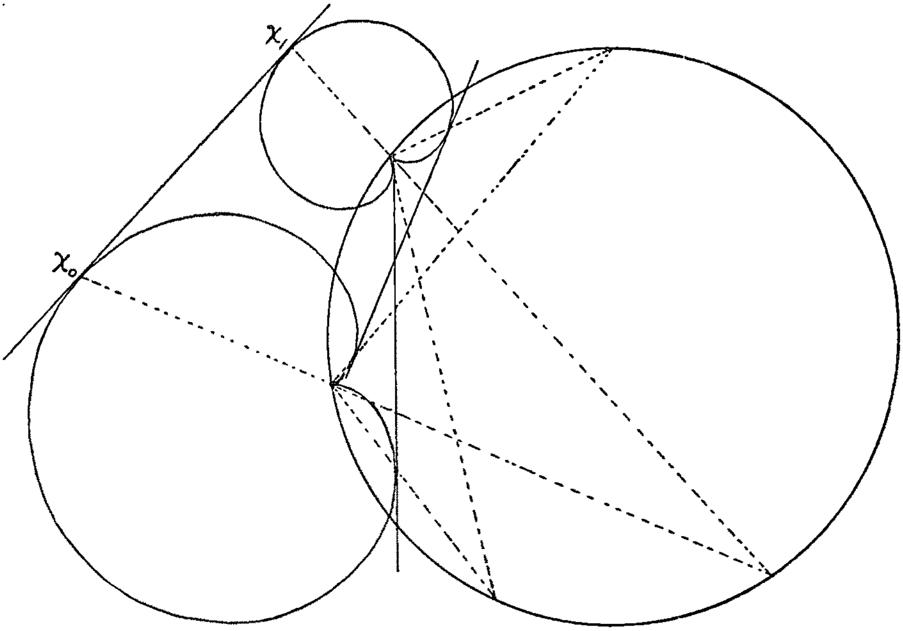

FIG. 1.

so that if θ_i and ϕ, respectively, are the angles which the lines t_i and the line of centers make with the axis of reals,

$$\theta_1 + \theta_2 + \theta_3 \equiv 3\phi \pmod{\pi}.$$

If we take the axis of reals for the line of centers, we have

$$\theta_1 + \theta_2 + \theta_3 \equiv 0 \pmod{\pi}.$$

Here, then, *any three tangents with the parameters* t_1, t_2, t_3 *such that* $t_1t_2t_3 = c/\bar{c}$, $\omega\, c/\bar{c}$, *or* $\omega^2 c/\bar{c}$ *are a set of tied tangents.*

2. *Method of selecting tied tangents.* The cardioid whose line equation is (1) has $x = 2t - t^2$ for its point equation. At a particular point, this

becomes $x_0 = 2t_1 - t_1^2$ and a line from x_0 through the cusp will have for its equation

$$(3) \qquad x = t_1^2 \bar{x} + t_1 + 1.$$

The cardioid (2), which has c for its center, has $x = -\bar{a}\tau^2 + 2a\tau + c$ for its point equation, and, at a particular point, $x_1 = -\bar{a}\tau_1^2 + 2a\tau_1 + c$. The equation of the line from x_1 through the cusp of (2) becomes

$$(4) \qquad x = a\tau_1^2\bar{x}/\bar{a} + f(\tau_1, a, \bar{a}, c, \bar{c}),$$

since the parameter of the cusp is $(a/\bar{a}) + c$ by virtue of the fact that, at the cusp, $dx/d\tau = 0$.

The relative clinant of lines (3) and (4) is $a\tau_1^2/\bar{a}t_1^2$. Accordingly as we let $\tau = t$, ωt or $\omega^2 t$, we have the relative clinant as $a/\bar{a}$, $\omega a/\bar{a}$ or $\omega^2 a/\bar{a}$, respectively.

If we let $\tau = t$, the line equation of (2) becomes

$$(x - c) - 3at + 3\bar{a}t^2 - (\bar{x} - \bar{c})t^3 = 0$$

and, as shown before, $t_1 t_2 t_3 = c/\bar{c}$ when we consider the common tangents with (1). But when $\tau = t$, x_0 and x_1 will lie on the same tangent line and the cuspidal chords (3) and (4) will have a relative clinant of $a/\bar{a}$. Hence, this selects three tangents whose cuspidal chords, respectively, form angles such that the relative clinant is $a/\bar{a}$, so that the three angles are equal, modulo π. Hence, *the three intersections of the cuspidal chords of these three tangents lie on a circle through the cusps.*

But the same is true for $\tau = \omega t$ and $\tau = \omega t^2$. Therefore, the nine possible *tangents of two cardioids are in three sets of three each such that in each set the cuspidal chords intersect on a circle through the cusps.* See fig. 1.

Since this puts five points on each circle, the cusps being on each circle, *one tangent is sufficient to definitely determine the other two associated with it to complete the set.*

The angle of intersection of these circles on the cusps is that of their tangents at the cusps, and hence the angle between a tangent and the join of the cusps of (1) and (2) is, in each case, equal to the angle between the corresponding cuspidal chords. But these angles differ by $2\pi/3$ so that *the three circles cut each other in angles of $2\pi/3$ each.*

3. *Special types of curves.* Two parabolas have at most three common tangents and their corresponding focal chords intersect on their circum-circle. See fig. 2. This can be shown as follows:

DAG[*] $+ AGD + GDA = \pi$ and $VLY + LYV + YVL = \pi$. But

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[*] Unless otherwise qualified *DAG*, *AGD*, etc. stand for the angles *DAG*, *AGD*, etc.

$YVL = GDA$ and $LYV = AGD$. Therefore, $DAG = T$ triangles FCS and JLT are similar for $SFC = TJL$ a Hence, $FCS = TLJ = \theta LI$. Also, triangles EBM and for $MEB = RIL$ and $BME = IRL$, so that $EBM = RL$ this it follows that $FCS = EBM$, $DAG + FCS = \pi$ and D Therefore the points A, B and C are concyclic with θ and θ

Expressing the equation of the cardioid in polar $r = a(1 - \cos\theta)/2$ or $r^{1/2} = a^{1/2} \sin(\theta/2)$. For the $r^{-1/2} = a^{-1/2} \sin(\theta/2)$ and for the circle $r = a \sin\theta$. All of under the general type[*]

$$r^{p/q} = a^{p/q} \sin(p\theta/q). \tag{5}$$

In the case of the cardioid $p/q = 1/2$, and if we set $p + q = 3$. For the parabola $p/q = -1/2$, so that if $p =$ $p + q = 1$. In the circle $p/q = 1$ and letting $p = 1$, $p + q = 2$. This seems to indicate that when p/q is in its $|p + q|$ is the number of sets of tied tangents between t type of (5). It is not meant that only curves of the abo tangent relationship.

4. *A theorem on the parabola.* If the parameters of t are t_2 and t_3, respectively, the equation of the tangent 23 is x and if τ is the parameter of the focus, the image in 23 is $x +$ The images of the same focus in the three tangents are on x which is the equation of the directrix of the parabola, with tangent to these three lines. This equation can be written m as $\tau^2 - x\tau = (\tau - t_1)(\tau - t_2)(\tau - t_3)/(\tau - t)$. The directrix is $(S_3/\tau)^{1/2}$ and that of any line perpendicular t If we consider the particular perpendicular which passes t of contact of $t_2 t_3$ with the parabola, we know that the lin angle which this makes with the focal chord. That is, if of the focal chord, then $[i(S_3/\tau)^{1/2}x]^{1/2} = i(t_2 t_3)^{1/2}$ and Now if the parameter of θ [see fig. 2] is τ_1, then $x = i(\tau_1 t$ the direction of GA. If the parameter of θ' is τ_2, then and this is the direction of DA. The difference of directio is $x/x' = (\tau_1/\tau_2)^{1/2}$ which is one-half the central angle τ_1/τ vertex A is on the circle of which this latter is a central $A, B, C, 1, 2, 3, \theta$ and θ' all lie on the same circle.

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[*] For another curve of this type, see F. Morley, "The Curve of A *Journal of Mathematics*, Vol. 46 (1924), pp. 193-200.

Since the direction of θA is $i(\tau_1 t_2 t_3/t_1)^{1/2}$ and the parameter of θ is τ_1, the parameter of A must be $t_2 t_3/t_1$. In like manner, B is $t_1 t_3/t_2$ and C is $t_1 t_2/t_3$. These evaluations are on the assumption that the parameters of 1, 2 and 3, respectively, are t_1, t_2 and t_3.

It follows that the direction of $1A$ is $i(t_2 t_3)^{1/2}$. But this is the direction of 23. Hence they are parallel. Since a single infinity of parameters satisfy the above condition we have the general theorem: *If a varying parabola has three fixed tangent lines, the focal chord of any one of them passes through a fixed point.*

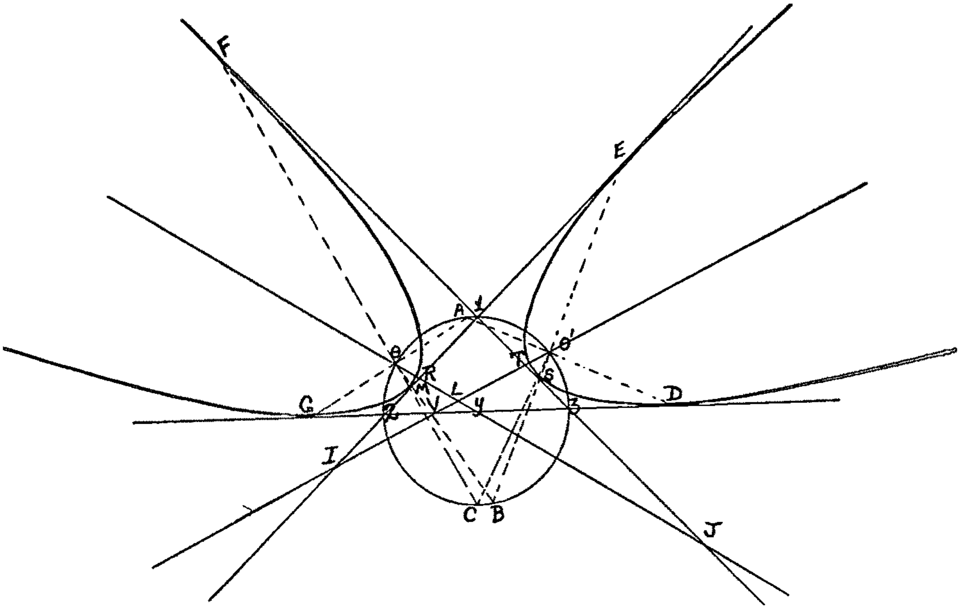

FIG. 2.

5. *Comparison with Professor Loud's work.* In the case of four lines, Professor Loud* showed that the center circles are in two sets, orthogonal systems, of four each obtained from the directed lines as follows:

System I				System II			
+	+	+	+	+	+	+	−
+	+	−	−	+	+	−	+
+	−	−	+	+	−	+	+
−	−	+	+	−	+	+	+

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* "Sundry metric theorems concerning *N*-lines in a plane," *Transactions of the American Mathematical Society*, Vol. 1 (1900), pp. 323-338.

where a change of sign indicates a reversal of direction. Now if we draw our sixteen circles tangent to the four-line, three at a time, and select them

System I	System II
X X X X or X < X <	< X X <
< < < <	X < < X
X X X X	< X X <
< < < <	X < < X

where < denotes an exterior and X an interior pair of tangents, we will obtain the identical grouping of Professor Loud. In each choice there is an even number of each type of tangents—an odd number will not complete the cycle for four points.

In the language of clinants, we can leave the center of any circle in two different ways, namely $t_1t_2 = c/\bar{c}$ and $t_1t_2 = -c/\bar{c}$ where the former may be a set of exterior tangents and the latter a set of interior ones. Hence, there are two centers on each circle which gives us a set of eight center circles. That these are in orthogonal pairs is shown by the algebra of Professor Loud's article by replacing t_m by $-t_m$. Hence, Professor Loud's algebra is also the algebra of tied tangents.

6. *Tied tangents applied to cardioids.* There are one hundred and thirty-five cardioids tangent to a five-line, four at a time. Our problem is to effect some systematic classification of these, if possible.

It has already been shown that two cardioids have three sets of three tangents such that if t_1, t_2 and t_3 are the parameters of any particular set, $t_1t_2t_3 = c/\bar{c}$, $\omega c/\bar{c}$ or $\omega^2 c/\bar{c}$. We now select the particular set $t_1t_2t_3 = c/\bar{c}$ and proceed by means of this direction to the center of another cardioid of which these three are a set of tied tangents. From this center we proceed by $t_2t_3t_4 = c'/\bar{c}'$ to another center, see fig. 3, etc.

We see the $a_1 = t_3t_4t_5/t_2t_3t_4 = t_5/t_2$ and in like manner $a_2 = t_1t_3$, $a_3 = t_2/t_4$, $a_4 = t_3/t_5$ and $a_5 = t_4/t_1$. Since

$$t_5/t_2 \cdot t_1/t_3 \cdot t_2/t_4 \cdot t_3/t_5 \cdot t_4/t_1 = 1;$$

the polygon is a closed polygon. It is also inscribed in a circle, as we shall show later. We have, then, when we remember that we can leave any center by three ways using $t_1t_2t_3 = \omega c/\bar{c}$ or $\omega^2 c/\bar{c}$ instead of $c/\bar{c}$, this significant theorem:

The centers of the cardioids tangent to a five line, four at a time, arrange themselves so that there are five centers on each circle and three circles on each center. Hence, the centers of the 135 *cardioids lie on* 81 *circles.*

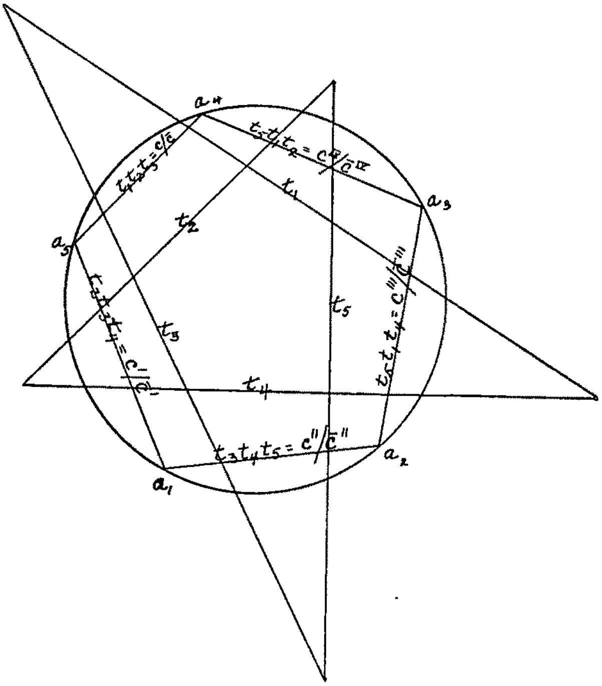
a_4
$t_1t_2t_3 = c/\bar{c}$
$t_5t_1t_2 = c^{iv}/\bar{c}^{iv}$
t_1
a_3
t_2
a_5
$t_2t_3t_4 = c'/\bar{c}'$
t_5
$t_4t_5t_1 = c'''/\bar{c}'''$
t_3
t_4
$t_3t_4t_5 = c''/\bar{c}''$
a_2
a_1

FIG. 3.

7. *The algebraic statement.* In the closing pages of a paper by Professor Morley [*] the equation of the tangent to a cardioid is given as

$$xt^3 + y = x_0t^3 + y_0 + at^2 + bt$$

which in another form,

$$(y - y_0) - bt - at^2 + (x - x_0)t^3 = 0,$$

is analogous to (2). In fact, if we replace t by $-1/\tau$, we obtain (2). If p is the distance from the origin to this tangent, then

$$p = (Ax + \bar{A}\bar{x} + p)/2(A\bar{A})^{1/2};$$

i. e., $$p_1 = [-t_1^3x - y + t_1^3x_0 + y_0 + bt_1 + at_1^2]/2t_1^{3/2}$$

and $$2p_1t_1^{3/2} = t_1^3x_0 + y_0 + bt_1 + at_1^2.$$

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[*] "Metric Geometry of the Plane N-line," *Transactions of the American Mathematical Society*, Vol. 1 (1900), pp. 97-115.

For four tangents, if x_0 is the center of the cardioid to which these lines are tangent, and if $x_1 = 2p_1 t_1^{3/2}$,

$$x_0 = \sum^{4} \frac{x_1}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)}.$$

If now we put

$$x = \sum \frac{x_1(t_1 - t)}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_1 - t_5)}, \tag{6}$$

we have a circle whose center is

$$a_1 = \sum^{5} \frac{x_1 t_1}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_1 - t_5)},$$

and whose radius is $| a_2 |$, where

$$a_2 = \sum^{5} \frac{x_1}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_1 - t_5)},$$

which is the circle on the centers of the five cardioids each one of which (center) is obtained by letting $t = t_1, t_2, t_3, t_4$ and t_5 successively.

The clinant of $a_4 a_5$ is $(a_5 - a_4)/\bar{a}_5 - \bar{a}_4)$ which we find to be $-1/t_1 t_2 t_3$. But bearing in mind that t_i of this equation is $-1/t_1$ of equation (2) we have $t_1 t_2 t_3$ as clinant of $a_4 a_5$. Hence $t_1^3 t_2^3 t_3^3 = [(a_5 - a_4)/(\bar{a}_5 - \bar{a}_4)]^3$ and $\theta_1 + \theta_2 + \theta_3 \equiv 3\phi \bmod \pi$. Hence, *the underlying relation of these cardioids whose centers are given by* (6) *is that they are connected by tied tangents.*

8. *The relation of the* 3^4 *circles.* The method used in establishing this relation is that of Professor Loud. If a_α is the characteristic constant, then

$$a_\alpha = \sum \frac{x_1 t_1^{n-\alpha-3}}{(t_1 - t_2)(t_1 - t_3) \cdots (t_1 - t_n)} \qquad (\alpha = 1, 2 \cdots n),$$

and

$$\bar{a}_\alpha = \sum \frac{x_1 t_1^{\alpha+3-n}}{(1/t_1 - 1/t_2) \cdots (1/t_1 - 1/t_n)}$$

$$= (-1)^{n-1} \tau \sum \frac{x_1 t_1^{-3} t_1^{n-1} t_1^{\alpha+3-n}}{(t_1 - t_2) \cdots (t_1 - t_n)}$$

where $\tau = t_1 t_2 t_3 \cdots t_n$. Therefore

$$\bar{a}_\alpha = (-1)^{n-1} \tau \sum \frac{x_1 t_1^{\alpha-2}}{(t_1 - t_2) \cdots (t_1 - t_n)}$$

$$= (-1)^{n-1} \tau a_{n-\alpha-1}.$$

When we omit the line x_n from the n-line, we have an $(n-1)$-line whose constants are[*]

$$a_1 - a_2t_n,\ a_2 - a_3t_n,\ \cdots,\ a_{n-1} - a_nt_n.$$

Now since $t_1t_2t_3 = c/\bar{c}$, or $\omega c/\bar{c}$, or $\omega^2 c/\bar{c}$ then $\omega t_1t_2t_3$ and $\omega^2 t_1t_2t_3$, on the assumption that $t_1t_2t_3 = c/\bar{c}$, take us to the other two centers. That is, this is equivalent to replacing t_n by ωt_n or $\omega^2 t_n$. Three circles derived on these respective assumptions that the n-th line is used or is replaced by ωt_n, or $\omega^2 t_n$, have the equations $x = a_1 - a_2t$, $x = a_1' - a_2't$ and $x = a_1'' - a_2''t$. These three circles coincide at a point which when the n-th line is omitted is called $a_1 - a_2t_n$, $a_1' - a_2't_n$ and $a_1'' - a_2''t_n$. Their angle of intersection is that of the strokes extending from their respective centers. Hence, it is the amplitude of $a_2t_n/a_2''t_n$, or a_2/a_2'', etc.

To obtain this amplitude separate from the ratio of the absolute values, we divide this expression by its conjugate and take the square root of the result. Since the conjugate of a_2 is $(-1)^{n-1}\tau a_{n-3}$ and of a_2'' is $\omega^2(-1)^{n-1}\tau a''_{n-3}$, we have

$$(a_2/a_2'')^{1/2}(\bar{a}_2''/\bar{a}_2)^{1/2} = (a_2/a_2'')^{1/2}(\omega^2 a''_{n-3}/a_{n-3})^{1/2},$$

and in the case of $n = 5$, this becomes ω or $2\pi/3$. Hence, *on each of the* 135 *centers there are three circles cutting each other at an angle of* $2\pi/3$.

9. *General conclusions.* As a complete statement, then, we may say that *a five line has* 135 *cardioids, each tangent to a four line, and their centers lie on* 3^4 *circles which cut each other in threes at these points at an angle of* $2\pi/3$ *so that there are five centers on each circle and three circles on each center.*

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[*] F. Morley, "Metric Geometry of a Plane *N*-line," *Transactions of the American Mathematical Society*, Vol. 1 (1900), pp. 97-115.

On the Group for a Class of Self-Dual Plane Rational Curves.

By Luther E. Wear.

1. *Introduction.* We are concerned in this paper with the group of transformations, including both collineations and correlations, under which a curve remains fixed. Self-duality implies that the curve must have the same number of cusps as inflexional tangents, and of double points as it has double lines.

Haskell (*Bulletin of the American Mathematical Society,* January, 1917), discussed the question of the maximum number of cusps of an algebraic plane curve, and enumerated the self-dual curves. The binomial curves,*

$$x_1{}^n = x_0{}^{n-r} x_2{}^r$$

have been studied and shown to be self-dual. The case of the quartic was considered in my dissertation at the Johns Hopkins University,† and the quintic in a later paper.‡

The equations of the Curve. Consider now the case of a rational plane curve of the nth order. Since the class of the curve must equal the order, we have

$$n = n(n-1) - 2d - 3c, \tag{1}$$

where d is the number of double points and c the number of cusps. Also, rational curves being of genus zero must have $(n-1)(n-2)/2$ double points and cusps together, hence,

$$d + c = (n-1)(n-2)/2. \tag{2}$$

Solving equations (1) and (2) for d and c, we find $c = n - 2$, $d = (n-2)(n-3)/2$, which are the number of cusps and double points, respectively of the self-dual rational curve.

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* Loria, *Spezielle Ebene Kurven,* page 308; Wieleitner, *Algebraische Kurven,* page 136; Snyder, *American Journal of Mathematics,* Vol. 30; Winger, *American Journal of Mathematics,* Vol. 36.

† *American Journal of Mathematics,* April, 1920.

‡ *Bulletin of the American Mathematical Society,* June, 1919.

Similarly for the inflexional tangents,

$$c' = 3n(n-2) - 6\,[(n-2)(n-3)/2] - 8(n-2),^{*}$$

which reduces to,

$$c' = n - 2.$$

Also

$$d' = [n(n-2)(n^2-9)/2] - (2d+3c)(n^2-n-6) + 2d(d-1) + [9c(c-1)/2] + 6dc,$$

which reduces to,

$$d' = (n-2)(n-3)/2$$

giving the number of double lines. Hence the self-dual curve has $n-2$ cusps and also inflexional tangents, while it has $\frac{1}{2}(n-2)(n-3)$ double points with the same number of double lines.

The curve, to be self-dual, must admit a correlation (polarity) or correlations whereby dual singularities are interchanged, and in particular cusps and inflexional tangents are interchanged. The class of curves to be considered here are those which admit of a sufficient number of correlations to send a given cusp into each of the $n-2$ inflexional tangents. The products of these correlations, two at a time, will generate a set of collineations which will permute the cusps. Hence the curve will be self-projective under a cyclic G_{n-2} or under a larger group containing the G_{n-2} as a subgroup. With the ternary group G_{n-2}, will go a corresponding binary group on the parameter, a g_{n-2}.† The transformations of the cyclic g_{n-2} can be written in the canonical form $t = \epsilon t'$, where $\epsilon^{n-2} = 1$.

The curve then can be written down at once in the following most general way:

$$(3) \qquad x_0 = t^n + at^2, \qquad x_1 = bt^{n-2} + 1, \qquad x_2 = t^{n-1} + t,$$

and with the g_{n-2} on the parameter, there goes a ternary G_{n-2} on the planes, viz.,

$$(4) \qquad x_0' = \epsilon^2 x_0, \qquad x_1' = x_1, \qquad x_2' = \epsilon x_2.$$

It is easily seen that the curve (3) remains unaltered when the transformations $t = \epsilon t'$ and (4) are made. All members of the ternary G_{n-2} can be written,

$$(5) \qquad x_0' = \epsilon^{2k} x_0, \qquad x_1' = x_1, \qquad x_2' = \epsilon^k x_2$$

where $\epsilon^{n-2} = 1$ and $k = 1, 2, \cdots, (n-2)$.

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[*] Salmon, *Higher Plane Curves*, page 93.

[†] See Winger, *American Journal of Mathematics*, January, 1916.

Let us now obtain the line equations of (3) by finding Jacobians two at a time. We have,

$$\xi_0 = -bnt^{2(n-2)} + [(b-1)n^2 + (1-3b)n]\, t^{n-2} - 1,$$
$$\xi_1 = -nt^2\{t^{2(n-2)} - [(a-1)n + 1 - 3a]\, t^{n-2} + a\},$$
$$\xi_2 = nt\,\{2bt^{2(n-2)} + [ab(4-n) + n]\, t^{n-2} + 2a\}.$$

Since the cusps are to be interchanged cyclically by the g_{n-2}, then they are given by the special set $t^{n-2} - 1$, and the other special set, $t^{n-2} + 1$, will give the inflexional tangents. Since the class must be reduced by the $n - 2$ cusps, then $t^{n-2} - 1$ must factor out of ξ_0, ξ_1, ξ_2. Dividing by this factor and equating the respective remainders to zero we have for ξ_0, $b = n/(n-4)$; for ξ_1, $a = n/(n-4)$; and for ξ_2, $2(a+b) + 4ab = nab - n$ which is satisfied by $a = b = n/(n-4)$. So the curve may now be written,

$$(3') \qquad x_0 = t^n + [n/(n-4)]\, t^2, \qquad x_1 = [n/(n-4)]\, t^{n-2} + 1,$$
$$x_2 = t^{n-1} + t, \qquad (n > 4),$$

and the line equations are,

$$(6) \qquad \xi_0 = [n/(n-4)]\, t^{n-2} - 1, \qquad \xi_1 = t^n - [n/(n-4)]\, t^2,$$
$$\xi_2 = -[2n/(n-4)]\, t(t^{n-2} - 1).$$

2. *The Self-Projective Group.* It is evident now that the curve is also invariant under the transformation $t = 1/t'$ when combined with the collineation,

$$(7) \qquad x_0' = x_1, \qquad x_1' = x_0, \qquad x_2' = x_2.$$

If the transformation $t = 1/t'$ be combined with the transformations $t = \epsilon t'$ we obtain those of the form $t = \epsilon/t'$ or in general $t = \epsilon^k/t'$, where $k = 1, 2, \cdots, (n-2)$. There is thus generated a dihedral group of order $2(n-2)$ on the parameter t.

If the collineation (7) be combined with (5) we obtain these new collineations:

$$(8) \qquad x_0' = \epsilon^{2k} x_1, \qquad x_1' = x_0, \qquad x_2' = \epsilon^k x_2.$$

There will be $n - 2$ of these and with the original $n - 2$ will constitute a collineation $G_{2(n-2)}$ of which the original cyclic G_{n-2} is a sub-group.

Collineations (4) will leave the reference triangle fixed. Those of the type of (8) are reflexions having as axes the lines,

$$x_0 \pm \epsilon x_1 = 0,$$

and as centers the points $\epsilon : -1 : 0$. The axes of reflexion meet on the point $\xi_2 = 0$ while, dually, the centers lie on the line $x_2 = 0$.

The cusp tangents are given by the determinant,

$$\begin{array}{ccc} x_0 & x_1 & x_2 \\ -[n/(n-4)]\,t^2, & [n/(n-4)]\,t^{n-2}+1, & t^{n-1}+t, \\ 1)\,t^{n-2}+[2n/(n-4)], & [n(n-2)(n-3)/(n-4)]\,t^{n-4}, & (n-1)(n-2) \end{array}$$

$$=0,$$

which reduces to,

$$x_0-\epsilon^2x_1=0.$$

The points of inflexion are $\eta_i^2 : -1 : 0$, where $i=1,2,\cdots,(n-2)$, and where η_i are the $(n-2)$-th roots of -1. Hence the cusp tangents meet at $\xi_2=0$, while, dually, the points of inflexion lie on the line $x_2=0$. Two cases arise according as n is odd or even: in the first case the flexes are distinct at points of intersection of the line $x_2=0$ with the curve, since η_i are distinct numbers: in the second case the numbers η_i^2 give only $(n-2)$ distinct values and that means points of inflexion will come together in pairs to form bi-flecnodes. Dually, when n is odd the cusp tangents will be distinct, but when n is even they will combine in pairs and we shall have double cusp tangents meeting at $\xi_2=0$.

We may summarize the results to this point by saying that in equation (3′) we have a curve which is self-projective under a dihedral $G_{2(n-2)}$ of collineations when combined with a dihedral $g_{2(n-2)}$ on the parameter. The elements of the groups may be set out as follows:

$$\begin{array}{ll} g_{2(n-2)} & G_{2(n-2)} \\ t=\epsilon^k t', & x_0'=\epsilon^{2k}x_0,\quad x_1'=x_1,\quad x_2'=\epsilon^k x_2,\quad \epsilon^{n-2}=1. \\ & \qquad k=1,2,\cdots,(n-1). \\ t=\epsilon^k/t', & x_0'=\epsilon^{2k}x_1,\quad x_1'=x_0,\quad x_2'=\epsilon^k x_2. \end{array}$$

The Self-Dual Group. Now let us write a transformation which will interchange dual singularities of the curve as, for example, $t\tau=\eta$ where $\eta^{n-2}=-1$. This will interchange the cusp parameter $t=1$ with the flex parameter $\tau=\eta$ and similarly with the other cusps and flexes. The equation of any line of the curve is,

$$\{-[n/(n-4)]\,\tau^{n-2}+1\}\,x_0+\{\tau^n-[n/(n-4]\,\tau^2\}\,x_1$$
$$+\{[2n/(n-4)]\tau^{n-1}-[2n/(n-4)]\,\tau\}\,x_2=0.$$

and if we make therein the transformation $t\tau=\eta$ we obtain

$$\{[n/(n-4)]t^{n-2}+1\}\,x_0+\{-\eta^2/t^n-[n/(n-4)]\,\eta^2/t^2\}\,x_1$$
$$+\{-[2n/(n-4)]\,\eta/t^{n-1}-[2n\eta/(n-4)t]\}\,x_2=0.$$

Any point of the curve is,

$$\{t^n + [n/(n-4)]\,t^2\}\,\xi_0 + \{[n/(n-4)]\,t^{n-2} + 1\}\,\xi_1 + [t^{n-1} + t]\,\xi_2 = 0.$$

If these two polynomials in t are identified there are obtained these equations:

$$(9) \qquad \xi_0 = x_0, \qquad \xi_1 = -\epsilon^2\eta^2 x_1, \qquad \xi_2 = -[2n/(n-4)]\,\epsilon\eta x_2.$$

These are the equations of a correlation sending any line of the curve into a point of the curve, and in particular exchanging dual singularities.

If now the correlation (9) is combined with the collineations (5) there arise the $n-2$ correlations,

$$\xi_0 = x_0, \qquad \xi_1 = -\epsilon^{2k}\eta^2 x_1, \qquad \xi_2 = -[2n/(n-4)]\,\epsilon^k\eta x_2,$$

where $k = 1, 2, \cdots, (n-2)$. Furthermore combining correlation (9) with collineations (8) we obtain a further set of $n-2$ correlations, vz.,

$$\xi_0 = x_1, \qquad \xi_1 = -\epsilon^{2k}\eta^2 x_0, \qquad \xi_2 = -[2n/(n-4)]\,\epsilon^k\eta x_2.$$

There have been obtained then the elements of a $G_{4(n-2)}$ of collineations and correlations with a corresponding $g_{4(n-2)}$ on the parameter. It is easily verifiable that the product of any two collineations is a third collineation of the set; that the product of a collineation and a correlation is a correlation of the set; that the product of any two correlations is one of the given collineations; and finally that the inverse of each element is included. We may now tabulate the elements of the group as follows:

Collineations

$$t = \epsilon^k t', \qquad x_0' = \epsilon^{2k} x_0, \quad x_1' = x_1, \quad x_2' = \epsilon^k x_2,$$

$$\text{where } \epsilon^{n-2} = 1,\ k = 1, 2, \cdots, (n-2),$$

$$t = \epsilon^k / t', \qquad x_0' = \epsilon^{2k} x_1, \quad x_1' = x_0, \quad x_2' = \epsilon^k x_2.$$

Correlations

$$t\tau = \epsilon^k\eta, \qquad \xi_0 = x_0, \quad \xi_1 = -\epsilon^{2k}\eta^2 x_1, \quad \xi_2 = -[2n/(n-4)]\,\epsilon^k\eta x_2,$$

$$\text{where } \eta^{n-2} = -1,$$

$$t = (\epsilon^k/\eta)\tau, \qquad \xi_0 = x_1, \quad \xi_1 = -\epsilon^{2k}\eta^2 x_0, \quad \xi_2 = -[2n/(n-4)]\,\epsilon^k\eta x_2.$$

Among the correlations only the first set written will be polarities.

4. *Special Case of the Sextic.* The foregoing general theory may be illustrated by the ρ^6. The equations of the curve are

$$x_0 = t^6 + 3t^2, \qquad x_1 = 3t^4 + 1, \qquad x_2 = t^5 + t.$$

The cusps are given by t^4-1 and the points of inflexion by t^4+1. On [illegible] we have double cusp tangents and on $x_2=0$ bi-flecnodes. Using the general results as given above we may write the self-dual group of the ρ as follows:

Collineations.

$$t=t' \qquad x_0'=x_0, \quad x_1'=x_1, \quad x_2'=x_2 \qquad (1)$$
$$t=it' \qquad x_0'=-x_0, \quad x_1'=x_1, \quad x_2'=ix_2 \qquad (S)$$
$$t=-t' \qquad x_0'=x_0, \quad x_1'=x_1, \quad x_2'=-x_2 \qquad (S^2)$$
$$t=-it' \qquad x_0'=-x_0, \quad x_1'=x_1, \quad x_2'=-ix_2 \qquad (S^3)$$
$$t=1/t' \qquad x_0'=x_1, \quad x_1'=x_0, \quad x_2'=x_2 \qquad (T)$$
$$t=i/t' \qquad x_0'=-x_1, \quad x_1'=x_0, \quad x_2'=ix_2 \qquad (ST)$$
$$t=-1/t' \qquad x_0'=x_1, \quad x_1'=x_0, \quad x_2'=-x_2 \qquad (S^2T)$$
$$t=-i/t' \qquad x_0'=-x_1, \quad x_1'=x_0, \quad x_2'=-ix_2 \qquad (S^3T)$$

Correlations

$$t=\tau, \qquad \xi_0=x_0, \quad \xi_1=ix_1, \quad \xi_2=-6(i)^{1/2}x_2, \qquad (\pi)$$
$$t=(-i)^{1/2}\tau, \qquad \xi_0=-x_0, \quad \xi_1=ix_1, \quad \xi_2=6(i)^{1/2}x_2, \qquad (\pi S)$$
$$t=-(i)^{1/2}\tau, \qquad \xi_0=x_0, \quad \xi_1=ix_1, \quad \xi_2=6(i)^{1/2}x_2, \qquad (\pi S^2)$$
$$t=(-i)^{1/2}\tau, \qquad \xi_0=-x_0, \quad \xi_1=ix_1, \quad \xi_2=6(-1)^{1/2}x_2, \qquad (\pi S^3)$$
$$t=i^{1/2}\tau, \qquad \xi_0=x_1, \quad \xi_1=ix_0, \quad \xi_2=-6(i)^{1/2}x_2, \qquad (\pi\cdot T)$$
$$t=-(-i)^{1/2}\tau, \qquad \xi_0=-x_1, \quad \xi_1=ix_0, \quad \xi_2=-6(-i)^{1/2}x_2, \qquad (\pi\cdot ST)$$
$$t=-(i)^{1/2}\tau, \qquad \xi_0=x_1, \quad \xi_1=ix_0, \quad \xi_2=6(i)^{1/2}x_2, \qquad (\pi\cdot S^2T)$$
$$t=(-i)^{1/2}\tau, \qquad \xi_0=-x_1, \quad \xi_1=ix_0, \quad \xi_2=6(-i)^{1/2}x_2, \qquad (\pi\cdot S^3T)$$

Perspective ρ^{n-1}. We proceed now to find the equations of a curve of the $(n-1)$-th order which shall be perspective to the ρ^n i. e. a curve such that a line t of the $(n-1)$-ic shall cut out the point t on the ρ. A line joining the points t_1 and t_2 of the ρ^n can be written

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ t_1^{n}+[n/(n-4)]\,t_1^2, & [n/(n-4)]\,t_1^{n-2}+1, & t_1^{n-1}+t_1 \\ t_2^{n}+[n/(n-4)]\,t_2^2, & [n/(n-4)]\,t_2^{n-2}+1, & t_2^{n-1}+t_2 \end{vmatrix}=0,$$

which becomes, after amplifying and factoring out t_1-t_2,

$$\begin{aligned} &\{[n/(n-4)]t_1^{n-2}t_2^{n-2}-[n/(n-4)]t_1t_2(t_1^{n-4}+t_1^{n-5}t_2+\cdots+t_2^{n-4}) \\ &+(t_1^{n-2}+t_1^{n-3}t_2\cdots+t_2^{n-2})+1\}\,x_0 \\ &+\{t_1^{n-1}t_2^{n-1}+t_1t_2(t_1^{n-2}+t_1^{n-3}t_2+\cdots+t_2^{n-2}) \\ &-[n/(n-4)]\,t_1^2t_2^2(t_1^{n-4}+t_1^{n-5}t_2\cdots+t_2^{n-4})+[(n/(n-4)]\,t_1t_2\}x_1 \\ &+\{-[n/(n-4)]\,t_1^{n-2}t_2^{n-2}(t_1+t_2)+[n/(n-4)]^2t_1^2t_2^2(t_1^{n-5} \\ &+t_1^{n-6}t_2+\cdots+t_2^{n-5})-(t_1^{n-1}+t_1^{n-2}t_2+\cdots+t_2^{n-1}) \\ &-[n/(n-4)]\,(t_1+t_2)\}\,x_2=0. \end{aligned}$$

This may be arranged as a polynomial in t_1 as follows:

$$\begin{aligned}
&\{(t_2^{n-1}+t_2)x_1-[n/(n-4)t_2^{n-2}+1]\,x_2\}t_1^{n-1}+\{[n/(n-4)]t_2^{n-2}x_0+x_0\\
&\quad-[n/(n-4)]\,t_2^2x_1+t_2^2x_1-[n/(n-4)]\,t_2^{n-1}x_2-t_2x_2\}t_1^{n-2}\\
&\quad+\{-[n/(n-4)]t_2x_0+t_2x_0-[n/(n-4)]\,t_2^3x_1+t_2^3x_1\\
&\quad+[n/(n-4)]^2t_2^2x_2-t_2^2x_2\}t_1^{n-3}+\{-[n/(n-4)]\,t_2^2x_0+t_2^2x_0\\
&\quad-[n/(n-4)]\,t_2^4x_1+t_2^4x_1+[n/(n-4)]^2t_2^3x_2-t_2^3x_2\}t_1^{n-4}\\
&\quad+\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\quad+\{-[n/(n-4)]\,t_2^{n-4}x_0+t_2^{n-4}x_0-[n/(n-4)]\,t_2^{n-2}x_1+t_2^{n-2}x_1\\
&\quad+[n/(n-4)]^2t_2^{n-3}x_2-t_2^{n-3}x_2\}t_1^2+\{-[n/(n-4)]\,t_2^{n-3}x_0\\
&\quad+t_2^{n-3}x_0+[n/(n-4)]\,t_2x_1+t_2^{n-1}x_1-[n/(n-4)]\,x_2-t_2^{n-2}x_2\}t_1\\
&\quad+\{x_0(t_2^{n-2}+1)-[n/(n-4)]\,t_2x_2-t_2^{n-1}x_2\}=0.
\end{aligned}\tag{10}$$

Given a point t_2 equation (10) gives the $n-1$ points t_1 in which the line through t_2 cuts the curve. Thus an involution, $I_{n-1,1}$ is set up on the curve, the fixed points of which may be written,

$$a_0t^{n-1}+a_1t^{n-2}+\cdots+a_{n-2}t+a_{n-1}=0.\tag{11}$$

The fixed points are apolar to every set; hence writing the condition that (11) be apolar to (10) we have,

$$\begin{aligned}
&a_0\,\{x_0(t^{n-2}+1)-t(t^{n-2}+[n/(n-4)])x_2\}\\
&\quad-a_1\{-[n/(n-4]\,t^{n-3}x_0+t^{n-3}x_0+[n/(n-4)]\,tx_1+t^{n-1}x_1\\
&\quad-[n/(n-4)]\,x_2-t^{n-2}x_2\}+a_2\{-[n/(n-4)]\,t^{n-4}x_0+t^{n-4}x_0\\
&\quad-[n/(n-4)]\,t^{n-2}x_1+t^{n-2}x_1+[n/(n-4)]^2t^{n-3}x_2-t^{n-3}x_2\}\\
&\quad+a_3\,\{-[n/(n-4)]\,t^{n-5}x_0+t^{n-5}x_0-[n/(n-4)]\,t^{n-3}x_1\\
&\quad+t^{n-3}x_1+[n/(n-4)]^2t^{n-4}x_2-t^{n-4}x_2\}\\
&\quad+\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\quad+(-1)^{n-1}\,a_{n-3}\,\{-[n/(n-4)]\,tx_0+tx_0-[n/(n-4)]\,t^3x_1\\
&\quad+t^3x_1+[n/(n-4)]^2t^2x_2-t^2x_2\}-(-1)^{n-1}\,a_{n-2}\{[n/(n-4)]t^{n-2}x_0\\
&\quad+x_0-[n/(n-4)]t^2x_1+t^2x_1-[n/(n-4)]t^{n-1}x_2-tx_2\}\\
&\quad+(-1)^{n-1}\,a_{n-1}\,\{(t^{n-1}+t)x_1-[n/(n-4)]\,t^{n-2}+1)x_2\}=0.
\end{aligned}$$

Since the parameter t runs over the curve, it is seen that the last term consists of the factors a_{n-1} and$x_2x_1-x_1x_2$, the latter of which vanishes identically. Collecting coefficients of x_0, x_1, x_2, and equating to ξ_0, ξ_1, ξ_2, respectively, we have,

(12)

$$\begin{aligned}
\xi_0={}&\{a_0-(-1)^{n-1}[na_{n-2}/(n-4)]\}t^{n-2}+[4a_1/(n-4)]t^{n-3}\\
&-[4a_2/(n-4)]t^{n-4}+\cdots+(-1)^n\,[4a_{n-3}/(n-4)]t+[a_0-(-1)^{n-1}a_{n-2}],
\end{aligned}$$

$$\begin{aligned}\xi_0 &= -a_1 t^{n-1} - [4a_2/n - 4)]t^{n-2} + [4a_3/(n-4)]t^{n-3}\\ &\quad + \cdots + (-1)^{n-1}[4a_{n-2}/(n-4)]\, t^2 - [na_1/(n-4)]\, t,\\ \xi_2 &= \{-a_0 + (-1)^{n-1}[na_{n-2}/(n-4)]\}t^{n-1} + a_1 t^{n-2}\\ &\quad + [8(n-2)/(n-4)^2]\, a_2 t^{n-3} - [8(n-2)/(n-4)^2]\, a_3 t^{n-4}\\ &\quad + \cdots + (-1)^{n-1}[8(n-2)/(n-4)^2]\, a_{n-3} t^2\\ &\quad + \{-[na_0/(n-4)] + (-1)^{n-1} a_{n-2}\}t + [na_1/(n-4)].\end{aligned}$$

Or the equations may be written more briefly as follows:

$$\begin{aligned}\xi_0 &= \{a_0 - (-1)^{n-1}[na_{n-2}/(n-4)]\}t^{n-2}\\ &\quad + [4/(n-4)]\, s\sum_{3}^{n-1}(-1)^{s-1}a_{s-2}\, t^{n-s} + [a_0 - (-1)^{n-1}a_{n-2}],\\ \xi_1 &= -a_1 t^{n-1} - [4/(n-4)]\, s\sum_{3}^{n-1}(-1)^{s-1}a_{s-1}\, t^{n-s+1} - [na_1/(n-4)]t,\\ \xi_2 &= \{-a_0 + (-1)^{n-1}[na_{n-2}/(n-4)]\}t^{n-1} + a_1 t^{n-2}\\ &\quad + [8(n-2)/(n-4)^2]\, s\sum_{3}^{n-2}(-1)^{s-1}a_{s-1}\, t^{n-s}\\ &\quad + \{-[na_0/(n-4)] + (-1)^{n-1}a_{n-2}\}t + [na_1/(n-4)].\end{aligned} \tag{12}$$

It is easily to be verified that, if we take the x's from the ρ^n and the ξ's as written for r^{n-1}, the row product $(x\xi)$ vanishes identically, which means incidence of a point of ρ^n with a line of r^{n-1}.

Among the indefinite number of curves represented by equations (12) there will be a unique curve invariant under the collineation group, i. e. self-projective. The ρ^n as given in lines by equations (6), is invariant under the transformations $t = \epsilon t'$ when combined with,

$$\xi_0' = \xi_0, \qquad \xi_1' = \epsilon^2\xi_1, \qquad \xi_2' = \epsilon\xi_2.$$

We require that the perspective curves (12) be invariant under the same transformations. Since the equation for ξ_0 is to be unchanged then ξ_0 can contain only a constant term and t^{n-2} since $\epsilon^{n-2} = 1$; also the equation for ξ_1 can contain only t^2 since ϵ^2 must factor out; and finally, ξ_2 must contain only t^{n-1} and t since ϵ factors out. Hence the curve may now be written,

$$\begin{aligned}\xi_0 &= \{a_0 - (-1)^{n-1}[na_{n-2}/(n-4)]\}t^{n-2} + [a_0 - (-1)^{n-1}a_{n-2}],\\ \xi_1 &= [4/(n-4)](-1)^{n-1}a_{n-2}\, t^2,\\ \xi_2 &= \{-a_0 + (-1)^{n-1}[na_{n-2}/(n-4)]\}t^{n-1}\\ &\qquad + \{-[na_0/(n-4)] + (-1)^{n-1}a_{n-2}\}t.\end{aligned} \tag{13}$$

Furthermore the curve is to be invariant under $t = 1/t'$, with which go the transformations,

$$\xi_0' = \xi_1, \qquad \xi_1' = \xi_0, \qquad \xi_2' = \xi_2. \tag{14}$$

Making the substitution $t = 1/t'$ in (13) and clearing of fractions we have,

$$\begin{aligned}
\xi_0' &= \{a_0 - (-1)^{n-1}[na_{n-2}/(n-4)]\}t + [a_0 - (-1)^{n-1}a_{n-2}]\,t^{n-1}, \\
\xi_1' &= [4/n-4)](-1)^{n-1}a_{n-2}\,t^{n-3}, \\
\xi_2' &= \{-a_0 + (-1)^{n-1}[na_{n-2}/(n-4)]\} \\
&\qquad + \{-[na_0/(n-4)] + (-1)^{n-2}a_{n-2}\}t^{n-2}.
\end{aligned}$$

It may be seen that conditions (14) are satisfied when the coefficient of t^{n-1} is made to vanish, i. e. $a_0 = (-1)^{n-1}a_{n-2}$. Hence the equations may be written,

$$\xi_0 = t^{n-3}, \qquad \xi_1 = -t, \qquad \xi_2 = -t^{n-2} + 1.$$

The curve has now become one of order $n-2$ instead of $n-1$.

To illustrate the matter of perspective curves consider first the ρ^5. The general set of perspective $(n-1)$-ics become when n equals 5,

$$\begin{aligned}
\xi_0 &= (a_0 - 5a_3)t^3 + 4a_1t^2 - 4a_2t + (a_0 - a_3), \\
\xi_1 &= -a_1t^4 - 4a_2t^3 + 4a_3t^2 - 5a_1t, \\
\xi_2 &= (-a_0 + 5a_3)t^4 + a_1t^3 + 24a_2t^2 + (-5a_0 + a_3)t + 5a_1.
\end{aligned}$$

The equations of the self-dual ρ^5 are,

$$x_0 = t^5 + 5t^2, \qquad x_1 = 5t^3 + 1, \qquad x_2 = t^4 + t.$$

It is easily verified that $(x\xi)$ vanishes. The unique perspective curve invariant under the group is,

$$\xi_0 = t^2, \qquad \xi_1 = -t, \qquad \xi_2 = -t^3 + 1.$$

In the case of the ρ^6 we have for the general set,

$$\begin{aligned}
\xi_0 &= (a_0 + 3a_4)t^4 + 2a_1t^3 - 2a_2t^2 + 2a_3t + (a_0 + a_4), \\
\xi_1 &= -a_1t^5 - 2a_2t^4 + 2a_3t^3 - 2a_4t^2 - 3a_1t, \\
\xi_2 &= (-a_0 - 3a_4)t^5 + a_1t^4 + 8a_2t^3 - 8a_3t^2 + (-3a_0 - a_4)t + 3a_1.
\end{aligned}$$

and the unique perspective curve invariant under the group is,

$$\xi_0 = t^3, \qquad \xi_1 = -t, \qquad \xi_2 = -t^4 + 1.$$

Determination of All the Abstract Groups of Order 72.

By G. A. Miller.

A group G of order 72 may contain either one or four subgroups of order 9, and the number of its subgroups of order 8 must be one of the following three numbers: 1, 3, 9. If it contains only one subgroup of each of the two orders 9 and 8, it must be a direct product of these two subgroups. Since there are two groups of order 9 and five groups of order 8, there are ten distinct such direct products. These ten groups include the six possible abelian groups of order 72. The four non-abelian groups included in this category are the direct products of the two possible groups of order 9 and either the octic group or the quaternion group.

I. *One subgroup of order 9 and three subgroups of order 8.* If G contains three subgroups of order 8, it must transform them according to the symmetric group of degree 3 since the Sylow subgroups of every group must be transformed according to a non-regular substitution group whose order is divisible by the prime number which divides the order of these Sylow subgroups. Hence such a G contains an invariant subgroup of index 2 corresponding to the subgroup of index 2 in the symmetric group of degree 3. As this invariant subgroup contains only one subgroup of each of the orders 9 and 4, it is abelian. Its Sylow subgroup of order 9 must be non-cyclic since it must admit an automorphism in which just three of the operators of the subgroup of order 9 correspond to themselves and one of the other operators of this subgroup is transformed into its inverse. Hence such a group is the direct product of a group of order 3 and a group of order 24 which involves three subgroups of order 8 and only one subgroup of order 3. As there are seven such groups of order 24, there are exactly seven groups of order 72 which contain only one subgroup of order 9 and three subgroups of order 8.

II. *One subgroup of order 9 and nine subgroups of order 8.* The nine subgroups of order 8 are transformed under G according to a transitive group of degree 9 whose order is 18, 36, or 72. If this order is 18, G contains an abelian invariant subgroup of order 36, and it may be constructed by adjoining to this subgroup an operator whose order is a power of 2 and which transforms every operator of the Sylow subgroup of order 9 into its inverse. There are eight such transformations, and two groups correspond to each of six of these while only one group corresponds to each of the other two. Hence there are 14 such distinct groups of order 72.

When the nine subgroups of order 8 are transformed under G according to a group of order 36, the Sylow subgroup of order 9 contained in G must be transformed according to a subgroup of order 4 in its group of isomorphisms, and hence this Sylow subgroup must be non-cyclic. The group of isomorphisms of this Sylow subgroup contains a cyclic and a non-cyclic set of conjugate subgroups of order 4 and each of these subgroups involves the operator of order 2 which transforms every operator into its inverse. Hence G involves either the generalized dihedral or the generalized dicyclic subgroup of order 36. The group of isomorphisms of each of these subgroups is known to be the holomorph of the abelian group of order 18 contained therein * and hence it contains 18 operators which transform the operators of this abelian subgroup in the same way.

As this abelian subgroup is transformed according to a group of order 2 under the group of inner isomorphisms of the said groups of order 36, and as the 18 isomorphisms of order 4 which transform this abelian subgroup in the same manner appear in two sets of conjugates, while the six isomorphisms of order 2 which transform this abelian group in the same way and are not inner isomorphisms also appear in two sets of conjugates under the group of isomorphisms of the given groups of order 36, we have to consider only three automorphisms of each of these two groups. In the case of the generalized dihedral group of order 36, the automorphism of order 2 under which six non-invariant operators of order 2 correspond to themselves evidently gives rise to two groups in which the subgroups of order 8 are abelian and of type (1, 1, 1) and (2, 1) respectively. The automorphism of order 2 under which no non-invariant operator of order 2 corresponds to itself gives rise to only one group, while the automorphism of order 4 gives rise to an additional group of order 72. Hence there are four such groups in this case. As there are also four such groups which involve the generalized dicyclic group of order 36, there are eight groups of order 72 which satisfy the conditions that each of them contains only one subgroup of order 9 and nine subgroups of order 8 and transforms these nine subgroups according to a group of order 36.

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* Miller, Blichfeldt, Dickson, *Finite Groups* (1916), pp. 169 and 170.

When the nine subgroups of order 8 are transformed according to a grou of order 72, the groups must be simply isomorphous with a transitive grou of degree 9 and of order 72. It is well known that there are three suc groups and that they are abstractly distinct.* Hence the total number groups of order 72 which separately involve only one subgroup of order and nine subgroups of order 8 is 25. It will be found that this is exactl [illegible] the number of the distinct groups of order 72.

II. *Four subgroups of order 9.* The four subgroups of order 9 ma be transformed under G either according to the alternating or according t the symmetric group of degree 4. In the former case, each subgroup order 8 contained in G must transform the four subgroups of order 9 ac cording to the non-cyclic regular group of order 4, while in the latter cas the four subgroups are transformed under a subgroup of order 8 accordi to the octic group. In each case, these four subgroups have a cross-cut order 3 which is invariant under G. When these subgroups are transform under G according to the alternating group of degree 4, then G invo an invariant subgroup of order 24 which contains only one subgroup order 3. If this subgroup contains also only one subgroup of order 8, th latter subgroup must admit an automorphism of order 3 and hence it either the quaternion group or the abelian group of type (1, 1, 1). Ther are four such groups of order 72. Two of them involve operators of orde [illegible] the other two do not have this property.

If the given subgroup of order 24 contains three subgroups of order 8 it must involve as an invariant subgroup the symmetric group of order 6. As this is a complete group, the subgroup of order 24 must be the direct product of this group of order 6 and the non-cyclic group of order 4. Hence it must involve a subgroup of order 12 composed of all its operators which are commutative with every operator of the said invariant symmetric group of order 6, and it must therefore be the direct product of the tetrahedral group and this symmetric group. It remains therefore to determine the groups of order 72 which involve four subgroups of order 9, and transform these subgroups according to the symmetric group of degree 4. If such a group contains no operator of order 9, it must involve two operators of order 3 whose product is of order 2, and hence it must contain as an invariant subgroup the direct product of the tetrahedral group and a group of order 3. To this direct product we may add two operators of order 2 which transform the tetrahedral group according to an outer isomorphism

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* F. N. Cole, *Bulletin of the American Mathematical Society*, Vol. 2 (1892), p. 250

One of these operators is commutative with the invariant operator of order 3 in the given direct product while the other transforms this operator into its inverse. Hence there are two distinct groups of order 72 which involve this direct product and transform its four subgroups of order 9 according to the symmetric group of order 24.

When the four subgroups of order 9 contained in G are cyclic and are transformed under G according to the symmetric group of degree 4, the subgroup of order 36 which corresponds to the alternating group of degree 4 involves two operators s_1, s_2 of order 9 which satisfy the following two conditions:

$$s_1^3 = s_2^{-3}, \qquad (s_1s_2)^2 = 1.$$

These equations represent a generalization of those which define the tetrahedral group. It is easy to prove that when s_1 and s_2 are two non-commutative operators which satisfy these two equations then 1, s_1s_2, s_2s_1, $s_1^{-1}s_2s_1^2$ is the non-cyclic group of order 4, and this is the commutator subgroup of the group generated by s_1, s_2. This subgroup of order 4 and s_1^3 generate an invariant abelian subgroup of index 3 under the group generated by s_1, s_2. Moreover, the commutator quotient group is always cyclic and these generators may be so chosen that the order of this invariant subgroup is an arbitrary multiple of 4. In the present case this order is 12 and the order of all the additional operators in the group generated by s_1, s_2 is 9. In general, there is one and only one such group whose order is an arbitrary multiple of 12. Since an operator of order 2 which is adjoined to the given group of order 36 to obtain the group of order 72 transforms two of the four cyclic subgroups of order 9 into themselves and is not commutative with their generators, it must transform their operators into their inverses, and hence the automorphism of the given subgroup of order 36 is completely determined. There are therefore eight groups of order 72 which contain separately four subgroups of order 9 and *the total number of groups of order 72 is 50.* Each of these groups is obviously solvable. They may be classified as follows:

Number of subgroups of order 8....	1	3	9	1	3
Number of subgroups of order 9....	1	1	1	4	4
Number of groups of order 72.....	10	7	25	4	4

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Geometric Aspects of the Abelian Modular Functions of Genus Four. Part II.

By Arthur B. Coble.

Introduction.

This article is a continuation of one which has appeared under the same title in this *Journal* [Vol. 46 (1924), pp. 143-192]. The sections are numbered consecutively with Part I. The notations of the earlier article are carried over without further explanation. In the meantime the author's book entitled *Algebraic Geometry and Theta Functions*[*] has appeared. This contains an exposition of some of the matter treated in Part I along with some further results (Chap. V, **54**, **55**, **56**). It contains also a sequence of lettered theorems which are unproved [**54** (*a*), $\cdots$, (*h*); **55** (*a*), (*b*), (*c*); **56** (*a*), (*b*)]. It is the purpose of this article to develop the general subject under consideration somewhat further, and incidentally to give proofs of the theorems mentioned.

11. The Stahl Quadric of the Symmetroid and the Perspective Curves of the Planar Rational Sextic.

Let the rational sextic locus of ∞^1 planes in space $S_3(z)$, $R_2(t)$, whose point sections are apolar to the line sections of the planar rational sextic, $\bar{S}_2(t)$, be expressed parametrically (with binary parameter t) as

$$(1) \qquad (\beta z)(bt)^6 = 0.$$

Then any linear pentad on $\bar{S}_2(t)$, with points determined by $t = t_2, \cdots, t_6$ or by $(qt)^5 = 0$, which contains the further point $t = t_1$ of $\bar{S}_2(t)$, will have, for polar as to (1), the value

$$(2) \qquad (\beta z)(bq)^5(bt) = \rho \cdot (z\zeta) \cdot (t_1 t).$$

For, $(qt)^5 \cdot (t_1 t)$ is apolar to (1) for every z whence the polar of $(qt)^5$ alone is either $(t_1 t)$ or zero. The apolarity must occur for points z on a plane ζ. The locus of planes ζ, each a locus of points z with point sections apolar to a fixed linear pentad on $\bar{S}_2(t)$, is the Stahl quadric $\bar{K}$. If the linear pentad q revolves about a further point t_1 of $\bar{S}_2(t)$, i. e., if q runs over a pencil, the plane ζ of $\bar{K}$ rotates about a generator $t = t_1$ of $\bar{K}$. The other set of generators τ of $\bar{K}$ arises from the perspective cubic curves of $\bar{S}_2(t)$. When these are written as in **5** (1),

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[*] *Colloquium Publications*, Vol. X, of the American Mathematical Society, New York, (1929); cited hereafter as "A. G."

$$(3) \qquad (\pi x)(a\tau)(\alpha t)^3 = 0,$$

with x, ξ as dual ternary variables, the sextic $\bar{S}_2(t)$ is

$$(4) \qquad (\pi\pi'\xi)(aa')(\alpha t)^3(\alpha' t)^3 = 0.$$

The line $t = t_1$ of the perspective cubic τ in (3) meets $\bar{S}_2(t)$ in (4) in the point $t = t_1$, and in the linear pentad $(qt)^5$ given by [cf. **5** (5)],

$$(5) \qquad (\pi\pi'\pi'')(aa')(a''\tau)(\alpha t)^3(\alpha' t)^3(\alpha'' t_1)^3$$
$$= (tt_1)\cdot f(\tau^1, t_1^2, t^5) = (tt_1)\cdot(qt).^5$$

For fixed τ and variable t_1 in $f(\tau^1, t_1^2, t^5) = 0$, the plane ζ of $\bar{K}$, which corresponds to the linear pentad in t, turns about the generator τ of $\bar{K}$ [cf. A. G., **54** (24)].

If a pencil of binary sextics has an apolar net of binary quintics (rather than a pencil), the sextics are the first polars of a binary septimic to which the quintics are apolar. For given τ, the linear pentads, $f(t^5, \tau^1, t_1^2) = 0$, on the lines t_1 of the perspective cubic τ are apolar to the pencil of point sections of $R(t)$ from points z on the generator τ of $\bar{K}$. For variable t_1, these pentads lie in a net, and therefore the point sections are polars of a septimic. As τ varies, this septimic,

$$(6) \qquad (c\tau)(\gamma t)^7 = 0,$$

must lie in a pencil, and therefore must contain the parameter τ linearly, since its polars,

$$(7) \qquad (c\tau)(\gamma t_1)(\gamma t)^6 = 0,$$

must be included in the linear system (∞^3) of point sections of $R_2(t)$. We prove that

(8) *The equation* (7) *is the parametric equation* (1) *of* $\bar{R}_2(t)$ *in which the coordinates* z *in* S_3 *are referred to* $\bar{K}$ *with generators* $\tau, t_1 = t$ *as in* **7** (1) [cf. A. G., **54** (a)].

Evidently (7) is the equation of $\bar{R}_2(t)$ referred to some quadric $\bar{K}'$. Let a plane ζ of $\bar{K}'$ contain generators τ_1, t_1. Then three independent points of ζ are τ_1, t_1; τ_1, t_2; and τ_2, t_1; and the corresponding point sections of $\bar{R}_2(t)$ are $(c\tau_1)(\gamma t_1)(\gamma t)^6 = 0$, $(c\tau_1)(\gamma t_2)(\gamma t)^6 = 0$, $(c\tau_2)(\gamma t_1)(\gamma t)^6 = 0$. Since the first two are polars of $(c\tau_1)(\gamma t)^7 = 0$, their common apolar quintics are those of the net apolar to this septimic. The polars of this net as to the [illegible] $(\epsilon\tau)(\gamma t)$ constitute a net of quadratics which [illegible]. [illegible] one quintic, $(at)^5$, apolar to the first two sextics [illegible] [illegible] [illegible] polar as to $(\epsilon\tau)(\gamma t)^7$; and this quintic [illegible] [illegible] $\gamma')(\gamma t)$. Thus the three-point sections have a common apolar [illegible] [illegible] plane ζ is a plane of $\bar{K}$, i. e., $\bar{K}'$ coincides with $\bar{K}$.

[illegible] the parameters of the double tangent of the perspective [illegible] [illegible] by giving τ in (3). The triad of cusps of this cubic envelope [illegible] [illegible] $(\bar{\alpha}'t)$ [illegible] (cf. **5**, pp. 163-4). Then t_1, t_2 are the [illegible] of this binary cubic in t, i. e.,

$$(\bar{\alpha}\bar{\alpha}')^2(\bar{\alpha}t)(\bar{\alpha}'t)(\alpha\tau)^3(\bar{\alpha}'\tau)^3 = k\cdot(tt_1)\cdot(tt_2)$$

[illegible] section of $\bar{S}_2(t)$ by this double tangent consist of th[illegible] [illegible] t_6 in addition to t_1, t_2. The three linear pentads, omitting [illegible] successively from this linear hexad, determine p[illegible] [illegible] parameters are, respectively, (τ, t_1), (τ, t_2), (τ', t_3), where τ' [illegible] [illegible] $(\tau x)(\alpha\tau)(\alpha t)$ [illegible] $= 0$ when $t = t_3$ and x is $x(t_3)$ on $S_2(t)$. [illegible] planes meet in the generator τ of $\bar{K}$ and the third cross [illegible] [illegible] at the point $z = (\tau, t_3)$ of $\bar{K}$. But these planes meet in t[illegible] [illegible] on Σ. On the other hand, the equation satisfied by a po[illegible] [illegible] K is obtained by setting the catalecticant of the binary sextic [illegible] [illegible], i. e.,

$$(e\tau)^2(\epsilon t)^4 = 0.$$

[illegible] [cf. A. G., **54** (b)].

The equation of the octavic curve of intersection of Σ and K [illegible] *of the generators* [illegible] *on $\bar{K}$ is given by* (10). *For* [illegible] *the quartic in t determines the further linear* [illegible] *out by the double tangent of the perspective* [illegible] *parameters are given by* (9).

Since the double tangent of the perspective cubic envelope [illegible] [illegible] to the cubic, it is projectively related to the sextic, $\bar{S}_2(t)$. [illegible] [illegible] joining corresponding points of the double tangent and th[illegible] [illegible] the perspective cubic envelope. Thus the four points [illegible] [illegible] double tangent and the sextic are self-corresponding and [illegible] [illegible]. Hence [cf. A. G., **54** (c)].

(12) *The only linear tetrads of $\bar{S}_2(t)$ such that the four points on their line are projective to their parameters on $\bar{S}_2(t)$ are the sets of four residual points of $\bar{S}_2(t)$ on the double tangents of the perspective cubics of $\bar{S}_2(t)$.*

The problem of determining linear pentads of $\bar{S}_2(t)$ whose parameters on their line are projective to their parameters on $\bar{S}_2(t)$ is poristic. The projectivity imposes only two conditions on the ∞^2 possible pentads. If, however, one pentad exists which satisfies the required projective relation, the locus of lines joining corresponding points of the line and sextic, which ordinarily would be a septimic, reduces to a conic, since five points are self-corresponding. The conic is necessarily a perspective conic of the sextic and the sextic satisfies one condition. If such a perspective conic exists, the problem has ∞^1 solutions, namely, the ∞^1 tangents of the perspective conic.

The parametric equation of the double tangent, or perspective line, of a cubic envelope is of degree five in the coefficients of the parametric equation of the envelope.* Being of degree five in the coefficients of (3), the parametric equation of the double tangent has the form, $g(\xi^1, \tau^5, t^1) = 0$; and the ternary equation has the form, $g(x^1, \tau^{10}) = 0$. Hence [cf. A. G., **54** (b)].

(13) *The double tangents of the perspective cubics of $\bar{S}_2(t)$ envelop a rational curve of class* 10, *$R^{10}(\tau)$. The ten tangents of $R^{10}(\tau)$ on a point of $\bar{S}_2(t)$ divide into a set of six, and a set of four, whose parameters τ are determined by* (9) *and* (10). *For the set of six, t belongs to the pair t_1, t_2; and, for the set of four, t belongs to the tetrad $t_3, \cdots, t_6$.*

The form, $g(x^1, \tau^{10})$, is of degree ten in the coefficients of (3). When x on $\bar{S}_2(t)$ is substituted from (4), the result is the product of the forms (9) and (10). Since (9) is of degree six in the coefficients of (3), there follows that (10) has the same degree, six.

In order to interpret the form (10) on the jacobian J we recall (cf. **10**, p. 188) that, if y is a point on J, the two trihedrals of planes of the cubic curves, $C_1(\tau)$, $C_2(t)$, on y (with parameters τ_1, τ_2, τ_3; t_1, t_2, t_3 respectively) are six planes of a quadric cone, $q'(y)$, with vertex at y. If y is so placed on J that this cone breaks up into two pencils of planes on lines L, L' through y, then L must contain planes τ_1, τ_2, t_3, and L' must contain planes τ_3, t_1, t_2. Thus y is a point where an axis τ_1, τ_2 in a plane t_3 of C_2 meets an axis t_1, t_2 of C_2 in a plane τ_3 of C_1. We recall from **5** (3) that, in the plane τ_3 of C_1,

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[*] Cf. A. B. Coble, "Symmetric Binary Forms and Involutions, III"; this *Journal*, Vol. 32 (1910), pp. 333-364; in particular, p. 353.

the perspective cubic ($\tau = \tau_3$) is cut out by the planes of $C_2(t)$; and $\bar{S}_2(t)$ is cut out by the axes of C_1 which lie in the planes of $C_2(t)$. Hence L' is the double tangent of the perspective cubic τ_3 with parameters t_1, t_2, which meets $\bar{S}_2(t)$ at $x(t = t_3)$. This is the situation which, as observed above, gives rise to a plane $\zeta(t_3, \tau_3)$ on the Stahl quadric $\bar{K}$ tangent at the point $z(t_3, \tau_3)$ which on Σ is the point $z(t_4, t_5, t_6)$. The birationally related point on J is $y(t_4, t_5, t_6)$, the image in I [cf. **10** (2)] of the point $y(t_1, t_2, t_3)$. Hence [cf. A. G., **54** (d)].

(14) *The locus on J which corresponds to the intersection of $\bar{K}$ with Σ is the image in I of the locus of points y for which the cone $q'(y)$ of* **10** (5) *breaks up into two pencils of planes,* i. e., *the points y in which the axis of C_2 in the plane τ of C_1 meets the axis of C_1 in the plane t of C_2. For such points y, the t, τ must satisfy* (10).

A perspective envelope, E, of $\bar{S}_2(t)$ of class $m + 3$ is a rational locus with parameter t such that line t of E cuts $\bar{S}_2(t)$ in the point t, and a further linear pentad which corresponds on $\bar{K}$ to a point t, τ. For every t there is just one pentad, and therefore one τ. For every τ, or perspective cubic, there are m tangents, common to the cubic and E, which have the same t since the two envelopes generate $\bar{S}_2(t)$ of order six. Hence the parameters t, τ of the pentads of E satisfy a relation of the form, $(k\tau)(\kappa t)^m = 0$. The parametric equation of E is obtained by the elimination of τ from this relation and $(\pi x)(a\tau)(\alpha t)^3 = 0$. Hence [cf. A. G., **54** (e)].

(15) *The family of perspective envelopes of $\bar{S}_2(t)$ of class $m + 3$ corresponds in* (1, 1) *fashion to the family of ∞^{2m+1} curves on $\bar{K}$ of order $m + 1$ and type $(k\tau)(\kappa t)^m = 0$. The parametric equation of the family is $(\pi x)(ak)(\kappa t)^m(\alpha t)^3 = 0$.*

In this last theorem it has seemed more convenient to think of t, τ as a point on $\bar{K}$. We shall continue this view throughout the following discussion of the perspective quartic envelopes of $\bar{S}_2(t)$. The symmetroid Σ is then the quartic envelope of planes which cut a rational space sextic curve, $R_2(t)$, in catalectic sextics; and $\bar{K}$ is the locus of points z such that plane sections of $\bar{R}_2(t)$ by planes on z have a common apolar quintic which determines a linear pentad on $\bar{S}_2(t)$. If $(m\tau)(\mu t) = 0$ is the equation of a generic plane section of $\bar{K}$, whose coordinates ζ are the coefficients $m_{ij} = m_i\mu_j$ [cf. **7** (2)], then

$$(\pi x)(am)(\mu t)(\alpha t)^3 = 0 \qquad (16)$$

is the generic perspective quartic envelope of $\bar{S}_2(t)$. If this is written in the form,

$$(17) \qquad \xi_0 = (\beta t)^4,\ \xi_1 = (\gamma t)^4,\ \xi_2 = (\delta t)^4,$$

where the coefficients β, γ, δ are linear in m_{ij}, the condition that the perspective envelope have a perspective line is (cf. *loc. cit.*)

$$(18) \qquad \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 \\ 0 & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & 0 \\ 0 & \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & 0 \\ 0 & \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{vmatrix} = 0.$$

This condition, of degree 6 in $\zeta = m_{ij}$, is the product of the equations of the quadric envelope $\bar{K}$, and of the quartic envelope Σ. For, the curve (16) has a perspective line, either if it is a proper quartic envelope which has a tritangent, or if it is a perspective cubic with a bitangent. The second case occurs when $(m\tau)(\mu t)$ factors into $(t_1 t)\cdot(\tau_1 \tau)$. Then (16) is the perspective cubic τ_1, i. e., $(\pi x)(a\tau_1)(\alpha t)^3 \cdot (t_1 t) = 0$; also the plane $\zeta = m_{ij}$ is a plane of $\bar{K}$. If, on the other hand, (16) is a proper quartic envelope with a tritangent ξ with parameters t_1, t_2, t_3 then ξ cuts $\bar{S}_2(t)$ in points $t_1, \cdots, t_6$, and the three pentads, obtained by dropping t_1, t_2, t_3 successively, correspond to three points on $\bar{K}$, and on its section by the plane $\zeta = m_{ij}$ which is also the plane $\zeta(t_4, t_5, t_6)$ of Σ. Hence

(19) *The tritangent condition of the curve* (16) *of class four factors into the equation,* $(mm')(\mu\mu') = 0$, *of* $\bar{K}$, *and into the equation of the symmetroid* Σ *of class four, and degree six in the coefficients of* (3).

If, in this equation of Σ, the m, μ are replaced by t, τ, the equation (10) of the section of Σ by $\bar{K}$ is obtained.

If $x' = x'(t')$ is a point of $\bar{S}_2(t)$ with parameter t', then

$$(\pi x')(a\tau)(\alpha t)^3 = (tt')\cdot f(\tau^1, t^2),$$

since, for any τ, one of the three tangents on x' of the perspective cubic has the parameter t'. If p_i $(i = 1, \cdots, 9, 0)$ is one of the ten nodes, $P_{10}{}^2$, of $\bar{S}_2(t)$, with nodal parameters $(q_i t)^2$, then

$$(20) \qquad (\pi p_i)(a\tau)(\alpha t)^3 = (\lambda_i t)(l_i \tau)\cdot(q_i t)^2.$$

Now the pencil of lines on p_i, covered four times, is a degenerate perspective quartic of $\bar{S}_2(t)$. This particular perspective curve is obtained when

$(m\tau)(\mu t)$ in (16) is $(\lambda_i t)(l_i\tau)$. For, if τ is eliminated between (20) and (3), the resulting line,

$$(\pi x)(\pi' p_i)(\alpha\alpha')(\alpha t)^3(\alpha' t)^3 = 0,$$

vanishes when $x = p_i$ for every t, since the left member then changes sign on interchange of equivalent symbols. Let the "incidence condition" of the perspective quartic be the condition that its tangents t_1, t_2, t_3 are on a point. Formed for (16), it has the degrees,

$$f(m^3, \mu^3, t_1^2, t_2^2, t_3^2) = 0. \tag{21}$$

The tritangent condition of (19) is of degree two in the coefficients of (21). But when the perspective curve is the four-fold point p_i, the condition (21) is satisfied for all values of t_1, t_2, t_3. Hence, for variation of t_1, t_2, t_3, the envelope (21) is in a linear system (∞^9) of cubic envelopes on the 10 planes, $(\lambda_i t)(l_i\tau) = 0$, of (20). Since, according to (19), $\Sigma\bar{K}$ is a quadratic in these cubic envelopes, the 10 planes must be the double planes, or tropes, of Σ. Hence

(22) *The ten bilinear equations,* $(\lambda_i t)(l_i\tau) = 0$, *in* (20) *are the sections of* $\bar{K}$ *by the ten tropes of* Σ. *If* A_i *is a common axis on* J *of the cubic curves,* $C_1(\tau)$, $C_2(t)$, *and if* A_i *is on the two planes* $(q_i\tau)^2$ *of* C_1 *and the two planes* $(q_i t)^2$ *of* C_2, *then, as* y *runs over* A_i, *the further planes* t, τ *of* C_1, C_2 *on* y *are related as in* $(\lambda_i t)(l_i\tau) = 0$. *For variation of* t_1, t_2, t_3 *in* (21), *the linear system of cubic adjoint surfaces of* Σ *is obtained.*

The statement concerning the axes A_i of J is an immediate consequence of the constructions given in **5** (3).

12. Loci Associated with a Division of the Nodes of the Rational Sextic and the Symmetroid.

Let $q(i, t)^0$ $(i = 1, \cdots, 9, 0)$ denote the binary quadratic in t which determines the pair of nodal parameters of $\bar{S}_2(t)$ at the node p_i; and $q(ij, t)^1$ the quadratic which determines the parameters of the two further intersections of $\bar{S}_2(t)$ with a line on the nodes $p_i p_j$. The corresponding quadratics for the paired sextic $\bar{S}_1(\tau)$ will be distinguished by the parameter τ.

The pencil of lines on the node p_1 of $\bar{S}_2(t)$ will cut $\bar{S}_2(t)$ in the fixed pair of points $q(1, t)^0$ and a variable tetrad t_3, t_4, t_5, t_6 which lies in an I_1^3 with equation

$$f(t_3^3, t_4^3) = 0. \tag{1}$$

The planes $t_3, \cdots, t_6$ of the variable tetrad in I_1^4 on the cubic curve $C_2(t)$ will determine a tetrahedron whose vertices will run over a twisted cubic curve N_1. To the particular vertex $y(t_4, t_5, t_6)$ of N_1 on J, opposite to the plane t_3, we may ascribe the binary parameter $t = t_3$; and thus there is established on N_1 a binary coordinate system $N_1(t)$. Then the axis t_3, t_4 of $C_2(t)$ is the bisecant t_5, t_6 of $N_1(t)$. Two cubic curves so placed that an infinity of bisecants of the one are axes of the other are called Hurwitz curves (cf. Meyer, Apolarität, p. 315). The particular line on p_1 which passes through p_2 cuts $\bar{S}_2(t)$ in the tetrad $q(2,t)^0 \cdot q(12,t)^1$. Hence the axis A_2, or $q(2,t)^0$, of $C_2(t)$, one of the ten common axes of $C_1(\tau)$, $C_2(t)$, is the bisecant $q(12,t)^1$ of $N_1(t)$. The point $t = t_3$ of $N_1(t)$ is the point $y(t_4, t_5, t_6)$ on J; and this is the image in the involution I of the point $y[q(1,t)^0, t_3]$ of A_1 on J.

In precisely the same way the curve N_1 arises from the I_1^4 cut out on $\bar{S}_1(\tau)$ by lines on its node q_1. The axis A_2 is the axis $q(2,\tau)^0$ of $C_1(\tau)$ as well as the bisecant $q(12,\tau)^1$ of N_1. The parameter τ on N_1 is that of the point of A_1 which corresponds under I to the point of N_1 whence the parameters t, τ on N_1 are related by the bilinear equation, $(\lambda_1 t)(l_1 \tau) = 0$, of **11** (22).

Since $q(12,t)^1$ and $q(12,\tau)^1$ are interchanged under the projectivity $(\lambda_1 t)(l_1 \tau) = 0$ and therefore also under the projectivity, $(\lambda_2 t)(l_2 \tau) = 0$, either quadratic must be fixed under the product of the projectivities. Hence [cf. A. G., **54** (f), (g)].

(2) *The cubic curve N_i on J, the transform of the line A_i on J under I, has the common axis A_j of $C_1(\tau), C_2(t)$ for bisecant with parameters $q(ij,\tau)^1$ or $q(ij,t)^1$ $(i, j = 1, \cdots, 9, 0; i \neq j)$. The parameters t, τ on N_i are related by $(\lambda_i t)(l_i \tau) = 0$. The 45 pairs of quadratics $q(ij,t)^1, q(ij,\tau)^1$ arise from the common solutions of the 45 pairs of projectivities, $(\lambda_i t)(l_i \tau) = 0$, $(\lambda_j t)(l_j \tau) = 0$, i. e., $q(ij,t)^1 = (\lambda_i t)(\lambda_j t)(l_i l_j)$, and $q(ij,\tau)^1 = (\lambda_i \lambda_j)(l_i \tau)(l_j \tau)$.*

We have seen [cf. **10** (14)] that the points of an axis A_i on J correspond to the directions at the node D_i of Σ. The curve N_i is the image of A_i in the involution I. This involution on J can be effected by the cubic Cremona involution determined by any net of quadrics in the web whose jacobian is J. The Cremona involution has a sextic F-curve which corresponds to a plane section of Σ. Since the involution transforms N_i back into A_i, N_i must meet the F-curve in 8 points. Hence N_i must correspond on Σ to a rational octavic m_i^8. Since N_i is cut by A_j in a pair of points with parameters $q(ij,t)^1$, the octavic m_i^8 must have a node at D_j with nodal parameters

$q(ij, t)^1$. The quadric M_i on the nine nodes other than D_i meets m_i^8 in 18 points at the nodes, and therefore contains it, i. e., m_i^8 is the intersection of Σ and M_i [cf. A. G., **54** (h)].

The equation satisfies by planes t, τ of the Stahl quadric $\bar{K}$ on the node D_i, or also the equation of the conic section of $\bar{K}$ by the polar plane of D_i, is $(\lambda_i t)(l_i \tau) = 0$ [cf. **11** (2) in dual form]. Hence the 45 quadratics in (2) are the parameters t (or parameters τ) of the pairs of planes of $\bar{K}$ on each of the 45 lines joining a pair of nodes D_i, D_j.

The following theorem is useful as a lemma [cf. A. G., **55** (a)]:

(3) *If two generic cubic space curves N_1, N_2, and eight of their ten common bisecants, say $A_3, \cdots, A_9, A_0$ are given, there exists a unique quartic surface J which contains N_1, N_2, and A_j $(j = 3, \cdots, 9, 0)$. This surface J is the jacobian of a web of quadrics apolar to the quadric envelopes on a unique pair of cubic curves C_1, C_2 for which the A_j are common axes. Either curve C with either curve N is a pair of Hurwitz curves. The relation between the pairs N_1, N_2 and C_1, C_2 is mutual but the transition from one pair to the other is reversed by a dual process.*

The equation of a quartic surface J contains 35 coefficients. To contain N_1, N_2 imposes 26 linear conditions upon these coefficients, and to contain the eight A_j imposes 8 more. There exists therefore at least one surface J on these curves. If there were a pencil of surfaces J, the base curve of the pencil would consist of N_1, N_2, A_j and a further conic B. This conic B could not meet every A_j in two points since the A_j are not coplanar. Suppose it meets A_3 in at most one point. Pass a plane π through A_3 to avoid the points common to two base curves which are not on A_3. Then π would meet the pencil in A_3, and in a pencil of cubic curves on a further point of N_1, of N_2, of $A_4, \cdots, A_9, A_0$, and of B. This pencil on ten points would have a common part which would belong to the base curve. But every plane section on A_3 could not contain an additional part of the base curve and thus J must be unique. The two cubic curves N_1, N_2 determine two paired planar rational sextics for which on either the eight common bisecants determine eight of the nodes. On N_1 (and similarly on N_2) the two further nodes determine two I_1^4's for which the eight pairs of parameters of A_j are pairs of the I_1^4. The planes of the triads of these I_1^4's envelope curves $C_1(\tau), C_2(t)$ for which the A_j are common axes. The jacobian J of the web of point quadrics apolar to the nets of quadric envelopes on $C_1(\tau), C_2(t)$ must contain the 8 common axes A_j of $C_1(\tau), C_2(t)$; and therefore must contain N_1, N_2, and must coincide with the J uniquely determined as above. Thus $C_1(\tau), C_2(t)$ are related to N_1, N_2 in a manner dual to that in which N_1, N_2 are related to $C_1(\tau), C_2(t)$. From this there follows [cf. A. G., **55** (b)]:

(4) *A planar rational sextic curve is uniquely determined, to within a projectivity, when the pairs of parameters of eight nodes are given.*

Naturally the eight pairs of parameters can not be selected at random. We take up later (cf. **14**) the relations among the further nodal pairs of the five rational sextics which are projectively determined when six nodal pairs are given at random. This theorem (4) is the basis for the proof (cf. A. G., p. 251) that the rational sextic, (N_1, N_2), whose nodal pairs are those cut out on N_1 by the ten common bisecants of N_1, N_2, is the transform of $\bar{S}_2(t)$, or $(\bar{C}_2, \bar{C}_1)$, by $J^5{}_{1,2}$, the Jonquières transformation of order five with fourfold F-point at the node p_1 of $\bar{S}_2(t)$ and simple F-points at $p_3, \cdots, p_9, p_0$. From this in turn there is derived a Cremona transformation, R_1, which will transform $\bar{S}_2(t)$ into its paired sextic $\bar{S}_1(\tau)$, i. e., will transform $(\bar{C}_2, \bar{C}_1)$ into $(\bar{C}_1, \bar{C}_2)$.

The symmetroid Σ with nodes at $D_1, \cdots, D_0$ is transformed by the cubic transformation A_{1234} with F-points at $D_1, \cdots, D_4$ into a symmetroid Σ' with nodes at $D_1', \cdots, D_0'$ [cf. A. G., **53** (7)]. We seek a Cremona transformation which will transform the J of Σ into the J' of Σ'. Such a transformation is not unique since J is transformed into itself by an infinite Cremona group. We prove that a simple transformation of the type required is the cubic transformation with F-curves consisting of N_1 and its bisecants A_2, A_3, A_4.

The generic spatial cubic transformation has sextic curves of genus three for direct and inverse F-curves such that the P-surface of either is the octavic locus of trisecants of the other. In the particular case when the one F-curve in $S_3(y)$ degenerates into a cubic curve c^3 and three bisecants b_1, b_2, b_3 the other in $S_3(Y)$ degenerates into a cubic curve C^3 and three bisecants B_1, B_2, B_3. The P-surface in $S_3(y)$ consists of four reguli r, r_1, r_2, r_3 whose lines correspond respectively to the points on C^3, B_1, B_2, B_3. The regulus r crosses b_1, b_2, b_3, and the regulus r_i crosses c^3, b_j, b_k $(i, j, k = 1, 2, 3)$. The P-surface in $S_3(Y)$ with reguli R, R_1, R_2, R_3 is similarly defined with reference to C^3, B_1, B_2, B_3. From the image in $S_3(Y)$ of a plane on b_i in $S_3(Y)$ there factors the regulus R_i leaving a plane on B_i. Thus the pencils of corresponding planes on b_i, B_i are projectively related.

The image of a generic line in $S_3(y)$ is a cubic curve bisecant to C^3, B_1, B_2, B_3. If the line is a bisecant of c^3, two rays of R separate from the image leaving a line bisecant to C^3. Thus the transformation sets up a correspondence between the bisecants of c^3 and the bisecants of C^3 which has been studied by Montesano (cf. A. G., p. 258 for references). If the parameters t, t' of the points of intersection of a bisecant of c^3 are used to determine a point p in a plane with reference to a norm conic, the bisecants of c^3 are mapped on the points p of a plane. With a similar mapping for the bisecants of C^3, the correspondence mentioned turns out to be quadratic with F-triangles determined by b_1, b_2, b_3 and B_1, B_2, B_3.

The image of a quartic surface on c^3, b_1, b_2, b_3 is a quartic surface on C^3, B_1, B_2, B_3. The jacobian J of a web of quadrics can be defined to be a quartic surface containing N_1, N_2 and the eight common bisecants $A_3, \cdots, A_9, A_0$. If N_1, A_3, A_4, A_5 are identified with c^3, b_1, b_2, b_3 the image of N_2 is a cubic curve N_2' bisecant to B_1, B_2, B_3. The images of $A_6, \cdots, A_9, A_0$ are common bisecants, $A_6', \cdots, A_9', A_0$ of N_2' and C^3. If C^3, B_1, B_2, B_3 are denoted by N_1', A_3', A_4', A_5' the image J' of J is a quartic surface on N_1', N_2', and eight of their common bisecants, $A_3', \cdots, A_0'$. Hence

(5) *The jacobian J of a web of quadrics is transformed by the cubic transformation whose sextic F-curve consists of N_i, A_j, A_k, A_l into a jacobian J'.*

The jacobian J is defined by the axes A_i, or by the sextic curve $\bar{S}_2(t) = (\bar{C}_2\ (t), \bar{C}_1(\tau))$. We seek the sextic curve which defines J'. The sextic curve (N_1, N_2) with eight common bisecants $A_3, \cdots, A_0$ is the transform of $\bar{S}_2(t)$ by $J^5{}_{1,2}$. Under the cubic transformation with sextic F-curve, N_1, A_3, A_4, A_5, the effect on the bisecants is obtained by first plotting their pairs of parameters with respect to a norm conic. This plotting operation on the bisecants which determine the nodes of (N_1, N_2) furnishes in the plane the nodes of the sextic paired with (N_1, N_2) [cf. **4** (4)]. But in the plane of the sextic the paired sextic is obtained by effecting the transformation R_1 (cf. A. G., p. 252). In the plane of the norm conic the quadratic transformation is then applied to obtain the pairs of parameters of the new set of nodes with respect to the new norm conic. These pairs are restored to the cubic curve N_1' by a second use of R_1. On N_1' the nodal set of (N_1', N_2') determines the lines A_i' of J'. These are axes for the sextic $\bar{S}_2'(t) = (\bar{C}_2'(t), \bar{C}_1'(\tau))$ which arises from the sextic (N_1', N_2') by a second application of $J^5{}_{1,2}$. Hence the sextic $\bar{S}_2'(t)$ which defines J' is the transform of the sextic $\bar{S}_2(t)$ which defines J by the Cremona transformation

$$T = J^5{}_{1,2} R_1 A_{345} R_1 J^5{}_{1,2}. \tag{6}$$

In order to prove that T is the quintic transformation A_{678902} with double F-points at nodes $p_2, p_6, \cdots, p_9, p_0$ we observe first that $J^5_{2,1} = (12)J^5_{1,2}(12)$, i. e., $J^5_{2,1}$ is the transform of $J^5_{1,2}$ by the transposition (12). Since $R_1 = J^5_{1,2}J^5_{2,1}(12)$ (cf. A. G., p. 252), we have, on eliminating $J^5_{2,1}$, that $R_1 = J^5_{1,2}(12)J^5_{1,2}$, whence

$$T = (12)J^5_{1,2}A_{345}J^5_{1,2}(12).$$

Furthermore

$$\begin{aligned} J^5_{1,2} &= (34)(56)(78)(90)A_{190}A_{178}A_{156}A_{134} \\ &= A_{134}A_{156}A_{178}A_{190}(90)(78)(56)(34). \end{aligned}$$

Hence

$$\begin{aligned} T = (12)(34)(56)(78)(90)&A_{190}A_{178}A_{156}A_{134}\cdot A_{345}\cdot \\ &A_{134}A_{156}A_{178}A_{190}(90)(78)(56)(34)(12). \end{aligned}$$

The following transforms are readily checked:

$$\begin{gathered} A_{134}A_{345}A_{134} = (15);\ A_{156}(15)A_{156} = (15); \\ A_{178}(15)A_{178} = A_{578};\ A_{190}A_{578}A_{190} = A_{157890}; \\ (12)(34)(56)(78)(90)A_{157890}(90)(78)(56)(34)(12) = A_{267890}. \end{gathered}$$

When the rational sextic is transformed into a congruent sextic by the quintic transformation, A_{267890}, the symmetroid of the one is transformed into the symmetroid of the other by the regular cubic transformation A_{1345}. Hence [cf. A. G., **55** (c)].

(7) *If J, Σ and the pair of rational sextics, $\bar{S}_2(t)$, $\bar{S}_1(\tau)$, a birationally related triad, are transformed by Cremona transformation into a similarly related J', Σ' and pair $\bar{S}_2'(t)$, $\bar{S}_1'(\tau)$, the transformations are such that the cubic transformation with sextic F-curve, N_1, A_2, A_3, A_4, on J, the regular cubic transformation with F-points at nodes $D_1, \cdots, D_4$ on Σ, and the quintic transformation $A_{5,\ldots,9,0}$ with double F-points at the nodes $p_5, \cdots, p_9, p_0$ of the sextic, correspond.*

By taking products of corresponding transformations of the types given in (7), more general corresponding transformations are obtained.

13. Linear Systems of Curves on J and Σ Referred to a Base.

A brief account of the Severi theory of a base for linear systems of curves on an algebraic surface is given by its author in Pascal's Repertorium.[*] This theory is applied by Snyder and Sharpe[*] to a discussion of certain involutions defined on J and Σ. We use their results to discuss some of the foregoing transformations.

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[*] Π_2, pp. 763-6; Teubner, Leipzig (1922).

In conformity with the notation introduced earlier (cf. **10**), let η denote the linear system of plane sections of J; B_6 (the $C_3'^6$ of **10**), the transform of η under I; C_6, the linear system of loci of nodes of nets of quadrics contained within the web which defines J; A_i $(i = 1, \cdots, 10)$, one of the ten lines on J; and N_i, the transform of A_i under I. The Snyder-Sharpe notation for these is $C_4, C_6, \bar{C}_6, \gamma_i, (\gamma_i)$, respectively. They show that η, B_6, and nine of the ten lines A_i constitute a minimum base for the surface and that the tenth line A is related to this base as in

$$7\eta \equiv 3B_6 + \Sigma_i A_i. \tag{1}$$

On account of the symmetry it is convenient either to retain all twelve of the curves, or to drop B_6. In the latter case the η, and the ten curves A_i, constitute a base, but not a minimum base, since B_6 can not be expressed integrally in terms of them.

If $[C_i, C_k]$ denotes the number of common points of the curves C_i, C_k; and if $[C_i, C_i]$ denotes the virtual grade of the linear system C_i, then (cf. S. S., pp. 287-8).

(2)
$$\begin{gathered}[\eta, \eta] = 4,\ [\eta, B_6] = 6,\ [\eta, A_i] = 1,\\ [B_6, B_6] = 4,\ [B_6, A_i] = 3,\ [A_i, A_i] = -2,\ [A_i, A_j] = 0.\end{gathered}$$

If two curves, C, C' are expressed in terms of the base by the relations,

$$C = \Sigma_i \lambda_i C_i, \quad C' = \Sigma_i \lambda_i' C_i, \tag{3}$$

the number of intersections of the two is given by the formula

$$\Sigma_{i,k} \lambda_i \lambda_k' \, [C_i, C_k]. \tag{4}$$

This is the polar of the *fundamental quadratic form* of the surface

$$\Sigma_{ik} \lambda_i \lambda_k \, [C_i, C_k]. \tag{5}$$

Cubic surfaces on a curve B_6 cut out the linear system C_6 [cf. **10** (17)], whence $B_6 + C_6 = 3\eta$. We then have from (2) and (4) that

$$[C_6, \eta] = 6,\ [C_6, C_6] = 4,\ [C_6, A_i] = 0. \tag{6}$$

It is convenient to use for a base on J the 11 curves C_6, A_i. The plane sections are connected with these by the relation

$$2\eta = 3C_6 - \Sigma_i A_i. \tag{7}$$

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[*] "Space involutions defined by a web of quadrics," *Transactions of the American Mathematical Society*, Vol. 19 (1918), pp. 275-290; cited hereafter as S. S.

With respective multiples λ_0 and λ_i $(i = 1, \cdots, 10)$ for these curves the fundamental quadratic form for J is

$$(8) \qquad 4\lambda_0^2 - 2\lambda_1^2 - \cdots - 2\lambda_{10}^2.$$

In passing to the birationally related symmetroid (cf. S. S., p. 290) C_6 on J becomes a plane section ζ of Σ, the lines A_i on J become the nodes D_i of Σ, and the quadric section 2η of J becomes the section of Σ by an adjoint cubic surface [cf. **10** (15)]. Thus the base finally retained for J becomes, on Σ, the base ζ, D_i.

If the surface J (or Σ) is birationally transformed into itself, the basal curves C_i are transformed into basal curves C_i' which in turn can be expressed in terms of the C_i. The linear transformation which thus arises must leave the fundamental quadratic form unaltered. Since, according to (3), the λ's behave contragrediently to the curves C this linear transformation on the curves C will have, as an invariant quadratic form, the dual of (8), i. e.,

$$(9) \qquad \begin{aligned} &C_6^2 - 2A_1^2 - \cdots - 2A_{10}^2 \text{ (on } J); \\ &\zeta^2 - 2D_1^2 - \cdots - 2D_{10}^2 \text{ (on } \Sigma). \end{aligned}$$

Furthermore if J (or Σ) is birationally transformed into a projectively similar, though not a projectively identical, surface, a like linear transformation appears which has the same quadratic invariant. If for example Σ is transformed into a symmetroid Σ' by the regular cubic transformation A_{1234} with F-points at $D_1, \cdots, D_4$, the plane section ζ' of Σ' arises from the section, $3\zeta - 2D_1 - 2D_2 - 2D_3 - 2D_4$, of Σ by the cubic surface with nodes at $D_1, \cdots, D_4$; and the node D_1' of Σ' arises from the quartic curve, $\zeta - D_2 - D_3 - D_4$, in which Σ is cut by the plane on D_2, D_3, D_4. The linear transformation then has the form

$$(10) \qquad \begin{aligned} &\zeta' \equiv \zeta + 2\pi_{1234} && (\pi_{1234} = \zeta - D_1 - \cdots - D_4), \\ &D_i' \equiv D_i + \pi_{1234} && (i = 1, \cdots, 4), \\ &D_j' \equiv D_j && (j = 5, \cdots, 10). \end{aligned}$$

But in this transformation to a surface with merely similar basal curves there is no reason why the order of the nodes of the one should have in advance any relation to the order of the nodes of the other so that any permutation π of the nodes of either might be adjoined in representing the transformation. By comparison with A. G. **15** it is clear that the linear group on the basal curves thus induced by regular transformation of Σ into Σ', and by permutation of the nodes, is the transposed form of the group $g_{10,3}$ determined by the congruent nodal sets of Σ, Σ'.

In terms of the base ζ, D_i on Σ, or C_6, A_i on J, certain involutions on

Σ, J take somewhat simpler forms than those given by Snyder-Sharpe (cf. S. S., pp. 288-9). The involution I of pairs on J apolar to the web is

$$(11)\qquad \begin{aligned} \zeta' &\equiv \zeta + 4\pi_{1,\ldots,10} && (\pi_{1,\ldots,10} = 2\zeta - D_1 - \cdots - D_{10}),\\ D_i' &\equiv D_i + 2\pi_{1,\ldots,10} && (i = 1, \cdots, 10). \end{aligned}$$

This is permutable with (10) whence if Σ is transformed into Σ' by regular Cremona transformation the pairs of I on Σ are transformed into the pairs of I' on Σ'.

The involution on Σ obtained by drawing lines through the node D_1, which on J is the involution whose pairs are bisecants of N_1, is

$$(12)\qquad \zeta' \equiv 3\zeta - 4D_1,\ D_1' \equiv 2\zeta - 3D_1,\ D_j' \equiv D_j \ \ (j = 2, \cdots, 10).$$

We determine finally the expression for the transformation of J by the cubic Cremona transformation whose sextic F-curve is N_1, A_2, A_3, A_4 [cf. **12** (7)]. The plane sections η' arise from the residual intersection of J with cubic surfaces on N_1, A_2, A_3, A_4, i. e., $\eta' \equiv 3\eta - N_1 - A_2 - A_3 - A_4$. Since $N_1 \equiv B_6 - \eta + A_1$ (cf. S. S.), $\eta' \equiv 4\eta - B_6 - A_1 - A_2 - A_3 - A_4$. The curve N_1' on J' arises from the residual intersection with J of the regulus on A_2, A_3, A_4 whence $B_6' - \eta_1' + A_1' \equiv 2\eta - A_2 - A_3 - A_4$. The line A_j' on J' arises from the residual intersection with J of the regulus on A_k, A_l, N_1 $(j, k, l = 2, 3, 4)$, whence $A_j' \equiv 2\eta - N_1 - A_k - A_l \equiv 3\eta - B_6 - A_1 - A_k - A_l$. The line A_i' on J' arises from the line A_i on J whence $A_i' \equiv A_i$ $(i = 5, \cdots, 10)$. The identity

$$7\eta' - 3B_6 - \Sigma_i A_i \equiv 0 \equiv 7\eta' - 3B_6' - \Sigma_i A_i'$$

then suffices to determine A_1'. On eliminating η and B_6 in favor of C_6, we have

$$(13)\qquad \begin{aligned} C_6' &\equiv C_6 + 2\rho_{1234} && (\rho_{1234} = 2C_6 - A_1 - \cdots - A_4),\\ A_i' &\equiv A_i + \rho_{1234} && (i = 1, \cdots, 4),\\ A_j' &\equiv A_j && (j = 5, \cdots, 10). \end{aligned}$$

Since this is the same transformation on J as (10) on Σ the theorem [12 (7)] is confirmed.

Furthermore, if Σ is transformed into Σ' by A_{1234} and Σ' into Σ'' by A'_{1234}, then Σ and Σ' are projective, and there exists a collineation C such that the product $A_{1234}A'_{1234}C$ is the identical collineation. Hence if J is transformed into J' by the Cremona transformation $T_{1,234}$ with F-curves N_1, A_2, A_3, A_4, and if J' is similarly transformed into J'' by $T_{2,134}$ there must exist a collineation C' which transforms J'' into J. Then the product $T_{1,234}T_{2,134}C'$ is a Cremona transformation of order five which on J is the identity. This order five is probably the lowest for a transformation having J as a locus of fixed points.

14. Projective Configurations Defined by Nodal Parameters of the Rational Sextic.

It is known (cf. Meyer, Apolarität, p. 317) that there are five projectively distinct planar rational sextics with six given pairs of nodal parameters; and that, if one of the five is selected, the remaining four are obtained from it by quadratic transformation with F-points at any three of the four remaining nodes.

Let the nodal parameters of $\bar{S}_2(t)$ at the nodes $p_5, \cdots, p_9, p_{10}$ be regarded as known, and let $r_{01}, r_{02}, r_{03}, r_{04}$ denote the binary quadratics which determine the nodal parameters at the nodes $p_1, \cdots, p_4$; furthermore, let r_{ij} denote the quadratic which determines the parameters of the two further points of $\bar{S}_2(t)$ on the line joining the nodes p_k, p_l $(i, j, k, l = 1, \cdots, 4)$. If $p_1, \cdots, p_4$ are taken in order to be the reference points and unit point, then the parametric equation of $\bar{S}_2(t)$ is

$$
(1) \qquad \begin{array}{ll}
x_1 = r_{02}r_{03}r_{14}, & x_2 - x_3 = r_{04}r_{01}r_{23}, \\
x_2 = r_{03}r_{01}r_{24}, & x_3 - x_1 = r_{04}r_{02}r_{31}, \\
x_3 = r_{01}r_{02}r_{34}, & x_1 - x_2 = r_{04}r_{03}r_{12}
\end{array}
\qquad (r_{ij} = -r_{ji}).
$$

Thus the four pairs of nodal parameters, in addition to the six given pairs, of these five rational sextics are

$$
(2) \qquad \begin{array}{rlllll}
\bar{S}_2(t) = S^{(0)}: & & r_{01}, & r_{02}, & r_{03}, & r_{04}; \\
S^{(1)}: & r_{10}, & & r_{12}, & r_{13}, & r_{14}; \\
S^{(2)}: & r_{20}, & r_{21}, & & r_{23}, & r_{24}; \\
S^{(3)}: & r_{30}, & r_{31}, & r_{32}, & & r_{34}; \\
S^{(4)}: & r_{40}, & r_{41}, & r_{42}, & r_{43}, &
\end{array}
$$

where the sextic $S^{(i)}$ arises from $S^{(0)}$ by the quadratic transformation with F-points at p_j, p_k, p_l.

The 10 quadratics r_{ij}, either in conjunction with the six given nodal pairs or by themselves define interesting projective configurations which we proceed to discuss. With reference to a norm-conic $\bar{K}(t)$ we plot the points r_{ij} determined by the ten quadratics r_{ij}; and denote by $R_6{}^2$ the six points similarly plotted from the six given pairs of nodal parameters common to the five rational sextics $S^{(i)}$ in (2). These five sextics constitute a symmetrical set in that the four nodes of $S^{(i)}$ have nodal parameters r_{ij} and that the further pair of points on the line joining nodes r_{ij}, r_{ik} has parameters r_{lm} $(i, \cdots, m = 0, 1, \cdots, 4)$. According to **4** (4), the six points $R_6{}^2$ and the four points r_{ij} $(j \neq i)$ constitute the nodes of a sextic $S^{(i)}(\tau)$ paired with the sextic $S^{(i)}(t)$ in (2). According to A. G., **55** (7), the sextic $S^{(i)}(\tau)$ arises from the sextic $S^{(j)}(\tau)$ by the Geiser involution with seven F-points at $R_6{}^2$ and r_{ij}. The pairs of points, r_{ik}, r_{jk}; r_{il}, r_{jl}; r_{im}, r_{jm} are pairs of this involution. Each pair together with the seven F-points are the base points of a pencil of cubics. The particular cubic $C^{(m)}$ common to the first two pencils contains $r_{ij}, r_{ik}, r_{jk}, r_{il}, r_{jl}$ and it must also contain r_{kl}. For, it is determined by the fact that it contains $r_{ik}, r_{jk}, r_{il}, r_{jl}$. Thus we find five cubics $C^{(m)}$ on $R_6{}^2$ such that $C^{(l)}, C^{(m)}$ meet in r_{ij}, r_{ik}, r_{jk}.

If the plane of $\bar{K}(t)$ is mapped upon a cubic surface Γ^3 by means of cubic curves on $R_6{}^2$, the five cubics $C^{(m)}$ give rise to an inscribed five-plane of Γ^3 whose ten vertices on Γ^3 are the images of the ten points r_{ij}. The five sextics $S^{(j)}(\tau)$ give rise to sections of Γ^3 by quadrics which touch Γ^3 at the vertices of one of the five tetrahedra contained in the inscribed five-plane.

Conversely, let a cubic surface Γ^3 contain an inscribed five-plane with planes $y_0, \cdots, y_4$ such that

$$(3) \qquad y_0 + y_1 + y_2 + y_3 + y_4 = 0.$$

The equation of Γ^3 is then

$$(4) \qquad \Gamma^3 = \Sigma a_{ijk} y_i y_j y_k = 0.$$

The tetrahedron, $y_0 = y_1 = y_2 = y_3 = 0$, has vertices

$$1, 0, 0, 0, -1;\ 0, 1, 0, 0, -1;\ \cdots\cdots;\ 0, 0, 0, 1, -1.$$

The tangent plane of Γ^3 at one of these four points is

$$a_{014} y_1 + a_{024} y_2 + a_{034} y_3 = 0.$$

The quadric,

$$(5) \qquad Q^{(4)} = a_{014} y_0 y_1 + a_{024} y_0 y_2 + \cdots + a_{234} y_2 y_3 = 0,$$

passes through these four points and has, at these points, the same tangent planes as Γ^3. One such quadric determines the entire set of five, $Q^{(i)}$. For, if $Q^{(4)}$ touches Γ^3 at the vertices of a tetrahedron, a plane y_4 can be determined in such wise that $\Gamma^3 - y_4 Q^{(4)}$ is four-nodal at the four vertices and therefore has the form

$$a_{012} y_0 y_1 y_2 + \cdots\cdots + a_{123} y_1 y_2 y_3 = 0.$$

Hence if a quadric touches Γ^3 at four vertices of a tetrahedron, the planes of the tetrahedron are part of an inscribed five-plane. But these planes are part of such a five-plane if the six edges of the tetrahedron meet Γ^3 again in six points on a plane. Hence [cf. A. G., **56** (a)].

(6) *Ten points $p_1, \cdots, p_4$; $p_5, \cdots, p_{10}$ are the nodes of a rational sextic if the four cubics on the last six points $P_6{}^2$, and respectively each three of the first four points, meet again by pairs in six points which also lie on a cubic curve containing $P_6{}^2$.*

An immediate proof is as follows: Let u be the canonical elliptic parameter on the cubic $C^{(1)}$; $u_{6,2}$, the sum of the parameters of P_6^2 on $C^{(1)}$; u_2, u_3, u_4, the parameters of p_2, p_3, p_4; and u_{12}, u_{13}, u_{14}, the parameters of the intersections with $C^{(1)}$ of $C^{(2)}, C^{(3)}, C^{(4)}$ respectively. Then the intersections of $C^{(1)}$ with $C^{(2)}, C^{(3)}, C^{(4)}$, and $C^{(0)}$ are all given; and

$$u_{6,2}+u_3+u_4+u_{12}\equiv 0,\quad u_{6,2}+u_4+u_2+u_{13}\equiv 0,$$
$$u_{6,2}+u_2+u_3+u_{14}\equiv 0,\quad u_{6,2}+u_{12}+u_{13}+u_{14}\equiv 0.$$

Hence $2(u_{6,2}+u_2+u_3+u_4)\equiv 0$, and the nine nodes on $C^{(1)}$ are a half-period set. If three such sets of nine are half-period sets, the set of ten is the nodal set of a rational sextic.

The configuration of five rational sextics $S^{(i)}(\tau)$ is thus projectively equivalent to an inscribed five-plane of a cubic surface Γ^3 together with an isolated line six on Γ^3. We consider further the configuration of ten points r_{ij} in the plane of $\bar{K}(t)$ without reference to the points R_6^2. From the linear relations which connect $x_1(x_2-x_3)$, $x_2(x_3-x_1)$, $x_3(x_1-x_2)$; (x_2-x_3), (x_3-x_1), (x_1-x_2); x_i, x_j, (x_j-x_i) $(i, j=1,2,3)$, respectively we find a symmetrical set of five quadratic relations connecting the binary quadratics r; namely

$$R^{(m)}\equiv r_{jk}r_{il}+r_{ki}r_{jl}+r_{ij}r_{kl}\equiv 0. \tag{7}$$

If these are satisfied by the ten quadratics, they will serve for the representation (1) of a rational sextic.

We wish to prove that the factors of proportionality in the ten quadratics r_{ij} can be chosen in such wise that the five quadratic relations (7) can be replaced by the five linear relations,

$$L^{(m)}\equiv r_{mi}+r_{mj}+r_{mk}+r_{ml}\equiv 0, \tag{8}$$

themselves linearly related, and by one of the five quadratic relations R. We may assume that these factors have been chosen so that

$$L^{(4)}\equiv r_{40}+r_{41}+r_{42}+r_{43}\equiv 0, \tag{9}$$

since any change in these factors can be compensated in (1) by a corresponding change in the factors attached to the remaining r_{ij}. We shall still have at disposal a factor common to these six. Let now

$$r_{30}+r_{31}+r_{32}\equiv r. \tag{10}$$

From this and $R^{(1)}, R^{(0)}$ by multiplying respectively by $r_{04}r_{42}, r_{24}+r_{14}, r_{04}$ and adding, the r_{31} and r_{32} are eliminated to yield

$$r_{30}[r_{04}r_{42}+r_{42}(r_{24}+r_{14})]\equiv r\, r_{04}r_{42}+r_{34}(r_{02}r_{24}+r_{02}r_{14}+r_{21}r_{04}).$$

On applying $R^{(3)}$ to bring into evidence the factor r_{24}, this becomes, after using (9),

$$(11) \qquad r_{43}(r_{30} + r_{20} + r_{10}) \equiv r\, r_{04}.$$

Since r_{04} and r_{43} have no common factor, r must be proportional to r_{43}, and the factor still undisposed in r_{30}, r_{31}, r_{32} in (10) may be so chosen that $r = r_{43}$. Then (10) reduces to $L^{(3)} \equiv 0$, and from (11) there follows that $L^{(4)} \equiv 0$. A similar elimination of r_{30}, r_{32}, rather than r_{31}, r_{32}, and the same choice of the factor in r, yields $L^{(1)} \equiv 0$. Then also $L^{(2)} \equiv 0$, since

$$(12) \qquad L^{(0)} + L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)} \equiv 0.$$

Conversely the five linear relations and one quadratic relation yield at once the entire set of five quadratic relations.

The ten points r_{ij}, as a planar set, are determined projectively by the linear relations (8), and then any one of the quadratic relations (7) determines the position of the norm-conic, $\bar{K}(t)$. For, the symmetric form,

$$\begin{aligned} r_{jk}(t) \cdot r_{il}(t') + r_{jk}(t') \cdot r_{il}(t) + r_{ki}(t) \cdot r_{jl}(t') \\ + r_{ki}(t') \cdot r_{jl}(t) + r_{ij}(t) \cdot r_{kl}(t') + r_{ij}(t') \cdot r_{kl}(t), \end{aligned}$$

vanishes when $t = t'$, and therefore is $k \cdot (tt')^2$, i. e., equated to zero, it is the equation of $\bar{K}(t)$. We denote the points r_{ij}, whose coordinates are the coefficients of the quadratics r_{ij}, by $q_1, \cdot\cdot\cdot, q_9, q_0$, and seek the conditions on the points which imply that such factors of proportionality can be attached to them that the linear relations (8) are satisfied when $r_{ij} = - r_{ji}$. Specifically let respectively

$$\begin{aligned} & r_{01}, r_{02}, r_{03}, r_{04}, r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34} \\ & \qquad = \lambda_1(q_1\xi), \lambda_2(q_2\xi), \cdot\cdot\cdot\cdot\cdot\cdot, \lambda_9(q_9\xi), \lambda_0(q_0\xi). \end{aligned}$$

Let (ijk) denote the determinant of the coordinates of the points q_i, q_j, q_k. Then the relations (8) that must be satisfied by the points q are

$$\begin{array}{lllllll} L^{(0)} \equiv & & (234)(q_1\xi) + & (143)(q_2\xi) + & (124)(q_3\xi) + & (132)(q_4\xi) & \equiv 0, \\ L^{(1)} \equiv & (567)(q_1\xi) & & + (176)(q_5\xi) + & (157)(q_6\xi) + & (165)(q_7\xi) & \equiv 0, \\ L^{(2)} \equiv & (589)(q_2\xi) + & (298)(q_5\xi) & & + (259)(q_8\xi) + & (285)(q_9\xi) & \equiv 0, \\ L^{(3)} \equiv & (680)(q_3\xi) + & (308)(q_6\xi) + & (360)(q_8\xi) & & + (386)(q_0\xi) & \equiv 0, \\ L^{(4)} \equiv & (790)(q_4\xi) + & (409)(q_7\xi) + & (470)(q_9\xi) + & (497)(q_0\xi) & & \equiv 0. \end{array}$$

We now multiply $L^{(0)}$ by (567), and $L^{(1)}$ by (243) to ensure that $r_{10} = - r_{01}$; then $L^{(0)}$ and $L^{(1)}$ by (589), and $L^{(2)}$ by (134)(567) to ensure that $r_{02} = - r_{20}$; then $L^{(0)}, L^{(1)}, L^{(2)}$ by (680), and $L^{(3)}$ by (142)(567)(589) to ensure that $r_{03} = - r_{30}$; and finally $L^{(0)}, L^{(1)}, L^{(2)}, L^{(3)}$ by (790), and $L^{(4)}$ by (123)(567)(589)(680) to ensure that $r_{04} = - r_{40}$. Then the condition $r_{12} = - r_{21}$ yields the relation (143)(567)(289) = (167)(589)(243), which expresses that the triangle whose vertices are q_1, q_2, q_5 is perspective to the triangle whose sides are q_8q_9, q_6q_7, q_3q_4. The other conditions have a similar translation, as might be inferred from the symmetr returning to the original notation [cf. A. G., **56** (b)]:

(13) *The ten points r_{ij} can be separated in* 10 *ways, by iso r_{ij}, into three points r_{lm}, r_{mk}, r_{kl}, and three pairs of $r_{ik}r_{jk}, r_{il}r_{jl}, r_{im}r_{jm}$ such that the triangles of the th the three lines are perspective.*

The perspectivity of these triangles, for isolated r_{34}, may b the quadratic identities $R^{(3)}$, $R^{(4)}$ in (7). We have remar of these furnishes the norm-conic, $\bar{K}(t)$. Let the first yield second $k_4\bar{K}$. From the combination $k_4R^{(3)} - k_3R^{(4)}$ we get identity connecting three pairs of points:

$$r_{12}(k_3r_{03} + k_4r_{04}) + r_{20}(k_3r_{13} + k_4r_{14}) + r_{01}(k_3r_{23} + k_4r$$

a ternary identity valid throughout the plane, and not merely o says that the three pairs are the vertices of a four-line, and tha

$$k_3r_{03} + k_4r_{04},\ k_3r_{13} + k_4r_{14},\ k_3r_{23} + k_4r_{24},$$

are three points of a line which is the axis of perspectivity. points, $r_{12}, r_{20}, k_3r_{23} + k_4r_{24}$ also are on a line, i. e., the lines r meet on the axis.

Since $r_{12}, r_{20}, k_3r_{23} + k_4r_{24}$ are on a line they are linearl $k_3 = k_4$ due to $L^{(2)} \equiv 0$. Similarly all the constants k are eq

(14) *If the points r_{ij} have such factors of proportionality t relations* (8) *are satisfied, then the left members of relations* (7) *are identical, and each represents the r*

A linear construction for the configuration r_{ij} is indicat of the points, say all except r_{24} and r_{34}, be selected arbitrarily the pair of perspective triangles determined by isolating r_{34}. meets the side $r_{03}r_{04}$ in a point a; the side $r_{10}r_{12}$ meets the si point b, whence ab is the axis ξ of perspection. The side $r_{21}r_{2}$ point c, and r_{24} must be a point on the line $r_{23}c$. After selecti line, a similar construction with the pairs of perspective tria by isolating first r_{24}, and then r_{23}, determines a pair of lines the configuration is subject to three independent projective c depends upon nine absolute projective constants.

If the sextic $\bar{S}_2(t)$ is mapped as in A. G., **56** (9) upon a r in space with nine actual nodes, then five of the pairs of nod and the ten pairs of parameters of further points on a plane co of the five nodes, yield a set of 15 quadratics which have algeb analogous to those of the 10 r_{ij}'s.

URBANA, ILLINOIS.

A Contribution to the Theory of Capacity.[*]

By Oliver D. Kellogg and Florin Vasilesco.

1. *Introduction.* The word *domain* will be used in the sense of open continuum. For definiteness, we shall restrict our considerations to space of three dimensions. A domain T is a *Dirichlet domain* if, corresponding to any continuous boundary values, there exists a function, harmonic in T (and this includes the demand that it shall vanish at infinity, if T contains the infinite region), and assuming the given boundary values. If T is infinite, but has a boundary t which is bounded, the *conductor potential* of t is the function V which is harmonic in T and assumes the boundary values 1. The *capacity* of t is the total charge on t producing the potential V, or

$$c = -\frac{1}{4\pi}\int\int_S \frac{\partial V}{\partial n}\,dS,$$

where S is any smooth surface enclosing t, and n is the outward normal.

The notion of capacity may be defined for any bounded set E.[†] If E' denotes the set consisting of E and its limit points, the complement of E' is open, and so consists of a denumerable set of domains. One and only one of these will be an infinite domain T. The complete boundary t of T will lie in E'. If $T_1, T_2, T_3, \cdots$ denotes an infinite set of nested Dirichlet domains in T, such that any point of T belongs ultimately to T_n, the conductor potential V_n of the boundary of T_n decreases monotonely, as n becomes infinite, to a function V which is harmonic in T. If a different sequence of Dirichlet domains $T'_1, T'_2, T'_3, \cdots$ is used, the limiting function V' is the same as V. For on the one hand, since $V' \leqq 1$ on the boundaries of $T_1, T_2, T_3, \cdots$, it cannot exceed the corresponding conductor potentials, and hence $V' \leqq V$. On the other hand, $V \leqq V'$, and so $V' = V$. Thus V is characteristic of the domain T, and not of the method by which it is obtained. We call it the *conductor potential* of E, and the corresponding charge, as given by Gauss' integral, the *capacity* of E. We remark that by a theorem of Harnack, the conductor potential V_n and its derivatives of any given order, converge uniformly to V and its corresponding derivatives, in any closed region in T. It follows that the capacity of t is the limit of the capacities of the boundaries of the domains $T_1, T_2, T_3, \cdots$.

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[*] Read before the American Mathematical Society, December 27, 1928.

[†] Wiener, *Journal of Mathematics and Physics, Massachusetts Institute of Technology*, Vol. 3, p. 49, p. 127.

The points of t at which V approaches 1 are called *regular* points of t. All other points of t are called *exceptional.* These points are also characterized as points at which Green's function, similarly defined as the limit of a sequence, does, or does not, approach 0.

Wiener[*] has given the following criterion for the regular or exceptional character of a boundary point p. Let λ denote a positive proper fraction, let $\lambda^n(p)$ denote the sphere about p of radius λ^n. Let Γ denote the complement of T, and $\Gamma_n(p)$ the points of Γ in the closed region between the two spheres $\lambda^{n+1}(p)$ and $\lambda^n(p)$. Let $\gamma_n(p)$ be the capacity of $\Gamma_n(p)$. *Then p is regular or exceptional according as the series*

$$\sum_0^\infty \frac{\gamma_n(p)}{\lambda^n} \tag{1}$$

diverges or converges.

If T is any domain whose boundary t is bounded, and if $F(p)$ is any function, defined and continuous on t, there exists a function U, bounded and harmonic in T, which takes on the values $F(p)$ at every regular point of t.[†] If T is a Dirichlet domain, U is uniquely determined. The question naturally arises as to whether this is true for any domain; that is, *can there exist more than one function, bounded and harmonic in T, and taking on the same preassigned continuous boundary values at every regular boundary point?*

This question would be definitely answered in the negative provided the following lemma[‡] were established:

Fundamental Lemma. Any bounded closed set of points of positive capacity contains at least one regular point.

Conversely, the theorem of uniqueness implies the validity of the lemma.[§] The lemma is of interest not only because of its connection with the theorem of uniqueness, but because many other properties of harmonic functions with respect to behavior on the boundary depend on it.[||] The corresponding lemma in two dimensions has been established.[*] Essential differences arise in three dimensions, and the truth of the lemma here is in doubt. The present paper offers results which reduce the scope of the problem, and which may well be of inherent interest.

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[*] *L. c.*, p. 130. See also Kellogg, *Bulletin of the American Mathematical Society*, Vol. 32 (1926), pp. 616-620.

[†] Kellogg, *Proceedings of the American Academy*, Vol. 58 (1923), pp. 528-529; Wiener, *l. c.*, p. 5.

[‡] Kellogg, *Proceedings of the National Academy of Sciences*, Vol. 12 (1926), p. 406.

[§] Kellogg, *Acta Universitatis, Szeged*, Vol. 4 (1928), pp. 4, 5. In this note it is regarded as part of the definition of a harmonic function in T that it shall vanish at infinity if T is infinite.

[||] See Vasilesco, *Journal de Mathématiques*, in a number soon to appear.

2. *Transformations of the Criterion of Wiener.* If a set of points is contained in a sphere, we may define the *relative capacity* of the set as the ratio of its capacity to that of the sphere—namely to the radius. Thus the relative capacity depends not only on the set, but on some sphere as well. Usually it will be clear what sphere is meant without explicit statement.

It will be noticed that the nth term of the series (1) is the relative capacity of $\Gamma_n(p)$. By a consideration of the conductor potential, one verifies at once that if a set of points and a containing sphere be subjected to a homothetic transformation, the relative capacity remains unchanged. We thus arrive at a form of Wiener's criterion given by Vasilesco:[†] let c_n denote the capacity of the set obtained by subjecting the set $\Gamma_n(p)$ to a homothetic transformation with center p and ratio of similitude $\lambda^n : 1$. *Then p is regular or irregular according as*

$$\sum_0^\infty c_n \qquad (2)$$

diverges or converges.

Suppose now that to a given bounded set E there corresponds a point p and a number λ, $0 < \lambda < 1$, such that the set homothetic to the portion of E in the sphere of radius λ about p contains E, the center and ratio of similitude being p and $\lambda : 1$, respectively. It follows that the same is true for the set homothetic to the portion of E in the sphere of radius λ^n about p, the ratio of similitude being $\lambda^n : 1$. Then *either p is regular, or E is of capacity 0.* For $c_0 \leqq c_1 \leqq c_2 \leqq \cdots$ (see the inequalities (4), below), so that the series (2) diverges, unless all its terms are 0. Thus the fundamental lemma holds for sets of this type, which includes such sets as the surfaces of cones and triangles, and sets formed after the analogy of Cantor's nowhere dense perfect set.[‡]

It is frequently convenient, in the application of Wiener's criterion to use, not the sets $\Gamma_n(p)$, but the sets of points $\Delta_n(p)$ of Γ in the closed spheres $\Sigma_n(p)$. Let $\delta_n(p)$ denote the capacity of $\Delta_n(p)$. *Then p is regular or irregular according as the series*

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[*] Kellogg, *Comptes Rendus*, Vol. 187 (1928), p. 526.

[†] *Comptes Rendus*, Vol. 186 (1928), p. 23.

[‡] See Kellogg, *l. c. Proceedings of the American Academy;* Wiener, *l. c.*, p. 14

$$(3) \qquad \sum_{0}^{\infty} \frac{\delta_n(p)}{\lambda^n}$$

diverges or converges.

To prove this, we remark that if $c(e)$ denotes the capacity of the set e, then for any two bounded sets e' and e'',

$$(4) \qquad c(e') \leqq c(e' + e'') \leqq c(e') + c(e'').$$

It follows that

$$\gamma_n(p) \leqq \delta_n(p), \text{ and } \delta_n(p) \leqq \gamma_n(p) + \delta_{n+1}(p).$$

From the first inequality it follows that (3) diverges if (1) does. If the second inequality be divided by λ^n and summed from 0 to n, we have, since a relative capacity never exceeds 1,

$$\sum_{0}^{n} \frac{\gamma_n(p)}{\lambda^n} \geqq (1-\lambda) \sum_{0}^{n} \frac{\delta_n(p)}{\lambda^n} - \lambda.$$

Accordingly, the series (3) converges if (1) does. This proves the theorem.

An advantage of this transformation of the criterion is that it throws into evidence the following fact: we may replace the spheres $\lambda^n(p)$ by regions $\sigma_n(p)$ of quite varied character. Thus if $\sigma_n(p)$ has the property that a positive integer m exists, such that from some n on $\sigma_n(p)$ is contained in $\lambda^{n-m}(p)$ and contains $\lambda^{n+m}(p)$, we may interpret $\delta_n(p)$ as the capacity of the portion of Γ in the closed region $\sigma_n(p)$ in the above theorem. Because of their space-filling property, it may be convenient to use cubes.

If λ is fixed, and if a bounded set E is given, determining a closed bounded set Γ, the series

$$(3) \qquad W(P) = \sum_{0}^{\infty} \frac{\delta_n(P)}{\lambda^n},$$

where $\delta_n(P)$ means the capacity of $\Delta_n(P)$, defines a function $W(P)$ at every point of space, if we admit $W(P) = \infty$ as a possible definition, characterizing a point at which the series diverges. With respect to this function, we shall prove a number of theorems, the first of which is as follows:

THEOREM I. *If $W(P_0) = \infty$, then given N, however large, there is a neighborhood of P_0 at every point of which $W(P) > N$.*

To show this, we select n so that

$$[\delta_0(P_0)/1] + [\delta_1(P_0)/\lambda] + \cdots + [\delta_n(P_0)/\lambda^n] > (N/\lambda) + 1.$$

Let $\rho = \lambda^n - \lambda^{n+1}$. Then if P is confined to the sphere of radius ρ about P_0, the points of Γ in the sphere $\lambda^n(P)$ include all those in the sphere $\lambda^{n+1}(P_0)$.

Hence $\delta_n(P) \geqq \delta_{n+1}(P_0)$, and similarly $\delta_{n-1}(P) \geqq \delta_n(P_0), \cdots, \delta_0(P) \geqq \delta_1(P_0)$. Hence

$$\begin{aligned} W(P) &\geqq [\delta_0(P)/1] + [\delta_1(P)/\lambda] + \cdots + [\delta_n(P)/\lambda^n] \\ &\geqq [\delta_1(P_0)/1] + [\delta_2(P_0)/\lambda] + \cdots + [\delta_{n+1}(P_0)/\lambda^n] \\ &\geqq \lambda[(N/\lambda) + 1 - \delta_0(P_0)] > N, \end{aligned}$$

as was to be proved. From this follows the

Corollary. If $W(P)$ is bounded on a set of points everywhere dense on E, E can have no regular points.

We close this section with an integral form for the criterion for regularity. Let $c(\rho)$ denote the capacity of the portion of Γ in the closed sphere of radius ρ about p. Then $c(\rho)$ is a monotone increasing bounded function of ρ, and so is integrable. Moreover,

$$\int_{\lambda^{n+1}}^{\lambda^n} \frac{c(\rho)}{\rho^2}\, d\rho \leqq c(\lambda^n) \int_{\lambda^{n+1}}^{\lambda^n} \frac{d\rho}{\rho^2} = \delta_n(p) \left(\frac{1}{\lambda^{n+1}} - \frac{1}{\lambda^n}\right) \leqq \frac{1}{\lambda} \cdot \frac{\delta_n(p)}{\lambda^n} .$$

On the other hand,

$$\int_{\lambda^{n+1}}^{\lambda^n} \frac{c(\rho)}{\rho^2}\, d\rho \geqq c(\lambda^{n+1}) \left(\frac{1}{\lambda^{n+1}} - \frac{1}{\lambda_n}\right) = (1-\lambda) \frac{\delta_{n+1}(p)}{\lambda^{n+1}} .$$

Hence, summing the inequalities, we obtain

$$(1-\lambda)\,[W(p) - \delta_0(p)] \leqq \int_0^1 \frac{c(\rho)}{\rho^2}\, d\rho \leqq (1/\lambda) W(p).$$

Thus p is regular or not according as the integral (in general improper)

$$\int_0^1 \frac{c(\rho)}{\rho^2}\, d\rho$$

is divergent or not.

The integral, like the series, is a function defined for any point P of space. With respect to either function, the fundamental lemma may be expressed: *given a closed bounded set E, the function is either infinite at some point of E, or else it is identically* 0. It may be remarked that if E either has interior points, or bounds any domains other than the infinite one—in other words, if Γ has interior points—the lemma holds for E. For it is obvious that a point p of the boundary of Γ, on a sphere in Γ, is a regular point.

3. *Capacity as a Function of Sets of Points.* For the sequel some further concepts will be useful.* A set E will be said to be *of capacity* 0 *at a*

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* Vasilesco, *Comptes Rendus*, Vol. 185 (1927), p. 1572; *Journal de Mathématiques, l. c.*

point P if there is a sphere about P the portion of E within which has the capacity 0. A closed set is said to be *reduced* provided it is non-empty and is of capacity 0 at none of its points. It has been shown* that every closed bounded set E of positive capacity contains a closed reduced set, and that it may be obtained by removing from E the points at which E is of capacity 0. This process does not change the capacity or the conductor potential of the set. Thus, for the purposes of establishing or disproving the fundamental lemma, one may restrict oneself to closed reduced sets.

The following measure of closeness of two sets is well adapted to the study of capacity.† We say that two closed bounded sets e and e' *differ by at most* ϵ, if e' is contained in the set I_ϵ consisting of the points within and on the spheres of radius ϵ, one about each point of e, and if e is contained in the corresponding set I_ϵ' formed for e'.

With these preliminaries, we establish the following theorem:

THEOREM II. *Let E denote a closed bounded set, and e any closed part of E. Then the capacity $c(e)$ of e is a function of sets which is upper semicontinuous. It is continuous only if $c(E) = 0$.*

To prove this, let any positive number η be given. We form the set of spheres $I_{1/n}$ of radius $1/n$ about the points of e. As we saw in connection with the definition of capacity in § 1, $c(I_{1/n})$ approaches the limit $c(e)$ as n becomes infinite. There is therefore an integer n such that

$$c(I_{1/n}) < c(e) + \eta.$$

If e' is any set in $I_{1/n}$, and, in particular, if e' is any set differing from e by less than $1/n$, $c(e') \leqq c(I_{1/n})$, by the inequalities (4), and hence

$$c(e') < c(e) + \eta,$$

so that $c(e)$ is upper-semicontinuous, as stated.

To establish the second part of the theorem, we notice that for any n there is always a set e' of capacity 0 differing from e by less than $1/n$. To see this, we note that by the Heine-Borel theorem, the points of e are interior to a finite number of the spheres of $I_{1/2n}$. If we choose for e' a single point of E in each of these spheres, e and e' will differ by at most $1/n$, and e' will have the capacity 0. Thus there are sets of capacity 0 arbitrarily near to e, and $c(e)$ can only be continuous if $c(e) = 0$ for every closed part of E, and in particular, for E itself. Thus the theorem is established.

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* Vasilesco, *Journal de Mathématiques*, *l. c.*

† Vasilesco, *Essai sur les fonctions multiformes de variables réelles*, Thèse, Paris, Gauthier-Villars (1925), p. 7.

4. *The Function Defined by the Wiener Series.* We begin by considering a typical term of the Wiener series, formed for a *reduced bounded set* E, and establish

THEOREM III. *The function* $\delta_n(P)$, *defined at every point of space, is upper semi-continuous, and in consequence, either continuous or of the first class, in the sense of Baire. It is not, in general, continuous.* The same holds for $\gamma_n(P)$.

For if P and η are given, and the set e of Theorem II is identified with $\Delta_n(P)$, then $\delta_n(P) = c(e)$, and so for all e' in $I_{1/m}$, for sufficiently large m,

$$c(e') < \delta_n(P) + \eta.$$

The set of points of E at a distance $1/m$ or more from the points of $\Delta_n(P)$ is a closed set with no points in $\lambda^n(P)$. If ρ' denotes its distance from $\lambda^n(P)$, and if ρ is the less of the two numbers $1/m$ and ρ', then for $P'P < \rho$, the set $\Delta_n(P')$ lies in $I_{1/m}$. Hence

$$\delta_n(P') < \delta_n(P) + \eta \text{ for } P'P < \rho,$$

and $\delta_n(P)$ is upper semi-continuous.

An example in which $\delta_n(P)$ is discontinuous is that in which E is the smaller portion of a sphere of radius λ^n bounded by a small circle. If P is the center of the sphere, $\delta_n(P)$ is positive, whereas for all points P' on the axis of the small circle and farther from the circle than P, $\delta_n(P') = 0$. Such a situation may arise in any case in which E, or the associated set Γ (consisting of E and any finite domains it may bound), has sub-sets of positive capacity on a sphere $\lambda^n(P)$ which are not limit points of points of Γ in the interior of the sphere. Thus if we increase the set E of the previous example by adjoining a small cone or cylinder along the diameter of the sphere which is the axis of the circle, thereby forming a new reduced set E, the function $\delta_n(P')$ will still fail to be continuous, even while P' is a point of E itself.

We shall revert to this question presently. First let us point out a corollary of the last theorem.

Corollary. The series of Wiener—(1) *or* (3)*—is a function of* P *of class* 2 *or lower, in the sense of Baire.*

For, since $\delta_n(P)$ is of class 1 or lower, the same is true of the separate terms of the series, and thus of the partial sums. Hence $W(P)$, the limit of the sum of the first n terms, is of class 2 or lower.

Nevertheless, there is a considerable category of sets E for which $W(P)$ is of class not greater than 1. This is shown in the following theorem.

THEOREM IV. *If E is such that the set of points common to the associated set Γ and the surface of any sphere $\lambda^n(P)$, P a point of t, is of capacity 0, then E, Γ, and t are identical, $\delta_n(P)$—and similarly $\gamma_n(P)$—is continuous, and $W(P)$ is at most of class one on E.*

Suppose that $\Gamma \not\equiv t$; let M be a point of Γ not belonging to t. As $t + T$ is a closed set of points, there exists a sphere of small radius about M whose points do not belong to $t + T$; they belong to Γ. Consider then a sphere (λ^n) of smaller radius λ^n (n large enough) about M and the straight line MM' joining M to a point M' of T. Since the sphere (λ^n) lies in Γ, the same is the case for all the spheres of the same radius about every point M'' near to M. As M'' moves away towards M' there will be a first point M_1 on MM' such that the corresponding sphere will have a point P of t on its surface. Then the sphere $\lambda^n(P)$ would contain on its surface a set of points of Γ of positive capacity. But this is contrary to hypothesis.

Accordingly $\Gamma \equiv t$, and *a fortiori* $E \equiv t$, for $\Gamma > E > t$.

To prove the remainder of the theorem, let P be any point of E, and $\lambda^n(P)$ any of the spheres of the theorem with P as center. Let $s_n(P)$ denote the set of points of E on the surface of $\lambda^n(P)$. As $s_n(P)$ has the capacity 0, by hypothesis, there corresponds to any $\eta > 0$ a set I_δ of spheres of radius δ about the points of $s_n(P)$, whose capacity is less than η. The points of E in $\lambda^n(P)$, but not interior to I_δ, form a closed set e^* with no points on the surface of $\lambda^n(P)$. Let ρ be the distance from e^* to the surface of $\lambda^n(P)$. Then if $P'P < \rho$, $\lambda^n(P')$ will contain all points of e^*, and hence

$$\delta_n(P') \geqq c(e^*).$$

Also, $e^* + I_\delta$ contains $\Delta_n(P)$, and hence

$$c(e^*) + \eta \geqq \delta_n(P).$$

It follows from the two inequalities that

$$\delta_n(P') \geqq \delta_n(P) - \eta,$$

and $\delta_n(P)$ is thus lower semi-continuous at P. Accordingly, by Theorem III, it is continuous at P. As P was any point, $\delta_n(P)$ is everywhere continuous. We remark for later use, that if, for a certain n and P, $s_n(P)$ has the capacity zero, the proof shows that $\delta_n(P)$ is continuous at the center of that sphere.

Sets of the type contemplated in the above theorem are easy to form. Among them are plane sets, sets on spheres, and sets on any other analytic surfaces. For the intersections of such sets with a sphere $\lambda^n(P)$ would lie on analytic curves, and these have the capacity 0. It is also possible that $\delta_n(P)$ is discontinuous and $W(P)$ of class 2 for some value of λ, whereas for a neighboring value $\delta_n(P)$ is continuous for all n, and $W(P)$ of class not greater than 1. A general method of constructing sets for which the $\delta_n(P)$ are continuous is given by the

Corollary. If for the closed bounded set E, the functions $\delta_n(P)$, P being a point of t, are not all continuous, then E contains a reduced subset E such that the functions $\delta_n(P)$, P being a point of E', are continuous and $W(P)$ is of class not greater than 1 *on E'.*

Suppose first that $\Gamma \equiv t$ and therefore $\Gamma \equiv E$; and suppose that for some fixed n, $\delta_n(P)$ is discontinuous at P_0. The part of E on the surface of $\lambda^n(P_0)$ is then of positive capacity. If this set is reduced, the resulting set E' is not of zero capacity and has the desired property, for it lies on a sphere of radius λ^n.

If now $\Gamma \not\equiv t$, the set Γ contains points which are not limiting points of T and are therefore interior points of Γ. Hence Γ contains at least one domain whose boundary e is a part of t. If P is a point of e and M a point of the domain just mentioned, the portion of e determined by the sphere of center P and radius PM is a set of positive capacity.* This portion e_1 of e is itself the complete boundary of a certain infinite domain T_1, such that e_1 coincides with its corresponding Γ. The proof of the first part of the theorem accordingly applies to e_1, which contains therefore a subset E' of the kind specified.

Since the function $\delta_n(P)$ is continuous on E', the function $W(P)$ will be of class not greater than 1 on E'.

If now the functions $\delta_n(P)$ are all continuous, $W(P)$ is not merely of class 1 or lower, but more, it is lower semi-continuous. This is a consequence of the fact that the terms of the Wiener series are never negative. It follows that if N is any positive number, the points where $W(P) \leqq N$ form a closed set. It is not true that the points where $W(P)$ is infinite always form a closed set, for the Lebesgue spine † gives a closed set for which $W(P)$ is infinite at all but a single point, namely the point of the spine. The fundamental lemma has to do with the possibility of bounded closed sets at which $W(P)$ is nowhere infinite and yet not identically 0. If such a set exists, it must coincide with the associated sets Γ and t. For the existence of an interior point of Γ would imply a regular point of E, and at it $W(P)$ would be infinite.*

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* Vasilesco, *Journal de Mathématiques, loc. cit.* No. 27.

† "Sur des cas d'impossibilité du problème de Dirichlet," *Comptes Rendus de la Société Mathématique de France* (1923), p. 17.

If a closed bounded set E without regular points exists, we may, by the corollary, assume that the sub-set, on which the corresponding $W(P)$ does not exceed any assigned positive number, is closed. If then E_n denote the sub-set on which $W(P) \leqq n$, we have, since $W(P)$ is nowhere infinite,

$$E = E_1 + E_2 + E_3 + \cdots.$$

It follows that for some n, $c(E_n) > 0$.† As $W(P)$, formed for E, is not greater than n at the points of E_n, it is certainly not greater than n at the points of the set E', obtained from E_n by reducing it according to the process of the last section. It follows that $W(P)$, formed for the sub-set E' of E, must also be not greater than n at the points of E'.

We shall now show that this function $W(P)$ is bounded. As it is bounded by n on E', it remains to show that it is bounded elsewhere. Let P be any point not in E', and ρ its distance from the nearest point p of E'. Let $\lambda^{k+1} < \rho \leqq \lambda^k$. We may assume k positive or 0, for otherwise $\Delta_0(P)$ would be empty, and $W(P) = 0$. Now $\Delta_{k+1}(P)$, $\Delta_{k+2}(P), \cdots$ are empty, so that $\delta_{k+1}(P) = \delta_{k+2}(P) = \cdots = 0$.

Moreover, for $i \leqq k$, $\Delta_i(P)$ is contained in the sphere about p of radius $\lambda^i + \lambda^k = \lambda^i(1 + \lambda^{k-i}) \leqq 2\lambda^i \leqq \lambda^{i-s}$, if s is the first integer for which $2 \leqq \lambda^{-s}$. That is, $\Delta_i(P)$ is a part of $\Delta_{i-s}(p)$. Hence if $s \leqq i \leqq k$, $\delta_i(P) \leqq \delta_{i-s}(p)$, and as $\delta_i(P) = 0$ for $i > k$, this inequality holds for all $i \geqq s$. We have, therefore, since relative capacities never exceed 1,

$$\begin{aligned} W(P) &= \frac{\delta_0(P)}{1} + \cdots + \frac{\delta_{s-1}(P)}{\lambda^{s-1}} + \frac{\delta_s(P)}{\lambda^s} + \frac{\delta_{s+1}(P)}{\lambda^{s+1}} + \cdots \\ &\leqq 1 + \cdots + 1 + 1/\lambda^s \left[\frac{\delta_0(p)}{1} + \frac{\delta_1(p)}{\lambda} + \cdots\right] \\ &\leqq s + n/\lambda^s, \end{aligned}$$

and so $W(P)$ is bounded.

Two functions $W(P)$, formed for the same set of points, but with

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[*] Vasilesco, *Journal de Mathématiques*, *loc. cit.*, No. 38.

[†] If the sum of a denumerable set of closed sets of capacity 0 is closed, it has the capacity 0. Vasilesco, *Journal de Mathématiques*, *l. c.*, § 8.

[illegible]ent values of λ, are bounded each by a linear function of the other, that if either is bounded, the other is also. Combining the results here a[illegible] [illegible] with the last corollary, we have

Theorem V. *If there exists a bounded closed set E, of positive capacity, [illegible] at regular points, then there exists a reduced sub-set E' for which [illegible] of Wiener is bounded, lower semi-continuous, and all of whose [illegible] are everywhere continuous.*

By means of a theorem of Baire, the last theorem admits an alternative [illegible]. If the set of the theorem exists, there is, as we have seen, a reduced [illegible] satisfying the hypothesis, which, by the last corollary, we may assume [illegible] such that $W(P)$ is of class not greater than 1. Accordingly, if we [illegible] this set by E, the points of continuity of $W(P)$, by the theorem of Baire, are everywhere dense on E. Let P_0 be a point of continuity of $W(P)$ on E. There is then a sphere σ about P_0 such that when P is in σ, $W(P) \leqq W(P_0) + 1$. Thus $W(P)$ is bounded on the portion of E in σ, and this [illegible] positive capacity, since E is reduced. If now this portion of E [illegible]d, the resulting set E' has the properties enunciated in the theorem.

[illegible]. *The Function Defined by the Integral in the criterion for Regularity.* The integral used in the criterion established in § 2 is

$$J(P) = \int_0^1 \frac{c(\rho, P)}{\rho^2}\, d\rho.$$

We consider, by the side of $J(P)$, the integral

$$J(\alpha, P) = \int_\alpha^1 \frac{c(\rho, P)}{\rho^2}\, d\rho,$$

where α is a fixed number, $0 < \alpha < 1$, and show that when based on a closed bounded set E, this function of P is continuous.

In fact, if $P'P < \delta < \alpha$,

$$c(\rho - \delta, P) \leqq c(\rho, P') \leqq c(\rho + \delta, P),$$

so that

$$\int_\alpha^1 \frac{c(\rho - \delta, P)}{\rho^2}\, d\rho \leqq \int_\alpha^1 \frac{c(\rho, P')}{\rho^2}\, d\rho \leqq \int_\alpha^1 \frac{c(\rho + \delta, P)}{\rho^2}\, d\rho.$$

Now

$$\int_\alpha^1 \frac{c(\rho + \delta, P)}{\rho^2}\, d\rho = \int_{\alpha+\delta}^{1+\delta} \frac{c(\rho, P)}{(\rho - \delta)^2}\, d\rho,$$

and as $c(\rho, P)$ is bounded, this is a continuous function of δ for $-\alpha < \delta < \alpha$. Hence $\delta > 0$ may be chosen so small that

$$| J(\alpha, P') - J(\alpha, P) |$$

is less than any given positive ϵ.

As $J(P) = \lim_{n \to \infty} J(1/n, P)$, it follows that $J(P)$ is lower semi-continuous. We have thus

THEOREM VI. *Given any bounded closed set E, the corresponding function $J(\alpha, P)$ is everywhere continuous, and the function $J(P)$ is lower semi-continuous.*

If the capacity of E is positive, and yet E has no regular points, we may, either by modifying appropriately the proof of Theorem V, or by applying that theorem and noticing that $J(P)$ is bounded by a linear function of $W(P)$, arrive at the following result:

THEOREM V'. *If there exists a closed bounded set E, of positive capacity, without regular points, there exists a reduced sub-set of E for which $J(P)$ is bounded.*

Collineations and Motions in Generalized Spaces.

By M. S. Knebelman.

Introduction. The first four paragraphs of this paper are devoted to the development of the tensor analysis of a space of paths whose connection—affine or projective—is a function of position and direction. The next five paragraphs deal with groups of collineations in a space of paths while the last three deal with groups of motions in a generalized metric space, that is, one in which the components of the fundamental tensor are functions of direction as well as of position.

The theory as developed below deals entirely with finite continuous groups, the fundamental result being the set of theorems which state the necessary and sufficient conditions that a space must satisfy in order to admit an r-parameter group of collineations or motions. Some theorems and properties of groups of motions in a Riemannian space have been generalized so as to apply to the spaces under discussion.

No attempt has been made here to solve the outstanding problem of classifying spaces according to the number and type of collineations that they may admit. The problem of determining spaces admitting an r-parameter translatory group is thus far unsolved as well as the problem of complete integrability of the equations of Killing for a generalized metric space (except for the case of two dimensions).

1. *General space of paths. The affine curvature tensor.* We consider an n-dimensional manifold whose points are represented in a given coordinate system x by the ordered set of numbers $(x^1, \cdots, x^n)$. We have given a system of differential equations

$$(1.1) \qquad (d^2x^i/dt^2) + H^i[x, (dx/dt)] = 0 \qquad (i = 1, \cdots, n),$$

where $H^i(x, dx/dt)$ are a set of functions of $x^1, \cdots, x^n$; $dx^1/dt, \cdots, dx^n/dt$ which are positively homogeneous of the second degree in dx^i/dt.

The curves $x^i = f^i(t)$ which satisfy the equations (1.1) are the *paths* of the manifold, the manifold together with the paths constituting a *general space of paths* defined by the functions $H^i(x, dx/dt)$.

The designation "general" is used to distinguish the space of paths as defined above from the case where $H^i(x, dx/dt)$ are polynomials of the second degree in dx/dt and also for the reason that equations (1.1) are the most general system of equations whose integral curves have the property that through any two points in general position, in a properly restricted neighborhood, there passes one and only one path.*

We shall denote dx^i/dt by $\dot{x}^i$ and assume that $H^i(x, \dot{x})$ are analytic in both sets of variables. By Euler's theorem on homogeneous functions we may write equations (1.1) in the form

$$(d\dot{x}^i/dt) + \Gamma^i_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k = 0, \tag{1.2}$$

the summation convention for repeated indices being used and Γ^i_{jk} being defined by

$$\Gamma^i_{jk}(x, \dot{x}) = \tfrac{1}{2}[\partial^2 H^i(x, \dot{x})/\partial\dot{x}^j\,\partial\dot{x}^k]. \tag{1.3}$$

From (1.3) and the definition of $H^i(x, \dot{x})$ it follows that $\Gamma^i_{jk}(x, \dot{x})$ are positively homogeneous of degree zero in $\dot{x}$, are symmetric in the subscripts and satisfy the equations

$$(\partial\Gamma^i_{jk}/\partial\dot{x}^l) = (\partial\Gamma^i_{jl}/\partial\dot{x}^k). \tag{1.4}$$

The invariant whose components in the given coordinate system x are $\Gamma^i_{jk}(x, \dot{x})$ is the *affine connection* of the space of paths, the functions Γ^i_{jk} being the *coefficients* of affine connection in the given coordinate system. The law of transformation of the Γ's under an arbitrary analytic transformation of coordinates can be obtained from the invariance of form of (1.2) when the affine parameter t is unaltered.† For let $x^i = f^i(\bar{x})$ define a transformation of coordinates; then

$$\dot{x}^i = \frac{\partial x^i}{\partial\bar{x}^\alpha}\dot{\bar{x}}^\alpha, \qquad \frac{d\dot{x}^i}{dt} = \frac{\partial^2 x^i}{\partial\bar{x}^\alpha\,\partial\bar{x}^\beta}\dot{\bar{x}}^\beta\dot{\bar{x}}^\alpha + \frac{\partial x^i}{\partial\bar{x}^\alpha}\frac{d\dot{\bar{x}}^\alpha}{dt}. \tag{1.5}$$

Substitution of (1.5) into (1.2) gives, since $(d\dot{\bar{x}}^\alpha/dt) + \bar{\Gamma}^\alpha_{\beta\gamma}(\bar{x}, \dot{\bar{x}})\,\dot{\bar{x}}^\beta\dot{\bar{x}}^\gamma = 0$,

$$\left(\Gamma^i_{jk}\frac{\partial x^j}{\partial\bar{x}^\beta}\frac{\partial x^k}{\partial\bar{x}^\gamma} + \frac{\partial^2 x^i}{\partial\bar{x}^\beta\,\partial\bar{x}^\gamma} - \bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial x^i}{\partial\bar{x}^\alpha}\right)\dot{\bar{x}}^\beta\dot{\bar{x}}^\gamma = 0, \tag{1.6}$$

$\bar{\Gamma}^\alpha_{\beta\gamma}$ being the coefficients of affine connection in the coordinate system $\bar{x}$.

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* Cf. O. Veblen, "Remarks on the Foundations of Geometry," *Bulletin of the American Mathematical Society*, Vol. 31 (1925), pp. 121-141; J. Douglass, *Annals of Mathematics*, Vol. 29 (1928), pp. 154-166.

† Cf. J. Douglas, *loc. cit.* for definition of "parameterized paths" and affine parameters.

To simplify (1.6), as well as for future use, we establish the following lemma: *Given a set of analytic functions $f^{(\alpha)}{}_{(\beta)i_1 i_2\ldots i_r}(y)$ which are homogeneous of degree zero in the n independent variables $y^1, \cdots, y^n$, symmetric in indices $i_1, i_2, \cdots, i_r$—(α) and (β) being any sets of indices and satisfying the relations*

$$(1.7)\qquad f^{(\alpha)}{}_{(\beta)i_1\ldots i_r} y^{i_1}\cdots y^{i_r} = 0;\quad \partial f^{(\alpha)}{}_{(\beta)i_1 i_2\ldots i_r}/\partial y^{\rho} = \partial f^{(\alpha)}{}_{(\beta)\rho i_2\ldots i_r}/\partial y^{i_1},$$

then

$$f^{(\alpha)}{}_{(\beta)i_1 i_2\ldots i_r}(y) = 0.$$

For if we differentiate the first identity of the lemma partially with respect to y^{j_r} we obtain

$$(1.8)\qquad (\partial f^{(\alpha)}{}_{(\beta)i_1\ldots i_r}/\partial y^{j_r})\, y^{i_1}\cdots y^{i_r} + r f^{(\alpha)}{}_{(\beta)i_1\ldots i_{r-1}j_r}\, y^{i_1}\cdots y^{i_{r-1}} = 0$$

because of the symmetry of f in $i_1\cdots i_r$. The first term of (1.8) vanishes because of (1.7) and Euler's theorem, hence if we differentiate partially with respect to $y^{j_{r-1}}, y^{j_{r-2}}\cdots y^{j_1}$ we shall obtain $r!f^{(\alpha)}{}_{(\beta)j_1\ldots j_r} = 0$, which establishes the lemma.

The quantities in parenthesis of (1.6) are of the form $f^i{}_{\beta\gamma}(\bar x, \dot{\bar x})$; they obviously satisfy the first two conditions of the lemma, $\dot{\bar x}$ being the variables. To show that they satisfy (1.7) we have from (1.5)

$$(1.9)\qquad \partial x^i/\partial\dot{\bar x}^{\beta} = 0,\qquad \partial\dot x^i/\partial\dot{\bar x}^{\beta} = \partial x^i/\partial\bar x^{\beta},\qquad \partial\dot x^i/\partial\bar x^{\beta} = (\partial^2 x^i/\partial\bar x^{\beta}\partial\bar x^{\gamma})$$

so that

$$(1.10)\qquad \frac{\partial f^i{}_{\beta\gamma}}{\partial\dot{\bar x}^{\delta}} = \frac{\partial\Gamma^i{}_{jk}}{\partial\dot x^l}\,\frac{\partial x^j}{\partial\bar x^{\beta}}\,\frac{\partial x^k}{\partial\bar x^{\gamma}}\,\frac{\partial\dot x^l}{\partial\dot{\bar x}^{\delta}} - \frac{\partial\bar\Gamma_{\beta\gamma}{}^{\alpha}}{\partial\dot{\bar x}^{\delta}}\,\frac{\partial x^i}{\partial\bar x^{\alpha}} = \frac{\partial f^i{}_{\beta\delta}}{\partial\dot{\bar x}^{\gamma}}$$

by virtue of (1.4). Therefore $f^i{}_{\beta\gamma}$ satisfy all conditions of the lemma and the law of transformation of the coefficients of affine connection may be written as

$$(1.11)\qquad \bar\Gamma^{\alpha}{}_{\beta\gamma}\,\frac{\partial x^i}{\partial\bar x^{\alpha}} = \Gamma^i{}_{jk}\,\frac{\partial x^j}{\partial\bar x^{\beta}}\,\frac{\partial x^k}{\partial\bar x^{\gamma}} + \frac{\partial^2 x^i}{\partial\bar x^{\beta}\,\partial\bar x^{\gamma}}.$$

Since $f^i{}_{\beta\gamma} = 0$ it follows from (1.10) that

$$(1.12)\qquad \bar\Gamma^{\alpha}{}_{\beta\gamma\delta} = \Gamma^i{}_{jkl}\,\frac{\partial x^j}{\partial\bar x^{\beta}}\,\frac{\partial x^k}{\partial\bar x^{\gamma}}\,\frac{\partial x^l}{\partial\bar x^{\delta}}\,\frac{\partial\bar x^{\alpha}}{\partial x^i},$$

where

$$\Gamma^i{}_{jkl} \equiv \partial\Gamma^i{}_{jk}/\partial\dot x^l.$$

Since $\Gamma^i{}_{jk}(x, \dot x)$ are homogeneous of degree zero in $\dot x$ we have

$$\Gamma^i{}_{jk}(x, \dot x) = \Gamma^i{}_{jk}(x, dx)$$

and it is more convenient in the problems where the affine connection is fundamental to regard the Γ's as functions of x and $dx, dx^1, \cdots, dx^n$ being a set of independent variables subject to the conditions[*] $d\bar{x}^i = (\partial \bar{x}^i/\partial x^j)dx^j$.

By a *tensor* we shall understand an invariant whose components in a given coordinate system x are functions of x and dx satisfying the tensor law of transformation; that is, we remove the restriction that the components of a tensor must be functions of the coordinates alone.[†] A tensor whose components depend on x alone will be said to be a *tensor of position* while one depending on both x and dx will be called a *tensor of position and direction.*[‡]

It thus appears from (1.12) that Γ^i_{jkl} is a tensor. Its vanishing is a necessary and sufficient condition that the affine connection be a function of position. Evidently Γ^i_{jkl} is homogeneous of degree -1 in dx and is symmetric in its subscripts.

By differentiating (1.11) partially with respect to $\bar{x}^\delta$ and eliminating third partial derivatives we get the equations

$$\bar{K}^\alpha{}_{\beta\gamma\delta} = K^i{}_{jkl} \frac{\partial x^j}{\partial \bar{x}^\beta} \frac{\partial x^k}{\partial \bar{x}^\gamma} \frac{\partial x^l}{\partial \bar{x}^\delta} \frac{\partial \bar{x}^\alpha}{\partial x^i},$$

where

$$(1.13) \qquad \begin{aligned} K^i{}_{jkl} = {} & (\partial \Gamma^i{}_{jk}/\partial x^l) - (\partial \Gamma^i{}_{jl}/\partial x^k) + \Gamma^h{}_{jk}\Gamma^i{}_{hl} - \Gamma^h{}_{jl}\Gamma^i{}_{hk} \\ & - (\Gamma^i{}_{jkh}\Gamma^h{}_{lm} - \Gamma^i{}_{jlh}\Gamma^h{}_{km})\,dx^m. \end{aligned}$$

$K^i{}_{jkl}$ are the components of the *affine curvature tensor*[§] of the affine connection. It is evident that $K^i{}_{jkl}$ are homogeneous of degree zero in dx.

2. *Covariant Differentiation. Fundamental Identities.* By means of the affine connection and its law of transformation it is possible to obtain from any tensor an infinite sequence of tensors. Thus if $\xi^i(x, dx)$ are the components of a vector in the coordinate system x, the components in the coordinate system $\bar{x}$ are given by $\bar{\xi}^i = \xi^\alpha \partial \bar{x}^i/\partial x^\alpha$ or

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[*] For a general definition of differentials cf. O. Veblen, *Cambridge Tract*, No. 24, sec. 5.

[†] Cf. O. Veblen, *loc. cit.*, sec. 8.

[‡] It is to be noted that being a tensor of position is a "tensor property" since a transformation of coordinate does not depend on dx. But being a tensor of direction is not a "tensor property," that is, the components of a tensor may be functions of direction in a particular coordinate system without being functions of direction in every coordinate system. There are no tensors whose components are functions of direction alone in every coordinate system.

[§] L. Berwald, *Mathematische Zeitschrift*, Vol. 25 (1926), pp. 40-74. Cf. sec. 5, where $K^i{}_{jkl}$ are obtained from the notion of parallel propagation of a vector around an infinitesimal circuit.

$$\xi^i = \bar{\xi}^\alpha \partial x^i / \partial \bar{x}^\alpha. \tag{2.1}$$

Henceforth we shall use the notation in which a subscript preceded by a period indicates partial differentiation with respect to dx. From (2.1) we have by means of (1.9)

$$\xi^i{}_{.j} = \bar{\xi}^\alpha{}_{.\beta} \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\beta}{\partial x^j} \tag{2.2}$$

from which it follows that $\xi^i{}_{.j}$ are the components of a tensor. From (2.1) we also have

$$\frac{\partial \xi^i}{\partial x^j} = \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial x^j} \bar{\xi}^\alpha + \frac{\partial x^i}{\partial \bar{x}^\alpha} \left(\frac{\partial \bar{\xi}}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial x^j} + \bar{\xi}^\alpha{}_{.\beta} \frac{\partial (d\bar{x}^\beta)}{\partial x^j} \right).$$

Making use of (1.11), (1.9) and (2.2) the last set of equations may be written as

$$\xi^i{}_{,j} = \bar{\xi}^\alpha{}_{,\beta} \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\beta}{\partial x^j}, \tag{2.3}$$

where

$$\xi^i{}_{,j} \equiv \frac{\partial \xi^i}{\partial x^j} + \xi^h \Gamma^i{}_{hj} - \xi^i{}_{.h} \Gamma^h{}_{jk} dx^k. \tag{2.4}$$

From (2.3) it is evident that $\xi^i{}_{,j}$ is a tensor which we call the *covariant derivative* of the vector ξ^i with respect to the affine connection $\Gamma^i{}_{jk}$. And in general if $T^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ are the components of a tensor, its covariant derivative[*] is given by

$$\begin{aligned} T^{(i)}{}_{(j),k} = {} & \frac{\partial T^{(i)}{}_{(j)}}{\partial x^k} - T^{(i)}{}_{(j).h} \Gamma^h{}_{mk} dx^m \\ & + \sum_{a=1}^{r} T_{(j)}{}^{\cdots i_{a-1} h i_{a+1} \cdots} \Gamma^{i_a}_{hk} - \sum_{a=1}^{s} T^{(i)}{}_{\cdots j_{a-1} h j_{a+1} \cdots} \Gamma^h{}_{j_a k}. \end{aligned} \tag{2.5}$$

In order to obtain the fundamental relations between the components of the curvature tensor we introduce geodesic coordinates defined for an arbitrary point x_0 as origin and arbitrary direction dx_0 by means of

$$x^i = x_0{}^i + \bar{x}^i - \tfrac{1}{2} \Gamma^i{}_{jk}(x_0, dx_0) \bar{x}^j \bar{x}^k + \psi^i(\bar{x}) \tag{2.6}$$

where $\psi^i(\bar{x})$ are a set of functions, differentiable at least twice and vanishing together with first and second partial derivatives for $\bar{x}^i = 0$. From (2.6) we have

$$(\partial x^i / \partial \bar{x}^\beta)_0 = \delta_\beta{}^i; \qquad (\partial^2 x^i / \partial \bar{x}^\beta \partial \bar{x}^\gamma)_0 = -\Gamma^i{}_{\beta\gamma}(x_0, dx_0)$$

---

[*] Cf. Berwald, *loc. cit.*, § 12, where he arrives at this definition of covariant differentiation from the notion of parallelism.

so that (1. 11) become $\bar{\Gamma}^i{}_{\beta\gamma}(x_0, d\bar{x}_0) = 0$ and (1. 13) become in geodesic coordinates

$$(\bar{K}^i{}_{jkl})_0 = [(\partial\bar{\Gamma}^i{}_{jk}/\partial\bar{x}^l) - (\partial\bar{\Gamma}^i{}_{jl}/\partial\bar{x}^k)]_0. \tag{2. 7}$$

From (2. 5) in geodesic coordinates, we get by differentiation

$$\begin{aligned}(T^{(i)}{}_{(j),k,l})_0 = (\partial^2 T^{(i)}{}_{(j)}/\partial x^k\partial x^l)_0 - [T^{(i)}{}_{(j).h}(\partial\Gamma^h{}_{mk}/\partial x^l)\,dx^m]_0 \\ + \sum_{a=1}^{r} [T_{(j)}{}^{\cdots h\cdots}(\partial\Gamma^{i_a}_{hk}/\partial x^l)]_0 - \sum_{a=1}^{s} [T^{(i)}{}_{\ldots h\ldots}(\partial\Gamma^h{}_{j_ak}/\partial x^l)]_0.\end{aligned}$$

When from this equation we subtract the result of interchanging k and l, we obtain, because of (2. 7),

$$\begin{aligned}T^{(i)}{}_{(j),k,l} - T^{(i)}{}_{(j),l,k} = - T^{(i)}{}_{(j).h}K^h{}_{mkl}\,dx^m \\ + \sum_{a=1}^{r} T_{(j)}{}^{\cdots h\cdots}K^i{}_{hkl} - \sum_{a=1}^{s} T^{(i)}{}_{\ldots h\ldots}K^h{}_{j_akl}.\end{aligned} \tag{2. 8}$$

Equations (2. 8) being true at an arbitrary point for an arbitrary direction are true at any point in any direction. In a similar manner we get from (2. 5) in geodesic coordinates

$$\begin{aligned}(T^{(i)}{}_{(j),k.l})_0 = [\partial^2 T^{(i)}{}_{(j)}/\partial x^k\partial(dx^l)]_0. \\ + \sum_{a=1}^{r} (T_{(j)}{}^{\cdots h\cdots}\Gamma^{i_a}_{hkl} - \sum_{a=1}^{s} (T^{(i)}{}_{\ldots h\ldots}\Gamma^h{}_{j_akl})_0\end{aligned}$$

and

$$(T^{(i)}{}_{(j).l,k})_0 = [\partial^2 T^{(i)}{}_{(j)}/\partial x^k\partial(dx^l)]_0$$

Hence

$$T^{(i)}{}_{(j),k.l} - T^{(i)}{}_{(j).l,k} = \sum_{a=1}^{r} T_{(j)}{}^{\cdots h\cdots}\Gamma^{i_a}_{hkl}\ )_0 - \sum_{a=1}^{s} T^{(i)}{}_{\ldots h\ldots}\Gamma^h{}_{j_akl} \tag{2. 9}$$

which are also true in general.*

From (2. 7) or (1. 11) we see that

$$K^i{}_{jkl} + K^i{}_{jlk} = 0 \tag{2. 10}$$

and

$$K^i{}_{jkl} + K^i{}_{klj} + K^i{}_{ljk} = 0. \tag{2. 11}$$

From (1. 13), in geodesic coordinates we have

$$\begin{aligned}(K^i{}_{jkl,m})_0 = \{(\partial^2\Gamma^i{}_{jk}/\partial x^l\partial x^m) - (\partial^2\Gamma^i{}_{jl}/\partial x^k\partial x^m) \\ - [\Gamma^i{}_{jkh}(\partial\Gamma^h{}_{pl}/\partial x^m) - \Gamma^i{}_{jlh}(\partial\Gamma^h{}_{pk}/\partial x^m]\,dx^p\}_0.\end{aligned}$$

Interchanging k, l and m cyclically and adding the three equations, we find

---

* Cf. Berwald, *loc. cit.*, p. 53.

$$
\begin{aligned}
(2.12)\qquad & K^i_{jkl,m} + K^i_{jlm,k} + K^i_{jmk,l} \\
& + (K^p_{jkl}\Gamma^i_{phm} + K^p_{jlm}\Gamma^i_{phk} + K^p_{jmk}\Gamma^i_{phl})\dot{x}^h
\end{aligned}
$$

Another identity that we shall need in our discussion and which can be obtained by the methods indicated above is

$$(2.13)\qquad \Gamma^i_{jkl,m} - \Gamma^i_{jkm,l} = K^i_{klm,j} = K^i_{jlm,k}.$$

From (2.13) it follows immediately that

$$(2.14)\qquad K^i_{jkl,m} + K^i_{jkm,l} + K^i_{jmk,l} = 0$$

and by contraction for i and j we have

$$(2.15)\qquad S_{kl,m} + S_{lm,k} + S_{mk,l} = 0.$$

S being the alternating contracted curvature tensor K^i_{ikl}. If K_{jk} is the Ricci tensor K^i_{jki}, it follows from (2.11) by contraction for i and l that

$$(2.16)\qquad K_{jk} - K_{kj} = -S_{jk}.$$

Hence K_{jk} will be symmetric if and only if $S_{jk} = 0$.

3. *Projective Invariants.* The equations of the paths of an affinely connected manifold with coefficients of connection defined by (1.3) may be written in the form*

$$(3.1)\qquad \dot{x}^j[(d\dot{x}^i/dt) + \Gamma^i_{kl}\dot{x}^k\dot{x}^l] - \dot{x}^i[(d\dot{x}^j/dt) + \Gamma^j_{kl}\dot{x}^k\dot{x}^l] = 0.$$

Equations (3.1) are invariant in form under arbitrary transformations of coordinates as well as under arbitrary transformations of the parameter t. We inquire whether it is possible for two different affine connections in the same continuum to have the same paths. Let $\Gamma^i_{jk}(x, \dot{x})$ and $\Gamma'^i_{jk}(x, \dot{x})$ be the two connections. When equations of the type (3.1) for each connection are subtracted one from the other we obtain

$$(3.2)\qquad (\Gamma'^i_{kl} - \Gamma^i_{kl})\dot{x}^j\dot{x}^k\dot{x}^l - (\Gamma'^j_{kl} - \Gamma^j_{kl})\dot{x}^i\dot{x}^k\dot{x}^l = 0.$$

From (1.11) it follows that $\Gamma'^i_{kl} - \Gamma^i_{kl}$ is a tensor, homogeneous of degree zero in $\dot{x}$ and symmetric in its subscripts; we designate it by a^i_{kl}, and write (3.2) in the form

$$(3.3)\qquad (a^i_{kl}\delta^j_h - a^j_{kl}\delta^i_h)\dot{x}^h\dot{x}^k\dot{x}^l = 0.$$

The quantities $a^i_{kl}\delta^j_h - a^j_{kl}\delta^i_h$ do not satisfy the second set of conditions of the lemma of § 1. We therefore differentiate (3.3) partially and successively with respect to $\mathring{x}^p$, $\mathring{x}^q$ and $\mathring{x}^r$, the result being

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* Cf. O. Veblen and T. Y. Thomas, *Transactions of the American Mathematical Society*, Vol. 25 (1923), pp. 557-560.

$$(a^i{}_{pq}\delta^j{}_r - a^j{}_{pq}\delta^i{}_r) + (a^i{}_{qr}\delta^j{}_p - a^j{}_{qr}\delta^i{}_p) + (a^i{}_{rp}\delta^j{}_q - a^j{}_{rp}\delta^i{}_q) + \mathring{x}^j a^i{}_{pqr} - \mathring{x}^i a^j{}_{pqr} = 0, \tag{3.4}$$

where

$$a^i{}_{pqr} \equiv a^i{}_{pq.r} = a^i{}_{pr.q}. \tag{3.5}$$

When (3.4) are contracted for i and r we get

$$a^i{}_{jk} = (\delta^i{}_j a^h{}_{hk} + \delta^i{}_k a^h{}_{jh} + \mathring{x}^i a^h{}_{jkh})/(n+1). \tag{3.6}$$

From the definition of $a^i{}_{jk}$ and (1.3) it follows that $a^h{}_{hk}$ is a gradiant in the variables $\mathring{x}$; hence we may let $a^h{}_{hk} = (n+1)\phi_{.k}$ so that (3.6) become

$$a^i{}_{jk} = \delta^i{}_j\phi_{.k} + \delta^i{}_k\phi_{.j} + \mathring{x}^i\phi_{.j.k}, \tag{3.7}$$

ϕ being a homogeneous function of the first degree in $\mathring{x}$. Conversely, if $\Gamma'^i{}_{jk} - \Gamma^i{}_{jk} = a^i{}_{jk}$ the latter being defined by (3.7) with ϕ entirely arbitrary—except as to homogeneity—the equations of the paths for the connection Γ' reduce to (3.1). Hence we have

A necessary and sufficient condition that two manifolds, with affine connections $\Gamma^i{}_{jk}(x, \mathring{x})$ *and* $\Gamma'^i{}_{jk}(x, \mathring{x})$ *shall have the same paths is that*

$$\Gamma'^i{}_{jk} = \Gamma^i{}_{jk} + \delta^i{}_j\phi_{.k} + \delta^i{}_k\phi_{.j} + \mathring{x}^i\phi_{.j.k}, \tag{3.8}$$

ϕ *being an arbitrary function homogeneous of the first degree in* $\mathring{x}$.

The connection Γ' in the above theorem is said to be obtained from the connection Γ by a *projective change* determined by the function ϕ. Or we may say, somewhat less definitely, that the connections Γ and Γ' are *projectively related.*

Contracting (3.8) for i and j we find $\phi_{.k} = (\Gamma'^h{}_{hk} - \Gamma^h{}_{hk})/(n+1)$ which when put into (3.8) give

$$\Pi'^i{}_{jk} = \Pi^i{}_{jk}$$

where

$$\Pi^i{}_{jk} = \Gamma^i{}_{jk} - (1/n+1)\ (\delta^i{}_j\Gamma^h{}_{hk} + \delta^i{}_k\Gamma^h{}_{hj} + \mathring{x}^i\Gamma^h{}_{hjk}). \tag{3.9}$$

Following T. Y. Thomas* we call the invariant defined by (3.9) the *projective connection,* $\Pi^i{}_{jk}$ being the *coefficients* of projective connection. They are obviously symmetric in the subscripts, are homogeneous of degree zero in $\dot{x}$ and have the property

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* Cf. *Proceedings of the National Academy of Sciences,* Vol. 11 (1925), pp. 199-203. See also J. Douglas, *Annals of Mathematics,* 1928.

$$\Pi^h{}_{hk} = 0. \tag{3.10}$$

By a *projective invariant* we shall mean one whose components are unaltered by a projective change of affine connection. It therefore follows immediately that

$$\begin{aligned} \Pi^i{}_{jkl} \equiv \Pi^i{}_{jk.l} = \Gamma^i{}_{jkl} \\ -(1/n+1)(\delta^i{}_j\Gamma^h{}_{hkl} + \delta^i{}_k\Gamma^h{}_{jhl} + \delta^i{}_l\Gamma^h{}_{jkh} + \mathring{x}^i\Gamma^h{}_{jkl.h}) \end{aligned} \tag{3.11}$$

is a projective tensor invariant, which is symmetric in j, k, l.

When we form the expression for the curvature tensor of the connection Γ', a simple but rather long calculation gives

$$\begin{aligned} K'^i{}_{jkl} = K^i{}_{jkl} + \delta^i{}_j(\phi_{.k,l} - \phi_{.l,k}) + \delta^i{}_k(\phi_{.j,l} - \phi_{.j}\phi_{.l} - \phi\phi_{.j.l}) \\ - \delta^i{}_l(\phi_{.j,k} - \phi_{.j}\phi_{.k} - \phi\phi_{.j.k}) + \mathring{x}^i(\phi_{.j.k,l} - \phi_{.j.l,k}). \end{aligned} \tag{3.12}$$

When (3.12) are contracted for i and l and for i and k we find

$$\begin{aligned} K'_{jk} = K_{jk} + (\phi_{.k,j} - \phi_{.j,k}) \\ -(n-1)(\phi_{.j,k} - \phi_{.j}\phi_{.k} - \phi\phi_{.j.k}) + \mathring{x}^h\phi_{.j.k,h} \\ S'_{kl} = S_{kl} + (n+1)(\phi_{.k,l} - \phi_{.l,k}). \end{aligned} \tag{3.13}$$

In order to eliminate ϕ and its derivatives from (3.12) we have from the second of (3.13)

$$\phi_{.k,l} - \phi_{.l,k} = [(S'_{kl} - S_{kl})/(n+1)] \tag{3.14}$$

and therefore

$$\phi_{.k,l,j} - \phi_{.l,k,j} = [(S'_{kl.j} - S_{kl.j})/(n+1)]. \tag{3.15}$$

Now by means of (2.9) we get

$$\phi_{.k,l.j} = \phi_{.j.k,l} - \phi_{.h}\Gamma^h{}_{jkl}$$

this result and (3.15) give

$$\phi_{.j.k,l} - \phi_{.j.l,k} = [(S'_{kl.j} - S_{kl.j})/(n+1)]. \tag{3.16}$$

When (3.15) are multiplied by $\mathring{x}^l$ and summed for l we obtain, since $\phi_{.j}$ is homogeneous of degree zero in $\mathring{x}$,

$$\mathring{x}^h\phi_{.j.k,h} = [\mathring{x}^h/(n+1)](S'_{kh.j} - S_{kh.j}). \tag{3.17}$$

We substitute (3.14) and (3.17) in the first of (3.13) and get

$$\begin{aligned} \phi_{.j,k} - \phi_{.j}\phi_{.k} - \phi\phi_{.j.k} = [\mathring{x}^h/(n+1)](S'_{kh.j} - S_{kh.j}) \\ -[1/(n-1)](K'_{jk} - K_{jk}) - [1/(n^2-1)](S'_{jk} - S_{jk}). \end{aligned} \tag{3.18}$$

When (3.14), (3.16) and (3.18) are substituted in (3.12) we obtain $W'^i{}_{jkl} = W^i{}_{jkl}$ where

$$
\begin{aligned}
(3.19)\quad W^i{}_{jkl} = K^i{}_{jkl} &+ [\delta_k{}^i/(n-1)]\, K_{jl} - [\delta_l{}^i/(n-1)]\, K_{jk} \\
&- [\delta_j{}^i/(n+1)]\, S_{kl} + [1/(n^2-1)]\, (\delta_k{}^i S_{jl} - \delta_l{}^i S_{jk}) \\
&- [\mathring{x}^i/(n+1)]\, S_{kl.j} - [\mathring{x}^h/(n^2-1)]\, (\delta^i{}_k S_{lh.j} - \delta^i{}_l S_{kh.j}),
\end{aligned}
$$

with a similar expression for $W'^i{}_{jkl}$. Following Weyl,[*] who discovered this tensor for affine connections of position, we call it the *projective curvature tensor,* since its components are unaltered by an arbitrary projective change of affine connection.

From (3.19) by means of (2.10), (2.11), (2.15) and (2.16) it follows that

$$
(3.20)\qquad
\begin{aligned}
&W^i{}_{jkl} = W^i{}_{jlk}, \\
&W^i{}_{jkl} + W^i{}_{klj} + W^i{}_{ljk} = 0, \\
&W^h{}_{hkl} = W^h{}_{klh} = 0.
\end{aligned}
$$

From (3.19) we have

$$
\begin{aligned}
W^i{}_{jkl.m} = K^i{}_{jkl.m} &+ [\delta^i{}_k/(n-1)]\, K_{jl.m} - [\delta^i{}_l/(n-1)]\, K_{jk.m} \\
&- [1/(n+1)]\, (\delta^i{}_j S_{kl.m} + \delta^i{}_m S_{kl.j}) - [\mathring{x}^i/(n+1)]\, S_{kl.j.m} \\
&- [\mathring{x}^h/(n^2-1)]\, (\delta^i{}_k S_{lh.j.m} - \delta^i{}_l S_{kh.j.m}) \\
&+ [\delta^i{}_k/(n^2-1)](S_{jl.m} + S_{ml.j}) - [\delta^i{}_l/n^2-1)](S_{jk.m} + S_{mk.j}).
\end{aligned}
$$

When (2.13) are contracted for i and m we find

$$K_{jk.l} = K_{lk.j};$$

hence $W^i{}_{jkl.m} = W^i{}_{mkl.j}$, since every group of terms in the expression for $W^i{}_{jkl.m}$ is symmetric in j and m. This fact and the second of (3.20) give

$$(3.21)\qquad W^i{}_{jkl.m} + W^i{}_{jlm.k} + W^i{}_{jmk.l} = 0.$$

From (3.8) and (1.3) we get by contraction

$$(3.22)\qquad \Gamma'^h{}_{hk} = \tfrac{1}{2} H^i{}_{.i.k} + (n+1)\phi_{.k}.$$

We choose for the components of the vector $\phi_{.k}$ in the given coordinate system the values $-[1/2(n+1)]H^i{}_{.i.k}$; then $\Gamma'^h{}_{hk} = 0$. An affine connection for which $\Gamma^h{}_{hk} = 0$, in some coordinate system, is said to be *normal.* From (3.22) we have the theorem

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[*] Cf. *Göttinger Nachrichten* (1921), pp. 99-112.

For a given affine connection there exists, in each coordinate system, a unique projectively related normal affine connection.[*]

From (1.13) it follows that for a normal affine connection $S_{kl}=0$ and therefore the Ricci tensor K_{ij} is symmetric. From (3.9) we see that in this case $\Pi^i{}_{jk}=\Gamma^i{}_{jk}$ and (3.19) become

$$W^i{}_{jkl}=K^i{}_{jkl}+[\delta^i{}_k/(n-1)]\,K_{jl}-[\delta^i{}_l/(n-1)]\,K_{jk}. \tag{3.23}$$

4. *Transformation of Projective Connections. Projective Derivatives.* When we apply an arbitrary analytic transformation of coordinates

$$x^i=f^i(\bar{x}) \tag{4.1}$$

to a given affine connection, (1.11) gives

$$\bar{\Gamma}^\mu{}_{\mu\gamma}=\Gamma^h{}_{hk}(\partial x^k/\partial\bar{x}^\gamma)+(\partial\log\Delta/\partial\bar{x}^\gamma), \tag{4.2}$$

where $\Delta=|\,(\partial x/\partial\bar{x})\,|$ is the jacobian of (4.1). When this is applied to (3.9) we get the law of transformation of the components of projective connection in the form

$$\bar{\Pi}^\alpha{}_{\beta\gamma}\frac{\partial x^i}{\partial\bar{x}^\alpha}=\Pi^i{}_{jk}\frac{\partial x^j}{\partial\bar{x}^\beta}\frac{\partial x^k}{\partial\bar{x}^\gamma}-\frac{\partial x^i}{\partial\bar{x}^\beta}\frac{\partial\theta}{\partial\bar{x}^\gamma}-\frac{\partial x^i}{\partial\bar{x}^\gamma}\frac{\partial\theta}{\partial\bar{x}^\beta}, \tag{4.3}$$

where

$$\theta=[1/(n+1)]\log\Delta. \tag{4.4}$$

When we set up the conditions of integrability of (4.3) we arrive at

$$\bar{B}^\mu{}_{\alpha\beta\gamma}+\delta^\mu{}_\gamma\bar{c}_{\alpha\beta}-\delta^\mu{}_\beta\bar{c}_{\alpha\gamma}=B^i{}_{jkl}\frac{\partial x^j}{\partial\bar{x}^\alpha}\frac{\partial x^k}{\partial\bar{x}^\beta}\frac{\partial x^l}{\partial\bar{x}^\gamma}\frac{\partial\bar{x}^\mu}{\partial x^i}, \tag{4.5}$$

where

$$\begin{aligned}B^i{}_{jkl}=(\partial\Pi^i{}_{jk}/\partial x^l)-(\partial\Pi^i{}_{jl}/\partial x^k)+\Pi^h{}_{jk}\Pi^i{}_{hl}\\ -\Pi^h{}_{jl}\Pi^i{}_{hk}-(\Pi^i{}_{jkh}\Pi^h{}_{ml}-\Pi^i{}_{jlh}\Pi^h{}_{mk})\,dx^m\end{aligned} \tag{4.6}$$

and

$$\bar{c}_{\alpha\beta}=\bar{\Pi}^\epsilon{}_{\alpha\beta}\frac{\partial\theta}{\partial\bar{x}^\epsilon}+\frac{\partial\theta}{\partial\bar{x}^\alpha}\frac{\partial\theta}{\partial\bar{x}^\beta}-\frac{\partial^2\theta}{\partial\bar{x}^\alpha\partial\bar{x}^\beta}. \tag{4.7}$$

When (4.5) are contracted for μ and γ, we get

$$\bar{c}_{\alpha\beta}=[1/(n-1)]\,\left(B^h{}_{jkh}\frac{\partial x^j}{\partial\bar{x}^\alpha}\frac{\partial x^k}{\partial\bar{x}^\beta}-B^\epsilon{}_{\alpha\beta\epsilon}\right)$$

and these values put back into (4.5) give

$$W^\mu{}_{\alpha\beta\gamma}=W^i{}_{jkl}\frac{\partial x^j}{\partial\bar{x}^\alpha}\frac{\partial x^k}{\partial\bar{x}^\beta}\frac{\partial x^l}{\partial\bar{x}^\gamma}\frac{\partial\bar{x}^\mu}{\partial x^i},$$

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[*] Cf. Eisenhart, L. P., *Non-Riemannian Geometry*, § 37.

where

$$W^i{}_{jkl} = B^i{}_{jkl} + [\delta^i{}_k/(n-1)]\, B^h{}_{jlh} - [\delta^i{}_l/(n-1)]\, B^h{}_{jkh}; \tag{4.8}$$

direct computation shows that the tensor defined by (4.8) is identical with the Weyl tensor (3.19).*

We introduce the following notation:

$$\begin{aligned} r_{jk} &= [1/(n-1)]\, B^h{}_{jkh} \\ &= [1/(n-1)]\, [(\partial \Pi^h{}_{jk}/\partial x^h) - \Pi^h{}_{jp}\Pi^p{}_{hk} - \Pi^p{}_{jkh}\Pi^h{}_{pm}dx^m], \\ \theta_l &= (\partial\theta/\partial x^l) \qquad\qquad u^i{}_j = (\partial x^i/\partial \bar{x}^j). \end{aligned} \tag{4.9}$$

Equations (4.7) may now be written as

$$(\partial\theta_\alpha/\partial\bar{x}^\beta) = \bar{r}_{\alpha\beta} - r_{jk}u^j{}_\alpha u^k{}_\beta + \bar{\Pi}^\epsilon{}_{\alpha\beta}\theta_\epsilon + \theta_\alpha\theta_\beta. \tag{4.10}$$

When (4.10) are differentiated partially with respect to $d\bar{x}^\gamma$ and we let $r_{jk.l} = r_{jkl}$, we obtain

$$\bar{r}_{\alpha\beta\gamma} = r_{jkl}u^j{}_\alpha u^k{}_\beta u^l{}_\gamma - \theta_\epsilon\bar{\Pi}^\epsilon{}_{\alpha\beta\gamma}. \tag{4.11}$$

From the first of (4.9) it follows that r_{jkl} is symmetric in all its subscripts. This invariant is not a tensor except when the projective connection is a function of position alone in which case $r_{jkl} = 0$. By using (4.3), (4.10) and (4.11) in the reduction, the other conditions of integrability of (4.10) can be written in the form

$$\rho_{\alpha\beta\gamma} = \rho_{jkl}u^j{}_\alpha u^k{}_\beta u^l{}_\gamma - \theta_\epsilon\bar{W}^\epsilon{}_{\alpha\beta\gamma}, \tag{4.12}$$

where

$$\begin{aligned} \rho_{jkl} = {} & (\partial r_{jk}/\partial x^l) - (\partial r_{jl}/\partial x^k) + \Pi^h{}_{jk}r_{hl} \\ & - \Pi^h{}_{jl}r_{hk} - (r_{jkh}\Pi^h{}_{ml} - r_{jlh}\Pi^h{}_{mk})\, dx^m. \end{aligned} \tag{4.13}$$

The invariant ρ_{jkl} will be called the *projective covariant.*†

From a given projective invariant we can build a sequence of other projective invariants by projective differentiation.‡ We define the *projective derivative* of a projective invariant in the same way as the covariant derivative of a tensor was defined, [cf. (2.5)], except that the coefficients of affine connection are replaced by those of projective connection. It therefore follows that the identities existing between $K^i{}_{jkl}$ and $\Gamma^i{}_{jkl}$ are true for $B^i{}_{jkl}$ and

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* J. M. Thomas, *Proceedings of the National Academy of Sciences*, Vol. 11 (1925), pp. 207-209.

† Cf. O. Veblen and J. M. Thomas, *Annals of Mathematics*, Vol. 27 (1926), p. 287.

‡ This term has recently been adopted by O. Veblen to designate a different operation. Cf. § 9. It is used here in its original sense as defined below.

$\Pi^i{}_{jkl}$ provided covariant derivatives are replaced by projective derivatives the latter derivative being designated by a subscript preceded by a solidus. Thus (2.13) become

$$\Pi^i{}_{jkl/m} - \Pi^i{}_{jkm/l} = B^i{}_{klm.j} = B^i{}_{jlm.k}. \tag{4.14}$$

When this is contracted for i and m we obtain

$$r_{jkl} = [1/(n-1)]\, \Pi^h{}_{jkl/h} \tag{4.15}$$

and by means of (4.13) we can easily show that

$$\rho_{jkl} = r_{jk/l} - r_{jl/k}.$$

The identity analogous to (2.12) is

$$\begin{aligned} B^i{}_{jkl/m} + B^i{}_{jlm/k} + B^i{}_{jmk/l} \\ + (B^h{}_{pkl}\Pi^i{}_{jhm} + B^h{}_{plm}\Pi^i{}_{jhk} + B^h{}_{pmk}\Pi^i{}_{jhl})\, dx^p = 0. \end{aligned} \tag{4.16}$$

By means of (4.8) we find

$$B^i{}_{jkl/m} = W^i{}_{jkl/m} - \delta^i{}_k r_{jl/m} + \delta^i{}_l r_{jk/m}$$

and (4.16) then become

$$\begin{aligned} W^i{}_{jkl/m} + W^i{}_{jlm/k} + W^i{}_{jmk/l} = \delta^i{}_k \rho_{jlm} + \delta^i{}_l \rho_{jmk} + \delta^i{}_m \rho_{jkl} \\ - (W^h{}_{pkl}\Pi^i{}_{jhm} + W^h{}_{plm}\Pi^i{}_{jhk} + W^h{}_{pmk}\Pi^i{}_{jhl})\, dx^p. \end{aligned} \tag{4.17}$$

By contraction for i and m the last equation becomes

$$W^h{}_{jkl/h} = (n-2)\rho_{jkl} - (W^h{}_{plm}\Pi^m{}_{jhk} - W^h{}_{pkm}\Pi^m{}_{jhl})\, dx^p. \tag{4.18}$$

Since when $n = 2$ $B^i{}_{jkl}$ has only three independent components r_{11}, r_{12} and r_{22}, it is easy to show that in that case the Weyl tensor vanishes identically. By (4.12) ρ_{jkl} becomes a tensor and (4.18) are satisfied independently of the value of ρ_{jkl}. We shall find that the projective geometry of a manifold of more than two dimensions is completely characterized by the tensors $W^i{}_{jkl}$ and $\Pi^i{}_{jkl}$ and their projective and partial derivatives with respect to dx. For $n = 2$ ρ_{jkl} and $\Pi^i{}_{jkl}$ with their derivatives play the same rôle.

5. *Collineations.* Let $\bar{x}^i = \phi^i(x)$ be a point transformation which carries the point $(x^1, \cdots, x^n)$ into the point of coordinates $(\bar{x}^1, \cdots, \bar{x}^n)$. Let $x^i = f^i(t)$ be the finite equations of a path C, the parameter t being affine, that is, the equations of C satisfying

$$\frac{d^2x^i}{dt} + \Gamma^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0. \tag{5.1}$$

The transformation $\bar{x}^i = \phi^i(x)$ will carry C into a curve $\bar{C}$ whose finite equations are of the form $\bar{x}^i = \psi^i(t)$. This transformation will be called a *projective collineation* if it carries each path C into a path $\bar{C}$. The collineation will be called affine if when C satisfies equations (5.1) so does $\bar{C}$.

In the present paragraph we shall confine ourselves to projective collineations. Since in this case the parameter on the path is not necessarily preserved it will be convenient to take the equations of the paths in the form (3.1) which are unchanged by arbitrary transformations of parameter and may therefore be written as

$$(5.2) \qquad (d^2x^i + \Gamma^i{}_{jk}dx^jdx^k)\,dx^h - (d^2x^h + \Gamma^h{}_{jk}dx^jdx^k)\,dx^i = 0.$$

Moreover, we shall consider only finite continuous groups of projective collineations and our problem will be to find the necessary and sufficient conditions that a space of paths must satisfy in order to admit such a group.

We consider the infinitesimal transformations of a one parameter group which may be written as

$$(5.3) \qquad \bar{x}^i = x^i + \xi^i(x)\delta u$$

where $\xi^i(x)$ are the components of a contravariant vector and δu is a differential of the parameter of the group. By differentiation we obtain from (5.3)

$$(5.4) \qquad d\bar{x}^i = dx^i + (\partial\xi^i/\partial x^j)\,dx^j\delta u$$

and

$$(5.5) \qquad d^2\bar{x}^i = d^2x^i + [(\partial^2\xi^i/\partial x^j\partial x^k)\,dx^jdx^k + (\partial\xi^i/\partial x^j)\,d^2x^j]\,\delta u.$$

If (5.3) is to define an infinitesimal collineation $\Gamma^i{}_{jk}(\bar{x}, d\bar{x})$ must be the same functions of $\bar{x}$ and $d\bar{x}$ as $\Gamma^i{}_{jk}(x, dx)$ are of x and dx. Therefore expanding into a Taylor series we have

$$(5.6) \qquad \Gamma^i{}_{jk}(\bar{x}, d\bar{x}) = \Gamma^i{}_{jk}(x, dx) + [(\partial\Gamma^i{}_{jk}/\partial x^l)\xi^l + \Gamma^i{}_{jkl}(\partial\xi^l/\partial x^m)\,dx^m]\delta u,$$

higher powers of δu than the first being neglected throughout the problem. When we write (5.2) in terms of $\bar{x}$ and apply the transformations (5.3), (5.4), (5.5), (5.6) and use the fact that the transformation is a collineation the transformed equations reduce to

$$\begin{aligned}
&[(\partial^2\xi^i/\partial x^j\partial x^k) - \Gamma^a{}_{jk}(\partial\xi^i/\partial x^a) + \Gamma^i{}_{ja}(\partial\xi^a/\partial x^k) + \Gamma^i{}_{ak}(\partial\xi^a/\partial x^j) \\
&\quad + \xi^a(\partial\Gamma^i{}_{jk}/\partial x^a) + \Gamma^i{}_{jkl}(\partial\xi^l/\partial x^a)\,dx^a]\;dx^hdx^jdx^k \\
&\quad - [(\partial^2\xi^h/\partial x^j\partial x^k) - \Gamma^a{}_{jk}(\partial\xi^h/\partial x^a) + \Gamma^h{}_{ja}(\partial\xi^a/\partial x^k + \Gamma^h{}_{ak}(\partial\xi^a/\partial x^j) \\
&\quad + \xi^a(\partial\Gamma^h{}_{jk}/\partial x^a) + \Gamma^h{}_{jkl}(\partial\xi^l/\partial x^a)\,dx^a]\;dx^idx^jdx^k = 0,
\end{aligned}$$

where the arbitrary δu has been dropped.

If we compare the first parenthesis in the above equations with $\xi^i_{,j,k}$ whose value is given by

$$
\begin{aligned}
(5.61)\qquad \xi^i_{,j,k} = {} & (\partial^2\xi^i/\partial x^j\partial x^k) - \Gamma^a_{jk}(\partial\xi^i/\partial x^a) + \Gamma^i_{ja}(\partial\xi^a/\partial x^k) \\
& + \Gamma^i_{ak}(\partial\xi^a/\partial x^j) + \xi^a(\partial\Gamma^i_{ja}/\partial x^k) - \xi^a\Gamma^i_{ja\beta}\Gamma^\beta_{km}dx^m \\
& + \xi^a\Gamma^\beta_{ja}\Gamma^i_{\beta k} - \xi^a\Gamma^\beta_{jk}\Gamma^i_{\beta a},
\end{aligned}
$$

we shall find that the equations of condition reduce to

$$(5.7)\qquad a^i_{jk}dx^h dx^j dx^k - a^h_{jk}dx^i dx^j dx^k = 0,$$

where

$$(5.8)\qquad a^i_{jk} = \xi^i_{,j,k} + \xi^l K^i_{jkl} + \Gamma^i_{jkl}\xi^l_{,m}dx^m.$$

Because ξ is a vector of position alone and in virtue of (2.8) and (2.11), a^i_{jk} is a tensor, symmetric in its subscripts, homogeneous of degree zero in dx. By using (2.9) twice and (2.13) we find

$$
\begin{aligned}
(5.9)\qquad a^i_{jk.l} = {} & \xi^h\Gamma^i_{jkl,h} + \xi^h_{,m}\Gamma^i_{jkl.h}dx^m - \xi^i_{,h}\Gamma^h_{jkl} \\
& + \xi^h_{,j}\Gamma^i_{hkl} + \xi^h_{,k}\Gamma^i_{jhl} + \xi^h_{,l}\Gamma^i_{jkh},
\end{aligned}
$$

from which it is obvious that $a^i_{jkl} = a^i_{jk.l}$, is symmetric in its subscripts. Hence if we write (5.7) in the form (3.3), it follows from (3.7) that (5.7) may be replaced by equations of the form

$$(5.10)\qquad \xi^i_{,j,k} + \xi^l K^i_{jkl} + \Gamma^i_{jkl}\xi^l_{,m}dx^m = \delta^i_j\phi_{.k} + \delta^i_k\phi_{.j} + dx^i\cdot\phi_{.j.k}$$

where ϕ is to be determined.

The above equations are a necessary condition that the vector ξ must satisfy in order to define an infinitesimal projective collineation. To show that it is sufficient we write (5.10) in the form

$$
\begin{aligned}
(5.11)\qquad & (\partial^2\xi^i/\partial x^j\partial x^k) - \Gamma^m_{jk}(\partial\xi^i/\partial x^m) + \Gamma^i_{jm}(\partial\xi^m/\partial x^k) + \Gamma^i_{mk}(\partial\xi^m/\partial x^j) \\
& + \xi^m(\partial\Gamma^i_{jk}/\partial x^m) + \Gamma^i_{jkl}(\partial\xi^l/\partial x^m)dx^m \\
& = \delta^i_j\phi_{.k} + \delta^i_k\phi_{.j} + dx^i\phi_{.j.k}.
\end{aligned}
$$

Contracting for i and j we shall find

$$
\begin{aligned}
\phi_{.k} = [1/(n+1)]\,[& (\partial^2\xi^h/\partial x^h\partial x^k) + \Gamma^h_{hm}(\partial\xi^m/\partial x^k) \\
& + \xi^m(\partial\Gamma^h_{hk}/\partial x^m) + \Gamma^h_{hkl}(\partial\xi^l/\partial x^m)dx^m]
\end{aligned}
$$

and when we evaluate $\phi_{.j.k}$ and put these results in (5.11) they become

(5.12)

$$
\begin{aligned}
& (\partial^2\xi^i/\partial x^j\partial x^k) - \Pi^m_{jk}(\partial\xi^i/\partial x^m) + \Pi^i_{jm}(\partial\xi^m/\partial x^k) \\
& + \Pi^i_{mk}(\partial\xi^m/\partial x^j) + \xi^m(\partial\Pi^i_{jk}/\partial x^m) + \Pi^i_{jkl}(\partial\xi^l/\partial x^m)dx^m \\
& = [1/(n+1)]\,[\delta^i_j(\partial^2\xi^h/\partial x^h\partial x^k) + \delta^i_k(\partial^2\xi^h/\partial x^h\partial x^j)].
\end{aligned}
$$

Now suppose that (5.12) admit a solution ξ^i. By a suitable transformation of coordinates* we can normalize the vector ξ^i so that its components are $\delta^i{}_1$. Equations (5.10) being in tensor form remain in the same form and so do (5.12), the latter reducing in this coordinate system to

$$(\partial \Pi^i{}_{jk}/\partial x^1) = 0. \tag{5.13}$$

The finite transformations determined by the above vector are

$$\bar{x}^i = x^i + a\delta^i{}_1 \tag{5.14}$$

and since the jacobian $\Delta = 1$, $\bar{\Pi}^i{}_{jk} = \Pi^i{}_{jk}$, cf. (4.3). Therefore the transformations (5.14) carry paths into paths. We therefore have

A necessary and sufficient condition that a vector ξ^i must satisfy in order to define an infinitesimal projective collineation is that it be a solution of (5.12).

From (5.13) we can conclude

The most general projective connection admitting an infinitesimal projective collineation may be obtained by choosing for the coefficients of projective connection functions of $n - 1$ of the coordinates.

From (5.14) we have

A manifold admitting an infinitesimal projective collineation admits the finite continuous group G_1 of projective collineations.

Returning to (5.12) we see that when the left-hand side of (5.12) is compared with $\xi^i{}_{/j/k}$, [cf. (5.61)] with Γ's replaced by Π's, we obtain

$$\xi^i{}_{/j/k} + \xi^l B^i{}_{jkl} + \xi^l{}_{/m} dx^m \Pi^i{}_{jkl} = \delta^i{}_j \psi_{/k} + \delta^i{}_k \psi_{/j}, \tag{5.15}$$

where,

$$\psi_{/j} = [1/(n+1)]\, \xi^h{}_{/h/j} \tag{5.16}$$

and it is to be noted that $\psi_{/j.k} = 0$.

To find the conditions of integrability of (5.15) we have

$$\begin{aligned}\xi^i{}_{/j/k/l} = \delta^i{}_j \psi_{/k/l} + \delta^i{}_k \psi_{/j/l} - dx^m \psi_{/m} \Pi^i{}_{jkl} + dx^m \xi^h \Pi^i{}_{jkp} B^p{}_{mlh} \\ - \xi^h{}_{/m} dx^m \Pi^i{}_{jkh/l} - \xi^h{}_{/l} B^i{}_{jkh} - \xi^h B^i{}_{jkh/l}.\end{aligned}$$

If from this equation we subtract the one obtained from it by the interchange of k and l, make use of the fact that

$$\xi^i{}_{/j.p} = \xi^h \Pi^i{}_{jhp}, \tag{5.17}$$

of (2.8) and (2.11) in projective form and of (4.14) and (4.16), we shall have as the result

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[*] Cf. L. P. Eisenhart, "Riemannian Geometry," p. 5.

$$(5.18) \qquad \xi^h B^i{}_{jkl/h} + \xi^h{}_{/m} B^i{}_{jkl.h} dx^m - \xi^i{}_{/h} B^h{}_{jkl} + \xi^h{}_{/j} B^i{}_{hkl} \\ + \xi^h{}_{/k} B^h{}_{jhl} + \xi^h{}_{/l} B^i{}_{jkh} = \delta^i{}_k \psi_{/j/l}.$$

The conditions of integrability of (5.15) with regard to dx are obtainable by the partial differentiation of (5.15). We note that by (2.9), written in projective form of course, we have

$$\xi^i{}_{/j/k.l} = \xi^i{}_{/j.l/k} + \xi^h{}_{/j} \Pi^i{}_{hkl} - \xi^i{}_{/h} \Pi^h{}_{jkl}$$

and by the use of (5.17) this becomes

$$(5.19) \qquad \xi^i{}_{/j/k.l} = \xi^h \Pi^i{}_{jlh/k} + \xi^h{}_{/k} \Pi^i{}_{jhl} + \xi^h{}_{/j} \Pi^i{}_{hkl} - \xi^i{}_{/h} \Pi^h{}_{jkl}.$$

We therefore have by (4.14), (cf. remark after (5.16)),

$$(5.20) \qquad \xi^h \Pi^i{}_{jkl/h} + \xi^h{}_{/m} dx^m \Pi^i{}_{jkl.h} - \xi^i{}_{/h} \Pi^h{}_{jkl} \\ + \xi^h{}_{/j} \Pi^i{}_{hkl} + \xi^h{}_{/k} \Pi^i{}_{jhl} + \xi^h{}_{/l} \Pi^i{}_{jkh} = 0.$$

By contraction in (5.18) for i and l we obtain

$$(5.21) \qquad \psi_{/j/k} = - (\xi^h r_{jk/h} + \xi^h{}_{/m} dx^m r_{jkh} + \xi^h{}_{/j} r_{hk} + \xi^h{}_{/k} r_{hj}).$$

When the last results are substituted in (5.18) we get because of (4.8) and the first of (4.9)

$$(5.22) \qquad \xi^h W^i{}_{jkl/h} + \xi^h{}_{/m} dx^m W^i{}_{jkl.h} - \xi^i{}_{/h} W^h{}_{jkl} \\ + \xi^h{}_{/j} W^i{}_{hkl} + \xi^h{}_{/k} W^i{}_{jhl} + \xi^h{}_{/l} W^i{}_{jkh} = 0.$$

That the above equations are invariant under an arbitrary projective change of affine connection is evident, since each term is independent of this change. When we express each projective derivative in terms of covariant derivatives, the form of (5.22) remains unaltered.* Hence (5.22) are tensor equations, i. e., they are invariant under an arbitrary analytic transformation of coordinates.

We must now examine the conditions of integrability of (5.21). To this end we differentiate (5.21) partially with respect to dx^l and use the fact that $\psi_{/j.l} = 0$, getting

$$(5.23) \qquad \psi_{/h} \Pi^h{}_{jkl} = \xi^h r_{jkl/h} + \xi^h{}_{/j} r_{hkl} + \xi^h{}_{/k} r_{jhl} + \xi^h{}_{/l} r_{jkh}.$$

If we differentiate (5.20) projectively and use (2.8) and (2.9) in projective form we shall get

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* Cf. M. S. Knebelman, *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), p. 397.

$$\begin{aligned}\xi^h\Pi^i{}_{jkl/m/h} &+ \xi^h{}_{/s}dx^s\Pi^i{}_{jkl/m.h} - \xi^i{}_{/h}\Pi^h{}_{jkl/m} + \xi^h{}_{/j}\Pi^i{}_{hkl/m} \\ &+ \xi^h{}_{/k}\Pi^i{}_{jhl/m} + \xi^h{}_{/l}\Pi^i{}_{jkh/m} + \xi^h{}_{/m}\Pi^i{}_{jkl/h} \\ &+ \Pi^i{}_{jkl}\psi_{/m} + \psi_{/s}dx^s\Pi^i{}_{jkl.m} - \delta^i{}_m\Pi^h{}_{jkl}\psi_{/h} \\ &+ \Pi^i{}_{mkl}\psi_{/j} + \Pi^i{}_{jml}\psi_{/k} + \Pi^i{}_{jkm}\psi_{/l} = 0.\end{aligned} \tag{5.24}$$

When (5.24) are contracted for i and m and use is made of (4.15) we obtain (5.23). Hence in studying the conditions that the vector ξ^i must satisfy we need not consider the equations arising from (5.21) by partial differentiation.

We now take the projective derivative of (5.21) obtaining after reduction by means of (5.15)

$$\begin{aligned}\psi_{/h}B^h{}_{jkl} = \xi^h\rho_{jkl/h} &+ \xi^h{}_{/m}dx^m\rho_{jkl.h} + \xi^h{}_{/j}\rho_{hkl} \\ &+ \xi^h{}_{/k}\rho_{jhl} + \xi^h{}_{/l}\rho_{jkh} + r_{jk}\psi_{/l} - r_{jl}\psi_{/k},\end{aligned}$$

which may be written as, cf. (4.8), (4.9),

$$\psi_{/h}W^h{}_{jkl} = \xi^h\rho_{jkl/h} + \xi^h{}_{/m}dx^m\rho_{jkl.h} + \xi^h{}_{/j}\rho_{hkl} + \xi^h{}_{/k}\rho_{jhl} + \xi^h{}_{/l}\rho_{jkh}. \tag{5.25}$$

Projective differentiation of (5.22) gives

$$\begin{aligned}\xi^hW^i{}_{jkl/m/s} &+ \xi^s{}_{/v}dx^vW^i{}_{jkl/m.s} - \xi^i{}_{/h}W^h{}_{jkl/m} + \xi^h{}_{/k}W^i{}_{jhl/m} \\ &+ \xi^h{}_{/l}W^i{}_{jkh/m} + \xi^h{}_{/m}W^i{}_{jkl/h} + 2W^i{}_{jkl}\psi_{/m} \\ &- \delta^i{}_mW^h{}_{jkl}\psi_{/h} + W^i{}_{mkl}\psi_{/j} + W^i{}_{jml}\psi_{/k} + W^i{}_{jkm}\psi_{/l} = 0.\end{aligned} \tag{5.26}$$

By contraction for i and m we get from (5.26)

$$\begin{aligned}(n-2)W^h{}_{jkl}\psi_{/h} = \xi^sW^h{}_{jkl/h/s} &+ \xi^s{}_{/v}dx^vW^h{}_{skl/h} \\ &+ \xi^s{}_{/k}W^h{}_{jsl/h} + \xi^s{}_{/l}W^h{}_{jks/h}.\end{aligned} \tag{5.27}$$

When $n > 2$ (5.27) can be reduced by means of (4.18), (5.20) and (5.22) to (5.25) thus showing that in this case the conditions of integrability of (5.21) are consequences of (5.20) and (5.22). In the case $n = 2$ (5.22) vanish identically and their place is taken by (5.25).

6. *Spaces of paths admitting an r-parameter group of projective collineations.* If a space is to admit an r-parameter group G_r of projective collineations equations (5.15) must admit r linearly independent solutions $\xi^i{}_{(1)}, \cdots, \xi^i{}_{(r)}$, each vector $\xi^i{}_{(a)}$ being a vector of position. The question of the existence and of the number of solutions of a system of partial differential equations is answered by a well-known theorem on mixed systems.*

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* L. Bianchi, *Teoria dei Gruppi Continui*, Pisa (1918), Chap. I; O. Veblen, "Invariants of Quadratic Differential Forms," *Cambridge Tract*, No. 24, pp. 73-76.

The theorem has to be somewhat modified in order to apply to equations (5.15). In the first place (5.15) are a pure system, that is, there are no conditions of the form $f(\xi, x) = 0$ imposed on the solutions and secondly it is obviously only a question of convenience whether we write our equations and their conditions of integrability in terms of the usual partial derivatives, covariant derivatives or projective derivatives.

Let " $;\alpha$ " denote projective differentiation when $\alpha = 1, \cdots, n$ and partial differentiation with respect x^α when $\alpha = n + 1, \cdots, 2n$ where x^{n+k} is used in place of dx^k. We may then write (5.15) as

$$Z^\mu{}_{;\alpha} = f^\mu{}_\alpha(x, Z) \begin{pmatrix} \mu = 1, \cdots, n^2 + 2n \\ \alpha = 1, \ldots, 2n \end{pmatrix}, \tag{6.1}$$

Z^μ being the dependent variables ξ^i, $\xi^i/_j$ and $\psi/_j$. In order to obtain the conditions of integrability we have (2.5) and (2.9) written in terms of projective derivatives which we can combine into

$$Z^\mu{}_{;\alpha;\beta} - Z^\mu{}_{;\beta;\alpha} = A^{\mu\gamma}{}_{\lambda\alpha\beta} Z^\lambda{}_{;\gamma}. \tag{6.2}$$

Thus by means of (6.2) we get from (6.1) a system of equations between x and Z which we denote by $F^{(1)}(x, Z)$. Differentiating each of the independent equations of the set $F^{(1)}$ and eliminating $Z^\mu{}_{;\alpha}$ by means of (6.1) we arrive at a set $F^{(2)}(x, Z)$ this process being repeated indefinitely. The theorem mentioned at beginning of this paragraph may be stated thus:

A necessary and sufficient condition that the functions $f^\mu{}_\alpha$ must satisfy in order that (6.2) *admit a solution is that the sets of equations $F^{(1)}$, $F^{(2)}$, $F^{(3)}, \cdots$ shall be compatible for the determination of the unknown quantities Z.*

In the particular problem before us $f^\mu{}_\alpha(x, Z)$ for the system (5.15) are linear and homogeneous in Z. Hence the above theorem may be stated in a more explicit form since in this case the sets of equations $F^{(1)}$, $F^{(2)}, \cdots$ are also linear homogeneous in Z. The theorem is

A necessary and sufficient condition that the sets $F^{(1)}$, $F^{(2)}, \cdots$ must satisfy in order to be compatible for the determination of the Z's is that there exist a positive integer $N(\geq 1)$ such that the matrix of the equations of the sets $F^{(1)}, \cdots, F^{(N)}$ and that of the equations of the sets $F^{(1)}, \cdots, F^{(N+1)}$ shall have the same rank. If M is the number of dependent functions Z

and $M-r$ $(1 \leq r \leq M)$[*] *is the common rank of the above matrices, the number of linearly independent solutions is* r.

If $r=M$ the conditions of integrability are satisfied identically and the equations (6.1) are then said to be completely integrable.

To apply the above results to the problem of projective collineations, we take for Z^a the functions ξ^i, $\xi^i{}_{/j}$, $\psi_{/j}$ and for the operation " $;i$ " we choose $;i = /i$ $(i=1,\cdots,n)$ and $;i = .i$ $(i=n+1,\cdots,2n)$, the independent variables being so chosen that $x^i = x^i$, $x^{n+i} = dx^i$ $(i=1,\cdots,n)$. The equations corresponding to (6.1) are then

$$(6.3) \qquad \xi^i{}_{.k}=0, \quad (\psi_{/j})_{.k}=0, \quad (\xi^i{}_{/j})_{.k}=\xi^h\Pi^i{}_{hjk}, \quad (\xi^i)_{/k}=\xi^i{}_{/k}$$

$$(5.21) \qquad (\psi_{/j})_{/k} = -(\xi^h r_{jk/h} + \xi^h{}_{/m} dx^m r_{jkh} + \xi^h{}_{/j} r_{hk} + \xi^h{}_{/k} r_{hj})$$

and

$$(6.4) \qquad (\xi^i{}_{/j})_{/k} = -(\xi^h B^i{}_{jkh} + \xi^h{}_{/m} dx^m \Pi^i{}_{jkh} - \delta^i_j \psi_{/k} - \delta^i_k \psi_{/j}).$$

The conditions of integrability of (6.3) are satisfied because of (6.3) and (2.9) and the conditions of integrability of (5.21) and (6.4) are (5.20) and (5.22). We therefore have

A necessary and sufficient condition that a projective connection must satisfy in order to admit r *linearly independent one parameter finite continuous groups of projective collineations* $(1 \leq r \leq n^2+2n)$ *is that there exist a positive integer* N *such that the sets of equations* $F^{(1)},\cdots,F^{(N)}$ *and the sets* $F^{(1)},\cdots,F^{(N+1)}$, *arising from the equations* (5.20) *and* (5.22), *shall be of rank* n^2+2n-r.

If equations (5.15) are to be completely integrable, (5.20) and (5.22) must vanish identically. If we write these equations in the form

$$(6.5) \qquad \begin{aligned} &\xi^h\Pi^i{}_{jkl/h} + \xi^h{}_{/m}(dx^m\Pi^i{}_{jkl.h} - \delta^i_h\Pi^m{}_{jkl} \\ &\qquad\qquad + \delta^m_j\Pi^i{}_{hkl} + \delta^m_k\Pi^i{}_{jhl} + \delta^m_l\Pi^i{}_{jkh}) = 0 \\ &\xi^h W^i{}_{jkl/h} + \xi^h{}_{/m}(dx^m W^i{}_{jkl.h} - \delta^i_h W^m{}_{jkl} \\ &\qquad\qquad + \delta^m_j W^i{}_{hkl} + \delta^m_k W^i{}_{jhl} + \delta^m_l W^i{}_{jkh}) = 0, \end{aligned}$$

it follows that

$$(6.6) \qquad \Pi^i{}_{jkl/h} = W^i{}_{jkl/h} = 0$$

and each parenthesis in (6.5) must vanish.

When these last two quantities are contracted for m and h we get

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[*] We do not consider the case $r=0$ since the equations are linear and homogeneous in the Z's. In the problem of equivalence $r=0$ implies equivalence under a particular transformation while $r>0$ implies equivalence under a finite continuous group of transformations. Cf. Eisenhart, *Non-Riemannian Geometry*, pp. 16-17.

$$\Pi^i{}_{jkl} = 0, \qquad 2W^i{}_{jkl} = 0. \tag{6.7}$$

From the first of (6.7) and the definition of $\Pi^i{}_{jkl}$, Cf. (3.11), it follows that the projective connection is a function of position and the second set of equations (6.7) for $n > 2$ are a necessary and sufficient condition for the space of paths to be projectively flat.[*] In case $n = 2$ the process applied above to (5.23) and (5.25), yields $p_{ijk} = 0$ and $r_{ijk} = 0$ which implies that the two-space is projectively flat. Hence

A necessary and sufficient condition that a projective connection must satisfy in order to admit $n^2 + 2n$ linearly independent projective collineations is that it be projectively flat.

7. *Group Property of Projective Collineations.* In § 5 it was proved that when a space of paths admits an infinitesimal collineation it admits the one parameter group G_1 generated by it. We shall now show that

If a space of paths admits r linearly independent collineations it admits the r-parameter group G_r generated by them.

For let ξ^i and η^i be any two of the r linearly independent solutions that equations (5.15) are assumed to admit and let $X \equiv \xi^i(\partial/\partial x^i)$, $Y \equiv \eta^i(\partial/\partial x^i)$ be the corresponding generators. In order to prove the above theorem it is necessary and sufficient to show that the Poisson operator (X, Y) is expressible linearly with constant coefficients in terms of the r generators. Let $(X, Y) = \lambda^i(\partial/\partial x^i)$. Then

$$\begin{aligned}(X, Y)f &= (XY - YX)f \\ &= [\xi^i(\partial\eta^j/\partial x^i) - \eta^i(\partial\xi^j/\partial x^i)]\ (\partial f/\partial x^j) = \lambda^j(\partial f/\partial x^j)\end{aligned} \tag{7.1}$$

and since (7.1) must hold for arbitrary functions we have

$$\lambda^i = \xi^k(\partial\eta^i/\partial x^k) - \eta^k(\partial\xi^i/\partial x^k)$$

or better

$$\lambda^i = \xi^k\eta^i{}_{/k} - \eta^k\xi^i{}_{/k}. \tag{7.2}$$

From (7.2) we obtain by projective differentiation

$$\lambda^i{}_{/j} = \xi^h\eta^i{}_{/h/j} - \eta^h\xi^i{}_{/h/j} + \xi^h{}_{/j}\eta^i{}_{/h} - \eta^h{}_{/j}\xi^i{}_{/h}$$

and since ξ^i and η^i satisfy (5.15), we get by means of these equations

$$\begin{aligned}\lambda^i{}_{/j} = {}& \xi^h{}_{/j}\eta^i{}_{/h} - \eta^h{}_{/j}\xi^i{}_{/h} + \xi^i\phi_{/j} - \eta^i\psi_{/j} + \delta^i_j(\xi^h\phi_{/h} - \eta^h\psi_{/h}) \\ & + \xi^h\eta^l B^i{}_{jlh} - \Pi^i{}_{jlh}(\xi^h\eta^l{}_{/m} - \eta^h\xi^l{}_{/m})\,dx^m,\end{aligned} \tag{7.3}$$

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[*] Cf. O. Veblen and J. M. Thomas, *Annals of Mathematics*, Vol. 27 (1926), p. 293-, § 11.

where in accordance with (5.16)

$$\psi_{/j} = [1/(n+1)]\, \xi^h{}_{/h/j}, \qquad \phi_{/j} = [1/(n+1)]\eta^h{}_{/h/j}.$$

When (7.3) are differentiated projectively and second derivatives of ξ^i and η^i are eliminated by means of equations (5.15) we obtain

$$\begin{aligned}\lambda^i{}_{/j/k} = {} & \xi^i{}_{/h}(\eta^l B^h{}_{jkl} + \eta^l{}_{/m} dx^m \Pi^h{}_{jkl}) + \xi^h{}_{/j}(\eta^l B^i{}_{hkl} + \eta^l{}_{/m} dx^m \Pi^i{}_{hkl}) \\ & - \eta^i{}_{/h}(\xi^l B^h{}_{jkl} + \xi^l{}_{/m} dx^m \Pi^h{}_{jkl}) - \eta^h{}_{/j}(\xi^l B^i{}_{hkl} + \xi^l{}_{/m} dx^m \Pi^i{}_{hkl}) \\ & + \xi^h \eta^l B^i{}_{jhl/k} + B^i{}_{jlh}(\eta^l \xi^h{}_{/k} - \xi^l \eta^h{}_{/k}) - (\xi^h \eta^l{}_{/m} - \eta^h \xi^l{}_{/m}) \Pi^i{}_{jlh/k} dx^m \\ & - \Pi^i{}_{jlh}\, [\xi^h{}_{/k}\eta^l{}_{/m} - \eta^h{}_{/k}\xi^l{}_{/m} + B^l{}_{mkp}(\eta^h \xi^p - \eta^p \xi^h)]\, dx^m \\ & + \xi^i \phi_{/j/k} - \eta^i \psi_{/j/k} + \delta^i_j (\xi^h \phi_{/h} - \eta^h \psi_{/h})_{/k}.\end{aligned}$$

When the values of $\Pi^i{}_{jkl}\lambda^l{}_{/m}dx^m$ and of $\lambda^h B^i{}_{jhk}$ computed by means of (7.2) and (7.3), are added to the above equations the result is reducible to six groups of terms; the first two of the form (5.18), the second two of the form (5.19) the remaining two being $\delta^i_j(\xi^h\phi_{/h} - \eta^h\psi_{/h})_{/k}$ and $\delta^i_k(\xi^h\phi_{/h} - \eta^h\psi_{/h})_{/j}$. That is, we find

$$\begin{aligned}(7.4) \qquad & \lambda^i{}_{/j/k} + \Pi^i{}_{jkl}\lambda^l{}_{/m}dx^m + \lambda^l B^i{}_{jkl} \\ & = \delta^i_k(\xi^h\phi_{/h} - \eta^h\psi_{/h})_{/j} + \delta^i_j(\xi^h\phi_{/h} - \eta^h\psi_{/h})_{/k}.\end{aligned}$$

When (7.3) are contrasted for i and j, the result is

$$\xi^h\phi_{/h} - \eta^h\psi_{/h} = [1/(n+1)]\, \lambda^h{}_{/h}$$

so that the right hand side of (7.4) can be written as $[1/(n+1)]\, (\delta^i_j\lambda^h{}_{/h/k} + \delta^i_k\lambda^h{}_{/h/j})$, showing that (7.4) are of the form (5.15). That is, λ^i is therefore a solution of (5.15) which are homogeneous of the first degree; this implies that if $\xi^i{}_{(a)}$ $(\alpha = 1, \cdots, r)$ are the r linearly independent solutions of (5.15), $\lambda^i = a^\alpha \xi^i{}_{(a)}$, where a^α are a set of constants. Hence if $X_a \equiv \xi^i{}_{(a)}(\partial/\partial x^i)$ we have

$$(X_\alpha, X_\beta)f = c^\gamma{}_{\alpha\beta} X_\gamma f$$

$c^\gamma{}_{\alpha\beta}$ being the constants of composition of the group.

8. *Affine Collineations.* By an affine collineation we shall understand a point transformation carrying paths into paths and preserving the affine parameter. That is, if a path is given in terms of an affine parameter, it is carried by an affine collineation into a path satisfying the more restricted equations (1.2). Using the notation of § 5, it follows from (5.7) that

$$(8.1) \qquad a^i{}_{jk}dx^j dx^k = 0.$$

From (5.10) it is evident that $a^i_{jk.l} = a^i_{jl.k}$ and since a^i_{jk} are homogeneous of degree zero in dx, they satisfy all the conditions of the lemma of § 1. Hence if the vector ξ^i is to define an infinitesimal affine collineation it is necessary and sufficient that it satisfy the equations

$$\xi^i_{,j,k} + \xi^l K^i_{jkl} + \xi^l_{,m} \Gamma^i_{jkl} dx^m = 0. \tag{8.2}$$

By a process very analogous to that used in § 5 we find the conditions of integrability of (8.2) to be

$$\xi^h K^i_{jkl,h} + \xi^h_{,m} dx^m K^i_{jkl.h} - \xi^i_{,h} K^h_{jkl} + \xi^h_{,j} K^i_{hkl} + \xi^h_{,k} K^i_{jhl} + \xi^h_{,l} K^i_{jl} \tag{8.3}$$

and

$$\begin{aligned} \xi^h \Gamma^i_{jkl,h} + \xi^h_{,m} dx^m \Gamma^i_{jkl.h} - \xi^i_{,h} \Gamma^h_{jkl} \\ + \xi^h_{,j} \Gamma^i_{hkl} + \xi^h_{,k} \Gamma^i_{jhl} + \xi^h_{,l} \Gamma^i_{jkh} = 0. \end{aligned} \tag{8.4}$$

By using the coordinate system employed to obtain (5.13) we see that if $\xi^i = a\delta^i_1$ is to satisfy (8.2) we must have $(\partial \Gamma^i_{jk}/\partial x^1) = 0$. Hence

The most general affine connection admitting a G_1 group of affine collineations may be obtained by taking $H^i(x, \dot{x})$ to be functions of $n - 1$ of *the coordinates x homogeneous of the second degree in $\dot{x}^1, \cdots, \dot{x}^n$.*

If equations (8.2) are to be completely integrable, equations (8.3) and (8.4) must be identities in the $n^2 + n$ quantities $\xi^i_{,j}$ and ξ^i. Therefore,

The greatest number of linearly independent affine collineations that a space of paths may admit is $n^2 + n$.

In this case we have from (8.3) and (8.4)

$$K^i_{jkl.h} dx^m - \delta^i_h K^m_{jkl} + \delta^m_j K^i_{hkl} + \delta^m_k K^i_{jhl} + \delta^m_l K^i_{jkh} = 0 \tag{8.5}$$

and

$$\Gamma^i_{jkl.h} dx^m - \delta^i_h \Gamma^m_{jkl} + \delta^m_j \Gamma^i_{hkl} + \delta^m_k {}^i_{jhl} + \delta^m_l \Gamma^i_{jkh} = 0. \tag{8.6}$$

Contracting each set of the above equations for m and h we find

$$K^i_{jkl} = 0, \qquad \Gamma^i_{jkl} = 0. \tag{8.7}$$

From the second set of (8.7) we see that the affine connection is a function of position (Cf. § 1) and the vanishing of the curvature tensor then shows that the space of paths is flat. Hence

A necessary and sufficient condition that a space of paths must satisfy in order to admit $n^2 + n$ affine collineations is that it be flat.

In a flat space there exists a class of coordinate systems in which the components of affine connection vanish. In one of these coordinate systems (8.2) become $(\partial^2\xi^i/\partial x^j\partial x^k)=0$ and therefore $\xi^i=a^i{}_h x^h+b^i$, the finite collineations being $\bar{x}^i=A^i{}_h x^h+B^h$.

If the space is not flat, the existence, and the number, of linearly independent affine collineations can be determined by differentiating (8.3) and (8.4) covariantly and partially with respect to dx and eliminating $\xi^i{}_{,j,k}$ by means of (8.2). We are thus led to a sequence of sets of equations $F^{(1)}, F^{(2)}, \cdots$ which must be compatible for the determination of the quantities ξ^i and $\xi^i{}_{,j}$.

Whether equations (8.2) admit one or more solutions, there exist a number of linear relations between the equations of the sets F so that the number of these equations can be greatly reduced by the following method: let the left hand sides of (8.3) and (8.4) be denoted by $T^i{}_{jkl}$ and $V^i{}_{jkl}$ respectively. Let $T^i{}_{jkl.m,p\ldots}$ and $V^i{}_{jkl,p.m\ldots}$ denote the results of performing the indicated operations and eliminating $\xi^i{}_{,j,k}$ by means of (8.2). Then, by means of (2.9) $V^i{}_{jkl.m}$ has the value

$$\begin{aligned}(8.8)\qquad V^i{}_{jkl.m}=&\ \xi^h\Gamma^i{}_{jkl.m,h}+\xi^h{}_{,p}\Gamma^i{}_{jkl.m.h}dx^p-\xi^i{}_{,h}\Gamma^h{}_{jkl.m}\\&+\xi^h{}_{,j}\Gamma^i{}_{hkl.m}+\xi^h{}_{,k}\Gamma^i{}_{jhl.m}+\xi^h{}_{,l}\Gamma^i{}_{jkh.m}+\xi^h{}_{,m}\Gamma^i{}_{jkl.h}.\end{aligned}$$

Evidently $V^i{}_{jkl.m}$ are a set of functions homogeneous of degree -2 in dx and symmetric in all subscripts. We assume

$$\begin{aligned}(8.9)\qquad V^i{}_{(P)}=&\ \xi^h\Gamma^i{}_{(P),h}+\xi^h{}_{,j}\Gamma^i{}_{(P).h}dx^j\\&-\xi^i{}_{,h}\Gamma^h{}_{(P)}+\sum_{a=1}^{p}\xi^h{}_{,j_a}\Gamma^i{}_{j_1\ldots j_{a-1}hj_{a+1}\ldots j_P}\end{aligned}$$

where (P) represents the subscripts $j_1,\cdots,j_P$. Partial differentiation of (8.9) with respect to dx and the use of (2.9) and (5.17) in covariant form shows that $V^i{}_{(P).j_{P+1}}=V^i{}_{(P+1)}$; hence reasoning by induction we see that (8.9) are true for any number of subscripts. Moreover $V^i{}_{(P)}$ are homogeneous of degree $-(P-2)$ in dx. Hence if we multiply (8.9) by dx^k and sum for k and any one of the subscripts, say j_P, we obtain

$$V^i{}_{(P)}dx^{j_P}=V^i{}_{(P-1).j_P}dx^{j_P}=-(P-3)V^i{}_{(P-1)}.$$

Therefore all the equations that can be obtained from (8.4) by partial differentiation with respect to dx 0, 1, 2, $\cdots$, r times, are equivalent to the last set. That is, out of $n\left[\binom{n+r+3}{n}-\binom{n+2}{2}\right]$ there are at most $n\left[\binom{n+r+2}{n+1}-\binom{n+1}{1}\right]$ independent equations in these sets.

Differentiating (8.4) covariantly, eliminating second covariant derivatives of ξ^i and using (2.8) and (2.9) we find

$$V^i{}_{jkl,m} = \xi^h \Gamma^i{}_{jkl,m,h} + \xi^h{}_{,r} \Gamma^i{}_{jkl,m.h} dx^r - \xi^i{}_{,h} \Gamma^h{}_{jkl,m}$$
$$+ \xi^h{}_{,j} \Gamma^i{}_{hkl,m} + \xi^h{}_{,k} \Gamma^i{}_{jhl,m} + \xi^h{}_{,l} \Gamma^i{}_{jkh,m} + \xi^h{}_{,m} \Gamma^i{}_{jkl,h}$$

and by a simple computation it can be shown that

$$(8.10) \quad V^i{}_{jkl,m.p} - V^i{}_{jkl.p,m} = V^h{}_{jkl} \Gamma^i{}_{hmp} - V^i{}_{hkl} \Gamma^h{}_{jmp} - V^i{}_{jhl} \Gamma^h{}_{kmp}$$
$$- V^i{}_{jkh} \Gamma^h{}_{lmp} + V^i{}_{hmp} \Gamma^h{}_{jkl} - V^h{}_{jmp} \Gamma^i{}_{hkl} - V^h{}_{kmp} \Gamma^i{}_{jhl} - V^h{}_{lmp} \Gamma^i{}_{jkh}.$$

Therefore the only equations we need consider arising out of (8.4) are those obtainable by covariant differentiation out of $V^i{}_{(r+2)}$, r being the greatest number of partial differentiations with respect to dx.

By means of (2.13) it can be shown that

$$(8.11) \qquad T^i{}_{jkl.m} = V^i{}_{jmk,l} - V^i{}_{jml,k}.$$

Therefore partial differentiation with respect to dx of equations (8.3) does not lead to equations which are linearly independent of those considered above. It can be shown also that

$$(8.12) \qquad T^i{}_{jkl,m.p} = T^i{}_{jkl.p,m} + T^h{}_{jkl} \Gamma^i{}_{hmp}$$
$$- T^i{}_{hkl} \Gamma^h{}_{jmp} - T^i{}_{jhl} \Gamma^h{}_{kmp} - T^i{}_{jkh} \Gamma^h{}_{lmp} + V^i{}_{hmp} K^h{}_{jkl}$$
$$- V^h{}_{jmp} K^i{}_{hkl} - V^h{}_{kmp} K^i{}_{jhl} - V^h{}_{lmp} K^i{}_{jkh}.$$

Hence the only equations arising out of (8.3) that we need consider, are those obtained by repeated covariant differentiation. In view of these facts we can state the following theorem:

A necessary and sufficient condition that an affine space of paths must satisfy in order to admit $r (0 < r \leq n^2 + n)$ *linearly independent affine collineations is that there exist two integers* $N (> 0)$ *and* $P (> 0)$ *such that the sets of equations* $F^{(1)}, \cdots, F^{(N)}$ *arising*[*] *out of* $T^i{}_{jkl} = 0$ and $V^i{}_{(P+2)} = 0$ *and the sets* $F^{(1)}, \cdots, F^{(N+1)}$ *arising out of* $T^i{}_{jkl} = 0$ *and* $V^i{}_{(P+3)} = 0$, *shall have matrices of rank* $n^2 + n - r$.

If $\xi^i{}_{(\alpha)}$ and $\xi^i{}_{(\beta)}$ define two linearly independent infinitesimal collineations in an affine space, the components of their alternant are (cf. § 7)

$$(8.13) \qquad \xi^i = \xi^m{}_{(\alpha)} \xi^i{}_{(\beta),m} - \xi^m{}_{(\beta)} \xi^i{}_{(\alpha),m}.$$

By means of (8.2) we obtain

$$(8.14) \qquad \xi^i{}_{,j} = \xi^m{}_{(\alpha),j} \xi^i{}_{(\beta),m} - \xi^m{}_{(\beta),j} \xi^i{}_{(\alpha),m} + \xi^l{}_{(\alpha)} \xi^m{}_{(\beta)} K^i{}_{jml}$$
$$+ \Gamma^i{}_{jlm} (\xi^m{}_{(\beta)} \xi^l{}_{(\alpha),h} - \xi^m{}_{(\alpha)} \xi^l{}_{(\beta),h}) dx^h$$

---

[*] By "arising" we mean "obtained by covariant differentiation *alone* and elimination of $\xi^i{}_{,j,k}$ by means of (8.2)."

and

$$
\begin{aligned}
\xi^i{}_{,j,k} = {} & \xi^i{}_{(a),m}(K^m{}_{jkh}\xi^h{}_{(\beta)} + \xi^h{}_{(\beta),l}dx^l\Gamma^m{}_{jkh}) \\
& - \xi^i{}_{(\beta),m}(K^m{}_{jkh}\xi^h{}_{(a)} + \xi^h{}_{(a),l}dx^l\Gamma^m{}_{jkh}) \\
& + \xi^m{}_{(\beta),j}(\xi^h{}_{(a)}K^i{}_{mkh} + \xi^h{}_{(a),l}dx^l\Gamma^i{}_{mkh}) \\
& - \xi^m{}_{(a),j}(\xi^h{}_{(\beta)}K^i{}_{mkh} + \xi^h{}_{(\beta),l}dx^l\Gamma^i{}_{mkh}) \\
& + \xi^l{}_{(a),k}\xi^m{}_{(\beta)}K^i{}_{jml} - \xi^l{}_{(a)}\xi^m{}_{(\beta),k}K^i{}_{jml} \\
& + dx^h\Gamma^i{}_{jml.k}(\xi^m{}_{(\beta)}\xi^l{}_{(a),h} - \xi^m{}_{(a)}\xi^l{}_{(\beta),h}) \\
& + dx^h\Gamma^i{}_{jml}(\xi^m{}_{(\beta),k}\xi^l{}_{(a),h} - \xi^m{}_{(a),k}\xi^l{}_{(\beta),h}) \\
& + (\xi^m{}_{(a)}\xi^p{}_{(\beta)} - \xi^m{}_{(\beta)}\xi^p{}_{(a)})K^l{}_{hkp}\Gamma^i{}_{jml}dx^h.
\end{aligned}
\tag{8.15}
$$

When to (8.15) we add the expressions for $\xi^l K^i{}_{jkl}$ and $\xi^l{}_{,h}dx^h\Gamma^i{}_{jkl}$ we obtain

$$
\begin{aligned}
\xi^i{}_{,j,k} + \xi^l K^i{}_{jkl} + \xi^l{}_{,h}dx^h\Gamma^i{}_{jkl} = {} & \xi^h{}_{(a)}T^i{}_{(\beta)jkh} - \xi^h{}_{(\beta)}T^i{}_{(a)jkh} \\
& + (\xi^h{}_{(a),l}V^i{}_{(\beta)jkh} - \xi^h{}_{(\beta),l}V^i{}_{(a)jkh})dx^l,
\end{aligned}
$$

the remaining terms vanishing because of (2.12) and (2.13). Now $\xi^i{}_{(a)}$ and $\xi^i{}_{(\beta)}$ are assumed to satisfy (8.2) and therefore also (8.3) and (8.4). It therefore follows that ξ^i satisfies (8.2) and we have

An affine space admitting r linearly independent infinitesimal collineations admits the finite continuous group G_r of collineations.

9. *Collineations in Spaces of Paths whose Connection is a Function of Position.* If $\Gamma^i{}_{jkl} = 0$ all functions used in the preceding paragraphs, which are homogeneous of degree zero become free of *dx* while those which are homogeneous of negative degree vanish. The equations for affine collineations are then

$$\xi^i{}_{,j,k} + \xi^l B^i{}_{jkl} = 0 \tag{9.1}$$

and for projective collineations[*]

$$\xi^i{}_{,j,k} + \xi^l B^i{}_{jkl} = \delta^i{}_j\phi_k + \delta^i{}_k\phi_j, \tag{9.2}$$

a point of difference between the two being in the fact that in the first case we are concerned with partial differential equations which are homogeneous in ξ^i while in the second case they are not. It is however possible to reduce the problem of projective collineations to the affine case in one of two ways; either by the use of the generalized projective connection of Veblen or by means of the associated space of T. Y. Thomas.[†]

We shall make use of the first mentioned method. Briefly stated the alized projective connection of Veblen is an invariant whose components transform according to the law

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[*] Cf. M. S. Knebelman, *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 396-400.

[†] Cf. O. Veblen, *Proceedings of the National Academy of Sciences*, Vol. 14 (1928), pp. 154-166; T. Y. Thomas, *Mathematische Zeitschrift*, Vol. 25 (1926), pp. 723-733.

$$(9.3)\qquad \bar\Pi^\alpha_{\beta\gamma} = \Pi^\rho_{\lambda\mu} v^\alpha_\rho u^\lambda_\beta u^\mu_\gamma + v^\alpha_\rho (\partial u^\rho_\beta / \partial \bar x^i)\delta^i_\gamma,$$

where the Greek indices are on the range 0, 1, $\cdots$, n while Latin ones are on the range 1, $\cdots$, n and

$$(9.4)\qquad u_0^\alpha = \delta_0^\alpha, \qquad u_j^i = (\partial x^i/\partial \bar x^j), \qquad u_j^0 = -[\partial \log u^{1/(n+1)}/\partial \bar x^j],$$

u being the jacobian of the transformation and v^α_β is the normalized cofactor of u^α_β, i. e., $u^\alpha_\beta v^\beta_\gamma = \delta^\alpha_\gamma$. This projective connection is more general than the one previously used as no conditions on the **Π**'s need be imposed; but in order to suit it to a geometry of paths the following conditions are imposed on the **Π**'s:

$$(9.5)\qquad \Pi^\alpha_{\beta\gamma} = \Pi^\alpha_{\gamma\beta}, \qquad \Pi^i_{ij} = 0, \qquad \Pi^\alpha_{\beta 0} = \delta^\alpha_\beta,$$
$$\Pi^0_{jk} = -[1/(n-1)]\,[(\partial \Pi^i_{jk}/\partial x^i) - \Pi^i_{lj}\Pi^l_{ik}].$$

The components Π^i_{jk} of the generalized connection transform according to the law (4.3), Π^0_{jk} being $-r_{jk}$ of § 4 for connections of position.

We may look upon a projective collineation as a transformation which on carrying a point x to the point $\bar x$ carries $\Pi^i_{jk}(x)$ into $\Pi^i_{jk}(\bar x)$. We must therefore consider the existence of solutions of equations (9.3)—with $\bar\Pi^\alpha_{\beta\gamma}(\bar x)$ replaced by $\Pi^\alpha_{\beta\gamma}(\bar x)$—regarded as second order partial differential equations in the n dependent variables x and the n independent variables $\bar x$. As previously we consider the infinitesimal transformation

$$(9.6)\qquad x^i = \bar x^i + \xi^i(\bar x)\,\delta t$$

from which we obtain

$$(9.7)\qquad (\partial x^i/\partial \bar x^j) = \delta_j^i + (\partial \xi^i/\partial \bar x^j)\delta t$$

and this gives*

$$(9.8)\qquad (\partial u/\partial \bar x^j) = (\partial^2 \xi^h/\partial \bar x^h \partial \bar x^j)\delta t.$$

The vector ξ^i in (9.6) is an affine vector †; to make it projective we introduce the component ξ^0 defined by

$$(9.9)\qquad \xi^0 = -\frac{1}{n+1}\,\frac{\partial \xi^h}{\partial \bar x^h}$$

so that (9.4) become, for our infinitesimal transformation

$$(9.10)\qquad u_\beta^\alpha = \delta_\beta^\alpha + (\partial \xi^\alpha/\partial \bar x^j)\cdot \delta_\beta^j \cdot \delta t.$$

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* Cf. Eisenhart and Knebelman, *loc. cit.*, p. 41.

† Cf. O. Veblen, *loc. cit.*

By Taylor's expansion—we assume of course that the Π's are analytic functions of the x's—we have

$$\Pi^\alpha{}_{\beta\gamma}(x) = \Pi^\alpha{}_{\beta\gamma}(\bar{x}) + [\partial\Pi^\alpha{}_{\beta\gamma}(\bar{x})/\partial\bar{x}^l]\xi^l\delta t + \cdots.$$

Substitution of these results into (9.3) gives (we drop the bars from over the x's and Π's)

$$(9.11)\qquad \delta^j{}_\beta\delta^k{}_\gamma(\partial^2\xi^\alpha/\partial x^j\partial x^k) + \Pi^\alpha{}_{\lambda\gamma}(\partial\xi^\lambda/\partial x^j)\delta_\beta{}^j + \Pi^\alpha{}_{\beta\lambda}(\partial\xi^\lambda/\partial x^j)\delta_\gamma{}^j$$
$$- \Pi^j{}_{\beta\gamma}(\partial\xi^\alpha/\partial x^j) + \xi^\lambda(\partial\Pi^\alpha{}_{\beta\gamma}/\partial\bar{x}^l) = 0.$$

If β or γ or both are zero (9.11) vanish identically and if $\alpha = 0$ we get (5.21) (the term $r_{jkh} \equiv 0$ for connections of position). When equations (9.11) are combined with the second *projective derivative* * of ξ^α—whose components are given by

$$\xi^\alpha{}_{,\beta,\gamma} = (\partial^2\xi^\alpha/\partial x^j\partial x^k)\delta^j{}_\beta\delta^k{}_\gamma + \Pi^\alpha{}_{\lambda\beta}(\partial\xi^\lambda/\partial x^k)\delta^k{}_\gamma + \xi^\lambda(\partial\Pi^\alpha{}_{\lambda\beta}/\partial x^k)\delta^k{}_\gamma$$
$$+ \Pi^\alpha{}_{\lambda\gamma}[(\partial\xi^\lambda/\partial x^j)\delta^j{}_\beta + \xi^\mu\Pi^\lambda{}_{\mu\beta}] - \Pi^\lambda{}_{\beta\gamma}[(\partial\xi^\alpha/\partial x^j)\delta^j{}_\lambda + \xi^\mu\Pi^\alpha{}_{\mu\lambda}]$$

we get

$$(9.12)\qquad \xi^\alpha{}_{,\beta,\gamma} + \xi^\mu B^\alpha{}_{\beta\gamma\mu} = 0,$$

$B^\alpha{}_{\beta\gamma\mu}$ being the components of the *projective curvature tensor.* Equations (9.12) are formally identical with (9.1) and the process of projective differentiation being formally identical with covariant differentiation as it is ordinarily defined, we have reduced the problem of projective collineations to one of affine collineations.

It may be of some interest to count the possible number of solutions of (9.12). Suppose the equations of the sets $F^{(1)}, \cdots, F^{(N)}$ of the last theorem but one of § 8 are of rank r; according to that theorem (9.12) will admit $(n+1)^2 + (n+1) - r$ linearly independent solutions. But in this case $(n+1)+1$ linearly independent conditions are imposed on each solution, namely the equations giving the values of $\xi^0{}_{,\alpha}$ and the one equation (9.9). Hence the number of solutions is $n^2 + 2n - r$ which is in agreement with the result obtained in § 6.

The basic advantage of this method of treating projective collineations is the fact that the equations used are *projective tensor* equations, which is not the case with most of the equations used in §§ 4-7. The generalization of this projective connection to one of position and direction should not prove difficult.

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* The term projective derivative is here used in Veblen's sense. Cf. *loc. cit.*, §§ 6 and 7.

10. *Generalized Metric Space. Equations of Motion.* In the preceding paragraphs we have been dealing with spaces of paths or affinely connected manifolds. We shall now consider an n-dimensional continuum in which there exists an absolute scalar differential invariant $F(x, dx)$, positively homogeneous of the second degree in dx and subject to the condition $| F_{.i.j} | \neq 0$. The invariant $F(x, dx)$ is called a *metric* and a space in which it exists *a generalized metric space.*[*]

An affinely connected space may be metric in more than one way; that is, it may admit more than one scalar invariant of the type of $F(x, dx)$. We are not concerned with that question here; we assume the existence of $F(x, dx)$ and obtain the affine properties of the space in terms of it.

We assume that the coordinates of a point are real and that a curve is given by $x^i = f^i(t)$, $f(t)$ being analytic functions of t not all constants. The element of arc ds is defined by

$$ds^2 = eF(x, \mathring{x})\,dt, \tag{10.1}$$

where e is plus or minus one according as F is positive or negative. By Euler's theorem on homogeneous functions we have

$$F(x, \mathring{x}) = \tfrac{1}{2} F_{.i.j} \mathring{x}^i \mathring{x}^j. \tag{10.2}$$

Hence if we let $\frac{1}{2} F_{.i.j} = g_{ij}$ (10.1) becomes

$$\mathring{s}^2 = e g_{ij} \mathring{x}^i \mathring{x}^j, \tag{10.3}$$

where, obviously, g_{ij} are symmetric in the subscripts, homogeneous of degree zero in $\mathring{x}$ and since F is a scalar, g_{ij} are the components of a tensor. It also follows easily from the law of transformation of a tensor that $g_{ijk} (\equiv g_{ij.k})$ is a tensor symmetric in its subscripts and homogeneous of degree -1 in $\mathring{x}$.

When we minimize $\int_{t_0}^{t_1} \dot{s}\,dt$ we obtain, as in Riemannian geometry, the equations of the extremals, or geodesics,

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{10.4}$$

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[*] The study of homogeneous differential forms other than quadratic polynomials dates back to R. Lipschitz, *Crelle*, Vol. 70 (1869), and Vol. 71, 72 (1870), where covariant differentiation first made its appearance. A complete system of invariants—the normal tensors—was developed by Emmy Noether in the *Göttinger Nachrichten*, Vol. 25 (1918); but the study of the geometry of a generalized metric space begins with P. Finsler's Dissertation, *Göttingen* (1918). It is probably for this reason that some authors call a space with a generalized metric a Finsler space.

where for simplicity's sake the arc s is used for a parameter, $\left\{ {i \atop jk} \right\}$ being the Christoffel symbols of the second kind formed out of $g_{ij}[x, (dx/ds)] \cdot \left\{ {i \atop jk} \right\}$ being homogeneous of degree zero in $\mathring{x}$, $\left\{ {i \atop jk} \right\} \mathring{x}^j \mathring{x}^k$ are the functions H^i of § 1; from this fact we get by partial differentiation *

$$\Gamma^i{}_{jk} = \left\{ {i \atop jk} \right\} + \mathring{x}^a \left\{ {i \atop ak} \right\}_{.j} + \mathring{x}^\beta \left\{ {i \atop j\beta} \right\}_{.k} + \tfrac{1}{2} \mathring{x}^a \mathring{x}^\beta \left\{ {i \atop a\beta} \right\}_{.j.k} \tag{10.5}$$

as the components of the fundamental affine connection of the metric space. From the definition of covariant differentiation of § 2, it follows that †

$$g_{ij,k} = g_{a\beta} \mathring{x}^\beta \Gamma^a{}_{ijk} \tag{10.6}$$

where it is to be understood that $\Gamma^a{}_{ij}$ are the components of the fundamental affine connection as defined by (10.5).

By a *motion* in a metric space we understand a member of a continuous group of point transformations which preserves the arc-length of every curve. We consider, as in the case of collineations, the effect of an infinitesimal transformation defined by a vector ξ^i on (10.3). From the definition of a geodesic it follows that if the point transformation is to be a motion it must carry geodesics into geodesics, and since the arc-length of every curve is preserved, the vector ξ^i must satisfy equations (8.2). Equations (8.2) however do not constitute a set of necessary and sufficient conditions that the vector ξ^i must satisfy in order to define an infinitesimal motion; to obtain these we have:

$$\begin{aligned}
&\dot{\bar{s}}^2 = e\bar{g}_{ij}(\bar{x}, \dot{\bar{x}}) \dot{\bar{x}}^i \dot{\bar{x}}^j \\
&\bar{x}^i = x^i + \xi^i(x)\,\delta u \\
&\dot{\bar{x}}^i = \mathring{x}^i + (\partial \xi^i / \partial x^j) \mathring{x}^j \,\delta u \\
&\bar{g}_{ij}(\bar{x}, \bar{x}) = g_{ij} + (\partial g_{ij} / \partial x^k) \xi^k \delta u + g_{ijk} (\partial \xi^k / \partial x^l) \mathring{x}^l \,\delta u + \cdots.
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{\bar{s}}^2 = e[g_{ij} + (\partial g_{ij} / \partial x^k) \xi^k \delta u + g_{ijk} (\partial \xi^k / \partial x^l) \mathring{x}^l \,\delta u + \cdots] \\
[\mathring{x}^i + (\partial \xi^i / \partial x^h) \mathring{x}^h \delta u] [\mathring{x}^j + (\partial \xi^j / \partial x^m) \mathring{x}^m \delta u]
\end{aligned}$$

from which we get, upon neglecting higher powers of δ_u than the first,

$$\dot{\bar{s}}^2 = \mathring{s}^2 + e h_{ij} \mathring{x}^i \mathring{x}^j \delta u$$

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[*] Cf. Berwald, *Jahresbericht der Deutschen Mathematiker Vereinigung*, Vol. 34 (1925), pp. 213-20.

[†] Cf. Berwald, *Mathematische Zeitschrift*, Vol. 25 (1926), p. 54.

where

$$h_{ij} = g_{ih}(\partial\xi^h/\partial x^j) + g_{hj}(\partial\xi^h/\partial x^i) + \xi^h(\partial g_{ij}/\partial x^h) + g_{ijh}(\partial\xi^h/\partial x^l)\dot{x}^l. \tag{10.7}$$

The vanishing of $h_{ij}\dot{x}^i\dot{x}^j$ is a necessary and sufficient condition that ξ^i must satisfy in order to define an infinitesimal motion. It is not difficult to verify the fact that the functions h_{ij} satisfy all conditions of the lemma of § 1. Hence we have

If a vector $\xi^i(x)$ *is to define an infinitesimal motion, it must satisfy the* $n(n+1)/2$ *equations*

$$g_{ih}(\partial\xi^h/\partial x^j) + g_{hj}(\partial\xi^h/\partial x^i) + \xi^h(\partial g_{ij}/\partial x^h) + g_{ijh}(\partial\xi^h/\partial x^l)\,\dot{x}^l = 0.$$

When in the above equations the partial derivatives are replaced in terms of covariant ones, these equations become

$$g_{ih}\xi^h{}_{,j} + g_{hj}\xi^h{}_{,i} + \xi^h g_{ij,h} + \xi^h{}_{,m}\dot{x}^m g_{ijh} = 0. \tag{10.8}$$

We shall refer to (10.8) as the equations of Killing who obtained them for a Riemann space.* From (10.8) it is evident that h_{ij} are the components of a tensor. We make use of this fact in obtaining metric spaces which admit a given infinitesimal motion. For the given vector $\xi^i(x)$ defining the motion can be normalized so that its components are, say, $\delta_1{}^i$. Then (10.8) become $\partial g_{ij}/\partial x^1 = 0$. Hence we have

The most general metric space admitting an infinitesimal motion may be obtained by choosing for $F(x,\dot{x})$ a function of $x^2,\cdots,x^n$ *homogeneous of the second degree in* $\dot{x}^1,\cdots,\dot{x}^n$, *such that* $F_{.1}\neq 0$.

Since the finite transformations generated by the infinitesimal one defined by $\delta_1{}^i$ are $\bar{x}^i = x^i + \delta_1{}^i a$ and g_{ij} is free of x^1, it follows that the arc is preserved under the finite transformations. Hence

A space admitting an infinitesimal motion admits the one parameter finite continuous group G_1 *of motions generated by the infinitesimal one.*

Two infinitesimal motions are said to be linearly independent if there exists no relation of the form $a\xi^i + b\eta^i = 0$ a and b being constants $(a, b\neq 0)$. By the *path-curves* of a motion we understand the curves of the congruence determined by the vector defining the motion. We then inquire whether two linearly independent motions may have the same path curves. To answer this question we normalize the vector defining one motion so that its components are $\delta_1{}^i$; then if the second motion is to have the same path-curves its components must be $\phi(x)\delta_1{}^i$, which when substituted into Killing's equations give

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* W. Killing, *Crelle's Journal,* Vol. 109, pp. 121-186.

$$g_{i1}(\partial\phi/\partial x^j) + g_{1j}(\partial\phi/\partial x^i) + g_{ij1}(\partial\phi/\partial x^l)\dot{x}^l = 0. \tag{10.9}$$

When these are multiplied by $\dot{x}^i\dot{x}^j$ and summed for i and for j we get $F_{.1}(d\phi/dt) = 0$ and since $F_{.1} \neq 0$ (cf. last theorem but one above) we must have $\phi = \text{const.}$ which makes the motions linearly dependent. Hence

Two linearly independent motions can not have the same path-curves.

In a general metric space the length of a vector and the angle between two vectors are defined with respect to a direction.[*] The length of a vector ξ^i relative to a direction dx is given by

$$l = \sqrt{e g_{ij}(x, dx)\xi^i\xi^j} \tag{10.10}$$

while the angle between two vectors ξ^i and η^i relative to a direction dx is given by

$$\cos\theta = \frac{g_{ij}(x, dx)\xi^i\eta^j}{\sqrt{e_1 g_{ij}\xi^i\xi^j}\ \sqrt{e_2 g_{kl}\eta^k\eta^l}} \tag{10.11}$$

where it is understood that neither ξ^i nor η^i is a null vector relative to the direction dx.

A motion will be said to be *translatory* or a translation if its path-curves are geodesics[†] and our next problem is to determine a method for obtaining a general metric space admitting a one parameter group of translations. We first consider the necessary and sufficient conditions that the metric F must satisfy in order that the curves of parameter x^1 (i. e. the curves $x^2 = c^2$, $x^3 = c^3, \cdots, x^n = c^n$) shall be geodesics. Our coordinate system may be so chosen that x^1 is the arc of the curves of parameter x^1 [*] and if these curves are to be geodesics we must have

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[*] Cf. L. Berwald, *Mathematische Zeitschrift*, Vol. 25 (1926), p. 56; J. L. Synge, *Transactions of the American Mathematical Society*, Vol. 29 (1925), pp. 61-68.

[†] A translation in Riemann spaces is defined as a motion in which each point moves through the same distance and our definition is proved as a theorem. It is just as easy to prove that if the path-curves of a motion are geodesics each point moves through the same distance, the main reason for the adopted definition being the fact that it applies to transatory collineations where no distance is involved. Cf. *Riemannian Geometry*, § 72.

$$\frac{d^2x^i}{dx^{12}} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{dx^1} \; \frac{dx^k}{dx^1} = 0,$$

which reduce to $\left\{ \begin{matrix} i \\ 11 \end{matrix} \right\} = 0$. From the last equation we have upon multiplying by g_{ih} and summing for i, $[11, h] = 0$ where $[ij, h]$ are the Christoffel symbols of the first kind. Hence our condition becomes

$$(10.12) \qquad \partial g_{1h}/\partial x^1 - \tfrac{1}{2}\partial g_{11}/\partial x^h = 0.$$

Since x^1 is to be the arc of these curves we have from (10.3) $g_{11} = e$. This and (10.12) gives $(\partial g_{1h}/\partial x^1) = 0$ that is, g_{1h} is free of x^1. From these results we see that $F (= g_{ij}\dot{x}^i\dot{x}^j)$ must be of the form $a(\dot{x}^1)^2 + \dot{x}^1\phi_1 + \phi_2$ where ϕ_1 is homogeneous of the first degree in $\dot{x}^2, \cdots, \dot{x}^n$ and is free of x^1 while ϕ_2 is homogeneous of degree two in $\dot{x}^2, \cdots, \dot{x}^n$. That this condition on F is sufficient may be proved very easily by showing that the curves $x^2 = c^2$, $x^3 = c^3, \cdots, x^n = c^n$ are geodesics of this metric space.

Combining this result with the second theorem of this section, we have

A space admitting a G_1 finite continuous group of translations may be obtained by taking F in the form $a(\dot{x}^1)^2 + \dot{x}^1\phi_1 + \phi_2$, ϕ_1 and ϕ_2 being functions of $x^2, \cdots, x^n$ and of $\dot{x}^2, \cdots, \dot{x}^n$ homogeneous in the last set of $n - 1$ variables of degrees one and two respectively, while a is an arbitrary constant.

We shall next prove a generalization of a theorem on translations in Riemann spaces,[†] which states that

The angle between the path-curves of a translation and any geodesic (with respect to the geodesic) does not vary along the geodesic.

We take the metric of the space admitting a translation, in accordance with the theorem established above, to be $F = (\dot{x}^1)^2 + \dot{x}^1\phi_1 + \phi_2$ so that

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[*] The truth of this statement may be briefly proved as follows: let $x^1 = t$, $x^2 = c^2, \cdots, x^n = c^n$ be the equations of the curve in terms of a parameter t. The element of arc is then $ds^2 = eg_{11}(dx')^2$ and since g_{11} is homogeneous of degree zero in the one variable dx' it is free of dx'. Hence $s = \int [eg_{11}(x')]^{1/2} dx' = f(x')$. We now apply a transformation of coordinates $\bar{x}^1 = f(x^1)$, $\bar{x}^2 = x^2, \cdots, x^n = x^n$. In this coordinate system the equations of our curves become $\bar{x}^1 = s$, $\bar{x}^2 = c^2, \cdots,$ $x^n = c^n$ where s is the arc.

[†] Cf. L. P. Eisenhart, *Riemannian Geometry*, § 72.

$g_{11} = 1$, $(\partial g_{ij}/\partial x^1) = 0$. Then the vector defining the translation has components δ_1^i and if $x^i = f^i(s)$ are the equations of a geodesic in terms of the arc, the angle in the theorem is given by

$$\cos\,\theta = g_{1j}\mathring{x}^j \tag{10.13}$$

as is obvious from (10.11). Now

$$(d\cos\theta/ds) = \mathring{x}^j\,[(\partial g_{1j}/\partial x^k)\mathring{x}^k + g_{1jk}\mathring{x}^{\cdot k}] + g_{1j}\mathring{x}^{\cdot j}$$

and since the curve is a geodesic we have $\mathring{x}^{\cdot j} = -\Gamma^j{}_{kl}\mathring{x}^k\mathring{x}^l$. Hence

$$(d\cos\theta/ds) = \mathring{x}^j\mathring{x}^k\,[(\partial g_{1j}/\partial x^k) - g_{1h}\Gamma^h{}_{jk} - g_{1jh}\Gamma^h{}_{kl}\mathring{x}^l].$$

Because $(\partial g_{ij}/\partial x^1) = 0$,

$$g_{hj}\Gamma^h{}_{1k}\mathring{x}^k = [1k, j]\,x^k = \tfrac{1}{2}[(\partial g_{1j}/\partial x_k) - (\partial g_{1k}/\partial x^j)]\,\mathring{x}^k \quad \text{(cf. (10.5))}.$$

Hence $g_{hj}\Gamma^h{}_{1k}\mathring{x}^j\mathring{x}^k = 0$ and $(d\cos\theta/ds) = g_{1j,k}\mathring{x}^j\mathring{x}^k = 0$ by (10.6) and therefore θ is constant along the geodesic.

If ξ^i is any vector and $x^i = f^i(s)$ are the equations of a curve, we define $\mu^i = \xi^i{}_{,j}\mathring{x}^j$ as the *associate* of ξ^i with respect to the curve.*

Let ξ^i be a vector determining a motion; its components satisfy Killing's equations and when these (cf. (10.8)) are multiplied by $\mathring{x}^i\mathring{x}^j$ and summed for i and for j we find, because of (10.6)

$$g_{hj}\xi^h{}_{,k}\mathring{x}^j\mathring{x}^k = 0, \tag{10.14}$$

from which we have $g_{ij}\mu^i\mathring{x}^j = 0$ which implies that μ^i is orthogonal to the curve with respect to the curve. Hence

The associate vector with respect to a curve, of a vector determining a motion, is orthogonal to the curve.

The converse of the above theorem is also true; that is, if in a metric space there exists a vector of position whose associate with respect to any curve, is orthogonal to the curve the vector defines a motion. For (10.14) is a necessary and sufficient condition for orthogonality and from it Killing's equations are easily obtainable by two successive partial differentiations with respect to $\mathring{x}^l$ and $\mathring{x}^m$.

11. *Integrability Conditions of Killing's Equations.* Because of the symmetry of the tensor g_{ij}, the conditions on the vector ξ^i defining a motion consist of $[n(n+1)/2]$ independent equations for the determination of the

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* Cf. L. P. Eisenhart, *Non-Riemannian Geometry*, § 16. The vector μ^i is also called the *derived* vector.

n^2 unknown functions $(\partial\xi^i/\partial x^j)$. We therefore can not solve Killing's equations for $\xi^i{}_{,j}$ in terms of g_{ij} and ξ^h. Since a motion carries a geodesic into a geodesic and preserves the arc, ξ^i must also satisfy equations (8.2). Hence the problem of obtaining the components of a motion reduces to the solution of

$$(11.1) \qquad \xi^i{}_{,j,k} + \xi^l K^i{}_{jkl} + \xi^l{}_{,h}\mathring{x}^h \Gamma^i{}_{jkl} = 0$$

subject to the conditions

$$(11.2) \qquad g_{ih}\xi^h{}_{,j} + g_{hj}\xi^h{}_{,i} + \xi^h g_{ij,h} + \xi^h{}_{,m}\mathring{x}^m g_{ijh} = 0.$$

The dependent variables in these equations are the $n^2 + n$ functions $\xi^i{}_{,j}$ and ξ^i subject to the $[(n^2+n)/2]$ conditions (11.2). Hence

The greatest number of linearly independent motions that a general metric space may admit is $[n(n+1)/2]$.

The conditions of integrability of (11.1) subject to (11.2) consist of all equations obtainable from (11.1) and (11.2) by partial and covariant differentiation and elimination of $\xi^i{}_{,j,k}$ by means of (11.1). As in the case of affine collineations we can discard a great many of these equations by proving them linearly dependent on the remaining ones.

Thus the conditions of integrability of (11.1) are (8.3) and (8.4). From (11.2) we get by covariant differentiation and the use of (11.1), (2.8) and (2.9),

$$(11.3) \qquad h_{ij,k} \equiv g_{ij,h}\xi^h{}_{,k} + g_{ih,k}\xi^h{}_{,j} + g_{hj,k}\xi^h{}_{,i} + \xi^h g_{ij,k,h} + \xi^h{}_{,m}\mathring{x}^m g_{ij,k.h} = 0.$$

When (8.4) are multiplied by $g_{ip}\mathring{x}^p$ and summed for i, the result reduces by means of (10.6) and (10.8) to (11.3). Hence (11.3) are linearly dependent on (8.4). Similarly by partial differentiation we get

$$(11.4) \qquad h_{ij.k} \equiv g_{ijh}\xi^h{}_{,k} + g_{ihk}\xi^h{}_{,j} + g_{hjk}\xi^h{}_{,i} + \xi^h g_{ijk,h} + \xi^l{}_{,m}\mathring{x}^m g_{ijk.h} = 0.$$

By means of (11.3) and (11.4) it can be shown that

$$h_{ij,k.l} - h_{ij.l,k} = g_{ip}V^p{}_{jkl} + g_{pj}V^p{}_{ikl} + \Gamma^p{}_{ikl}h_{pj} + \Gamma^p{}_{jkl}h_{pi},$$

where $V^p{}_{jkl}$ has the significance of § 8. Therefore, since the equations arising from (11.3) are equivalent to those arising from (8.4), the equations arising from (11.4) by covariant differentiation are also equivalent to those arising from (8.4). Hence we have

A necessary and sufficient condition that a general metric space must satisfy in order to admit r linearly independent motions is that there exist three whole numbers M, N and P (≥ 0) *such that the rank of the matrix of*

$T, T_1 \cdots T_M$; $V_N, V_{N,1} \cdots V_{N,M}$; $h, h_1 \cdots, h_p$ and that of $T, \cdots T_{M+1}$; $V_{N+1}, \cdots, V_{N+1,M+1}$ $h, \cdots, h_{p+1}$ shall be $[n(n+1)/2] - r$.

We have shown in § 8 that if $\xi^i{}_{(\alpha)}$ $(\alpha = 1, \cdots, r)$ are the components of r linearly independent motions, the components of the alternant of any two of them satisfy (11. 1). We therefore need only prove that the components of the alternant satisfy Killing's equations in order to prove that these motions form a G_r finite continuous group of motions. When the values of ξ^h and $\xi^h{}_{,m}$ given by (8. 13) and (8. 14) are put into (11. 2) we find

$$\begin{aligned} H_{ij} \equiv g_{ih}\xi^h{}_{,j} + g_{hj}\xi^h{}_{,i} + \xi^h g_{ij,h} + \xi^h{}_{,m}\mathring{x}^m g_{ijh} = g_{ih}(\xi^m{}_{(\alpha),j}\xi^h{}_{(\beta),m} - \xi^m{}_{(\beta),j}\xi^h{}_{(\alpha),m}) \\ + g_{hj}(\xi^m{}_{(\alpha),i}\xi^h{}_{(\beta),m} - \xi^m{}_{(\beta),i}\xi^h{}_{(\alpha),m}) \\ + \xi^l{}_{(\alpha)}\xi^m{}_{(\beta)}(g_{ih}K^h{}_{jml} + g_{hj}K^h{}_{iml} + g_{ijh}\mathring{x}^p K^h{}_{pml}) \\ + \mathring{x}^p(\xi^m{}_{(\beta)}\xi^l{}_{(\alpha),p} - \xi^m{}_{(\alpha)}\xi^l{}_{(\beta),p})(g_{ih}\Gamma^h{}_{jlm} + g_{hj}\Gamma^h{}_{ilm}) \\ + \mathring{x}^p g_{ijh}(\xi^m{}_{(\alpha),p}\xi^h{}_{(\beta),m} - \xi^m{}_{(\beta),p}\xi^h{}_{(\alpha),m}). \end{aligned}$$

From (2. 8) and (2. 9) we have

$$g_{ih}K^h{}_{jml} + g_{hj}K^h{}_{iml} + g_{ijh}\mathring{x}^p K^h{}_{pml} = g_{ij,l,m} - g_{ij,m,l}$$

and

$$g_{ih}\Gamma^h{}_{jlm} + g_{hj}\Gamma^h{}_{ilm} = g_{ijl,m} - g_{ij,m,l}.$$

Therefore, by adding and subtracting suitable terms,

$$\begin{aligned} H_{ij} = \xi^l{}_{(\alpha)}h_{(\beta)ij,l} - \xi^l{}_{(\beta)}h_{(\alpha)ij,l} + \xi^l{}_{(\alpha),p}\mathring{x}^p h_{(\beta)ij.l} - \xi^l{}_{(\beta),p}\mathring{x}^p h_{(\alpha)ij.l} \\ + \xi^l{}_{(\alpha),i}h_{(\beta)jl} - \xi^l{}_{(\beta),i}h_{(\alpha)jl} + \xi^l{}_{(\alpha),j}h_{(\beta)il} - \xi^l{}_{(\beta),j}h_{(\alpha)il} = 0 \end{aligned}$$

because of (11. 2), (11. 3) and (11. 4). This concludes the proof of the statement made above.

12. *Simply transitive groups as groups of motion.* Let* $\xi_\alpha{}^i$ be n linearly independent vectors of position only. Let these vectors be the components of the infinitesimal transformations of a finite continuous group G_n whose constants of composition are $C^\alpha{}_{\beta\gamma}$. Since $|\xi_\alpha{}^i| \neq 0$ the group is simply transitive. We shall prove that

Any simply transitive group in n variables is the group of motions of an infinity of general metric spaces.†

Let $\eta_i{}^\alpha$ be the set of covariant vectors conjugate ‡ to the set $\xi_\alpha{}^i$, that is,

$$(12.1) \qquad \xi_\alpha{}^i\eta_j{}^\alpha = \delta_j{}^i, \qquad \xi_\alpha{}^i\eta_i{}^\beta = \delta_\alpha{}^\beta.$$

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* In this section Greek indices will indicate the vector and Latin ones the component. The summation convention applies to both sets of indices.

† For Riemannian spaces this theorem is due to Bianchi, *loc. cit.*, p. 517.

‡ *Non-Riemannian Geometry*, p. 45.

Killing's equations may be written as

$$(12.2)\qquad \frac{\partial g_{ij}}{\partial x^k} = -\eta_k{}^a\left(g_{ih}\frac{\partial \xi_a{}^h}{\partial x^j} + g_{hj}\frac{\partial \xi_a{}^h}{\partial x^i} + \frac{\partial g_{ij}}{\partial \mathring{x}^m}\frac{\partial \xi_a{}^m}{\partial x^h}\mathring{x}^h\right).$$

We define an invariant J by means of its components

$$(12.3)\qquad J^h{}_{pq} = -\eta_q{}^a(\partial \xi_a{}^h/\partial x^p) = \xi_a{}^h(\partial \eta_q{}^a/\partial x^p).$$

Because $\xi_a{}^i$ generate a group, we have

$$(12.4)\qquad J^h{}_{pq} - J^h{}_{qp} = -c^\gamma{}_{a\beta}\xi_\gamma{}^h\eta_p{}^a\eta_q{}^\beta$$

and by repeated application of (12.3), (12.4) and the relations between the constants of composition of a group, we obtain

$$(12.5)\qquad (\partial J^h{}_{jk}/\partial x^l) - (\partial J^h{}_{jl}/\partial x^k) = J^h{}_{pk}J^p{}_{jl} - J^h{}_{pl}J^p{}_{jk}.\text{[*]}$$

From the first of (12.3) we have

$$\partial \xi_a{}^h/\partial x^k = -\xi_a{}^j J^h{}_{kj}, \qquad \partial^2 \xi_a{}^h/\partial x^k \partial x^l = -\xi_a{}^j(\partial J^h{}_{kj}/\partial x^l) + \xi_a{}^p J^j{}_{lp} J^h{}_{kj}$$

and therefore

$$(12.6)\qquad (\partial J^h{}_{kj}/\partial x^l) - (\partial J^h{}_{lj}/\partial x^k) = J^h{}_{kp}J^p{}_{lj} - J^h{}_{lp}J^p{}_{kj}.$$

Equations (12.2) may now be put in the form

$$(12.7)\qquad \partial g_{ij}/\partial x^k = g_{ih}J^h{}_{jk} + g_{hj}J^h{}_{ik} + (\partial g_{ij}/\partial \mathring{x}^m)\mathring{x}^h J^m{}_{hk}.$$

The above system of partial differential equations in the n^2 dependent variables g_{ij} is of a type studied by König,[†] the conditions of complete integrability of (12.7) being

$$(12.8)\qquad \frac{\partial f_{ijk}}{\partial x^l} - \frac{\partial f_{ijl}}{\partial x^k} + f_{pql}\frac{\partial f_{ijk}}{\partial g_{pq}} - f_{pqk}\frac{\partial f_{ijl}}{\partial g_{pq}} + \frac{\partial f_{ijk}}{\partial g_{pqr}}\frac{\partial f_{pql}}{\partial \mathring{x}^r} - \frac{\partial f_{ijl}}{\partial g_{pqr}}\frac{\partial f_{pqk}}{\partial \mathring{x}^r}$$
$$+\left(\frac{\partial f_{ijk}}{\partial g_{pqs}}\frac{\partial f_{pql}}{\partial g_{tu}} - \frac{\partial f_{ijl}}{\partial g_{pqs}}\frac{\partial f_{pqk}}{\partial g_{tu}}\right)g_{tus} = 0$$

and

$$(12.9)\qquad \frac{\partial f_{ijk}}{\partial g_{pqr}}\frac{\partial f_{pql}}{\partial g_{hms}} - \frac{\partial f_{ijl}}{\partial g_{pqr}}\frac{\partial f_{pqk}}{\partial g_{hms}} + \frac{\partial f_{ijk}}{\partial g_{pqs}}\frac{\partial f_{pql}}{\partial g_{hmr}} - \frac{\partial f_{ijl}}{\partial g_{pqs}}\frac{\partial f_{pqk}}{\partial g_{hmr}} = 0,$$

where f_{ijk} is the right-hand side of (12.7) and x^i, $\mathring{x}^i$, g_{ij} and g_{ijk} are regarded as independent variables in the partial derivatives of (12.8) and (12.9). From (12.7) we have

$$(12.10)\qquad \partial f_{ijk}/\partial g_{pqr} = \delta_i^p\delta_j^q J^r{}_{tk}\mathring{x}^t$$

and when this is put into (12.9) it is found that these conditions are satisfied identically. Again

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[*] Cf. *Riemannian Geometry*, pp. 247-49.

[†] J. König, *Mathematische Annalen*, Vol. 23 (1884), pp. 520-27.

$$(12.11)\quad \begin{aligned} &\partial f_{ijk}/\partial x^l = g_{ih}(\partial J^h{}_{jk}/\partial x^l) + g_{hj}(\partial J^h{}_{ik}/\partial x^l) + g_{ijm}(\partial J^m{}_{hk}/\partial x^l)\dot{x}^h, \\ &\partial f_{pqk}/\partial \dot{x}^r = g_{pqm}J^m{}_{rk}, \qquad \partial f_{ijk}/\partial g_{pq} = \delta_i{}^p J^q{}_{jk} + \delta_j{}^q J^p{}_{ik}. \end{aligned}$$

When these values of the partial derivatives of f are put into (12.8) the result is

$$\begin{aligned} &g_{ih}[(\partial J^h{}_{jk}/\partial x^l) - (\partial J^h{}_{jl}/\partial x^k) + J^h{}_{pl}J^p{}_{jk} - J^h{}_{pk}J^p{}_{jl}] \\ &\quad + g_{hj}[(\partial J^h{}_{ik}/\partial x^l) - (\partial J^h{}_{il}/\partial x^k) + J^h{}_{pl}J^p{}_{ik} - J^h{}_{pk}J^p{}_{il}] \\ &\quad + g_{ijm}[(\partial J^m{}_{hk}/\partial x^l) - (\partial J^m{}_{hl}/\partial x^k) + J^m{}_{pl}J^p{}_{hk} - J^m{}_{pk}J^p{}_{hl})\dot{x}^h = 0 \end{aligned}$$

which is satisfied identically because of (12.5). Hence (12.2) are completely integrable, that is, they admit a solution which for $x^i = x_0{}^i$ are arbitrary functions of $\dot{x}^i$, which proves the theorem.

From the fact that a vector defining a motion satisfies not only (12.2) but also (8.2) it follows that the above theorem will be true if we replace in it the words " motion " and " metric space " by " collineations " and " space of paths " respectively.

The Complete System of Two Quaternary Quadratics.

By J. WILLIAMSON.

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1. *Introduction.* The system of concomitants for two quaternary quadratic forms was worked out by Gordan in the *Mathematische Annalen,* Bd. 56. Gordan's list of 580 forms was reduced by Turnbull to 125 forms.[*] This paper, while only reducing the number from 125 to 122, finds the concomitants by quite a different method. The method shows the fundamental rôle played by the four quadratic covariants in determining the complete system and gives very simply all the formulae of reduction used by Turnbull. Further, it seems likely that without much difficulty the method could be extended to the case of two quadratics in five or six variables.

2. List A.

The Prepared System.

$$1_x = a_x, \qquad 2_x = (A\beta x), \qquad 3_x = (B\alpha x), \qquad 4_x = b_x.$$

$$(12) = a_\beta (Ap), \qquad (23) = (AB)(\alpha\beta p),$$

$$(43) = b_\alpha (Bp), \qquad (14) = (abp),$$

$$(13) = (aB\alpha p) = \dot{b}_\alpha (a\dot{b}'p), \qquad (42) = (bA\beta p).$$

$$(123) = -a_\beta (AB) u_\alpha, \qquad (432) = -b_\alpha (AB) u_\beta,$$

$$(124) = a_\beta (Abu), \qquad (431) = b_\alpha (Bau).$$

$$(1234) = -a_\beta (AB) b_\alpha, \qquad (12, 43) = a_\beta b_\alpha (ABux).$$

List B.

The 122 unreduced forms.

5 Invariants	$a_\alpha^2,\ b_\alpha^2,\ (AB)^2,\ a_\beta^2,\ b_\beta^2.$
5 Covariants	$i_x^2, \qquad (1234)\, 1_x 2_x 3_x 4_x.$
5 Contravariants	$(ijk)^2, \quad (1234)(123)(234)(124)(134).$
16 Complexes	$(ij)^2 \quad 6, \quad (ij)(jk)(ki) \quad 4,$ $(1234)(12)(34), \quad (1234)(14)(23),$ $(1234)(ij)(ik)(im) \qquad 4.$

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[*] H. W. Turnbull, "The Simultaneous System of Two Quadratic Quaternary Forms," *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 18, Parts 1 and 2, pp. 70-94. Further reference to this paper will be indicated simply by "Turnbull's paper."

19 Mixed Forms containing x and u

$(1234)(123)4_x$, $(1234)(234)1_x$,
$(ijk)i_xj_xk_x$ 4
$(1234)(ijk)(ijm)i_xj_x$ 6
$(ijk)(ijm)(ikm)i_x$ 4
$(12, 43)^2$, $(1234)(12, 43)$
$(12, 43)(124)(134)1_x4_x$.

14 Mixed Forms containing x and p

$i_x(ij)j_x$ 6, $(1234)(23)1_x4_x$,

$$\left.\begin{array}{l}(1234)(21)(14)1_x3_x\\ (1234)(23)(34)1_x3_x\\ (1234)(34)1_x2_x\end{array}\right\}\text{and three similar forms.}$$

$(1234)(12)(23)(34)2_x3_x$.

14 Mixed Forms containing p and u

$(1234)(ijk)(ijm)(ij)$ 6, $(123)(234)(14)$,

$$\left.\begin{array}{l}(134)(342)(12)\\ (1234)(123)(134)(12)(23)\\ (1234)(123)(134)(14)(43)\end{array}\right\}\text{and three similar forms.}$$

$(124)(134)(12)(14)(34)$.

44 Mixed Forms containing x, p and u

$(123)(12)3_x$, $(123)(23)1_x$, $(124)(12)4_x$,
$(124)(14)2_x$, $(1234)(123)(41)1_x$
$(1234)(123)(43)3_x$, $(1234)(124)(32)2_x$,
$(1234)(124)(34)4_x$, $(132)(23)(14)4_x$, and nine similar forms.
$(143)(23)(12)4_x$,
$(ijk)(ij)(ik)i_x$ 12,

$$\left.\begin{array}{l}(1234)(124)(14)(34)1_x,\\ (1234)(123)(23)(34)2_x,\\ (1234)(124)(12)(23)1_x,\\ (1234)(234)(42)(21)4_x,\\ (1234)(124)(24)(43)2_x,\end{array}\right\}\text{and five similar forms.}$$

$(12, 43)(12)(34)$,
$(12, 43)(124)(34)4_x$, and a similar form.

In this list i, j, k, m take the values 1, 2, 3, 4. The number appearing on the right of a concomitant is the number of distinct concomitants of that type. A similar form means a form in which the symbols 1, 4 and 2, 3 are interchanged. The actual complete system is obtained by removing from each form any invariant factor which may appear. For example,

$$(12)^2 = a_\beta^2(Ap)^2.$$

We therefore drop the factor a_β^2 and take $(Ap)^2$ as the unreduced form.

3. *Notation.* Let

$$(1) \qquad f = a_x^2 = a'^2_x = \cdots, \qquad g = b_x^2 = b'^2_x = \cdots,$$

be the two quadratics. Further let A, B denote a convolution of two equivalent symbols a and a', b and b' respectively, so that the factor $(aa'cd)$ is written as (Acd). Similarly, let a convolution of three equivalent symbols a, a', a'' be written as α and of three equivalent symbols b, b', b'' as β, so that $(daa'a'')$ is written d_α or $(d\alpha)$.

If x, y, z, t are four sets of cogredient variables, the following notation may be used;

$$(2) \qquad \begin{cases} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} = (ab \mid xy) = (abxy), \\[2ex] \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = (abc \mid xyz) = (abcxyz), \\[2ex] \mid a_x \quad b_y \quad c_z \quad d_t \mid = (abcd \mid xyzt) = (abcd)(xyzt). \end{cases}$$

Furthermore we have the fundamental identities;

$$(3) \quad \begin{cases} (abcd)e_x = (abce)d_x + (abed)c_x + (aecd)b_x + (ebcd)a_x, \\ (ab \mid xy) = a_x b_y - a_y b_x, \\ (abc \mid xyz) = (ab \mid xy)c_z - (ac \mid xy)b_z - (bc \mid xy)a_z, \\ \qquad\qquad\quad = (ab \mid xy)c_z - (ab \mid xz)c_y - (ab \mid yz)c_x, \\ (abcd \mid xyzt) = (abc \mid xyz)d_t - (abd \mid xyz)c_t - (adc \mid xyz)b_t - (dbc \mid xyz) \end{cases}$$

These identities may be written more shortly in the forms,

$$(\mathring{a}\dot{b}\dot{c}\mathring{d})\dot{e}_x = 0,$$
$$(ab \mid xy) = \dot{a}_x \dot{b}_y = a_{\dot{x}} b_{\dot{y}},$$
$$(abc \mid xyz) = (\mathring{a}\dot{b} \mid xy)\dot{c}_z = (ab \mid \mathring{x}\dot{y})c_{\dot{z}},$$
$$(abcd \mid xyzt) = (\dot{a}\dot{b}\dot{c} \mid xyz)\dot{d}_t,$$

where the dots above the letters indicate the determinantal permutations that must be made.

4. *Co-ordinates.* Let the four vertices of a tetrahedron be the points

$$(4) \qquad x, \quad y, \quad z, \quad t$$

and the four planes of the tetrahedron be

$$(5) \qquad q, \quad w, \quad v, \quad u$$

where $x = (x_1x_2x_3x_4)$, $q = (q_1q_2q_3q_4)$ with similar expressions for the other points and planes, so that x, y, z, t denote cogredient quaternary symbols each contragredient to any one of q, w, v, u. Let the three planes u, v, w meet at x and so on, as is indicated by the order of writing (4) and (5). Then, if (uvw) denotes the set of four determinants $(u_2v_3\,w_4,\ u_1v_4w_3,\ u_1v_2w_4,\ u_1v_3w_2)$,

$$(6)\qquad \begin{cases} x = (uvw), & \Delta^2 u = (xyz), \\ \Delta^3 = (xyzt), & \Delta(uv)_{12} = -\,(xy)_{34} \text{ etc.,} \end{cases}$$

where $(qwvu) = \Delta$.

Hence, if d, e, f, g are quaternary symbols cogredient with u,

$$(7)\qquad \begin{cases} \qquad\qquad\qquad\ \Delta^2(defu) = (def \mid xyz) \\ \Delta(dep) = \Delta(deuv) = -\,(de \mid xy) \\ \qquad\qquad\qquad\ \Delta^3(defg) = (defg)(xyzt) = (defg \mid xyzt). \end{cases}$$

5. *Determination of the prepared system.* By the fundamental theorem every concomitant of the two quadratics (1) is a sum of products of factors of four types, d_x, $(deuv)$, $(defu)$ and $(defg)$, where each of the symbols d, e, f, g stands for a symbol a or b and must occur exactly twice in each product. The factor Δ might also occur but as it is a concomitant as it stands, we neglect it.

Hence by (7) every concomitant multiplied by a suitable power of Δ can be represented as a sum of products of factors of the four types d_x, $(de \mid xy)$, $(def \mid xyz)$ and $(defg \mid xyzt)$, where a power of Δ is introduced with each variable y, z, and t.

Let K be such a concomitant multiplied by a suitable power of Δ. If K has the factor $(aa_1a_2a_3 \mid xyzt)$, then by (2) $K = \sum(aa_1a_2a_3 \mid xyzt)a_\xi a_{1\eta}a_{2\zeta}a_{3\delta}M$, where ξ, η, ζ, δ are variables x, y, z, t but not necessarily distinct. By determinantally permuting the equivalent symbols a, a_1, a_2, a_3, we have

$$4!\,K = \sum N$$

where each term N has the factor $(aa_1a_2a_3)^2(xyzt)^2$ or else is zero. Hence, if K has a factor a_α (or b_β), K is either zero or has the invariant factor $a_\alpha{}^2$ (or $b_\beta{}^2$). In either case K may be considered to be reducible, and we denote this by writing $K \equiv 0 \bmod a_\alpha$.

A similar proof shows that, if $a_1a_2a_i$ or $b_1b_2b_i$ $(i = 2 \text{ or } 3)$ are convolved in K, the complementary symbols may also be convolved or finally:

LEMMA I. *If K is not reducible,*

$$mK = \sum M,$$

where each M is a product of factors of the types a_ξ, b_ξ, $(A \mid \xi\eta)$, $(B \mid \xi\eta)$, $(\alpha \mid \xi\eta\zeta)$, $(\beta \mid \xi\eta\zeta)$ and each symbol a, b, A, B, α, β occurs exactly twice in each product, while m is an integer.

Now two quadratics in n variables have $n+1$ invariants which form a complete system of invariants.[*] For our case these invariants are $a_\alpha{}^2$, $b_\beta{}^2$, $(AB)^2 = \Theta$, $a_\beta{}^2 = \Phi$, $b_\alpha{}^2 = \Psi$. There are also n quadratic covariants belonging to the complete system of two quadratics in n variables.[†] Let us denote the four quadratics in our case by,

$$\begin{aligned} &1_x{}^2 = a_x{}^2, \quad 4_x{}^2 = b_x{}^2, \quad \text{(i. e. the two original forms)},\\ &2_x{}^2 = (A\beta x)^2 = (a_\beta a'_x - a'_\beta a_x)^2 \text{ where } A = aa',\\ &3_x{}^2 = (B\alpha x)^2 = (b_\alpha b'_x - b'_\alpha b_x)^2 \text{ where } B = bb'. \end{aligned}$$

Now

$$\begin{aligned} (12 \mid \xi\eta) &= a'_\beta(aa'' \mid \xi\eta) - a''_\beta(aa' \mid \xi\eta) = a_\beta(A \mid \xi\eta) + (aa''a' \mid \xi\eta\beta) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A = a'a''), \\ &\equiv a_\beta(A \mid \xi\eta) \text{ neglecting the last term.} \end{aligned} \tag{8}$$

Similarly

$$\begin{aligned} (123 \mid \xi\eta\zeta) &\equiv a_\beta(A\dot{a}_1 \mid \xi\eta\zeta)(Ba_2\dot{a}_3) \qquad (\alpha = a_1a_2a_3), \\ &\equiv a_\beta(\alpha \mid \xi\eta\zeta)(AB) \qquad \text{mod } a_\alpha, \end{aligned} \tag{9}$$

and

$$(1234 \mid xyzt) \equiv a_\beta(AB)b_\alpha(xyzt). \tag{10}$$

We have also the corresponding results

$$\begin{cases} (43 \mid \xi\eta) \equiv b_\alpha(B \mid \xi\eta), \\ (432 \mid \xi\eta\zeta) \equiv b_\alpha(BA)(\beta \mid \xi\eta\zeta). \end{cases} \tag{11}$$

Hence, if K be multiplied by a suitable factor I composed of powers of Θ, Φ, and Ψ, it follows by lemma I that mIK can be expressed as a sum of terms, where each term is a product of factors of the one type $i_\xi (i = 1, 2, 3, 4:$ $\xi = x, y, z, t)$. But in K the variables x, y, z, t were originally convolved as x, (xy), (xyz), and $(xyzt)$. Now mIK involves the same variables as K, the only difference being that each symbol a or b occurring in K has been replaced by a symbol 1, 2, 3 or 4. Hence we may convolve the variables in mIK back again, exactly as they were originally convolved in K, and therefore have the result that $mIK = \sum M$ where each M is a product of factors of the four types i_x, $(ij \mid xy)$, $(ijk \mid xyz)$ and $(ijkm \mid xyzt) = (ijkm)(xyzt)$ and the symbols A, α, B, β, which occurred in K, also occur in M. We have now proved the theorem:

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[*] H. W. Turnbull, *Theory of Determinants, Matrices and Invariants*, Chap. XX, § 6, page 304.

[†] Turnbull and Williamson, "The Minimum System of Two Quadratics in n Variables," *Proceedings of the Royal Society of Edinburgh*, Vol. 45 (1924-25), pp. 149-165.

THEOREM I. *Every concomitant K, is either reducible or else, when multiplied by a suitable invariant factor composed of powers of Θ, Φ and Ψ, it can be expressed as a sum of terms, where each term is a product of factors of the four types mentioned above. Further, each symbol i, j, k, m must occur exactly twice in each product and the symbols A, α, B, β which are convolved in K appear explicitly in the symbols i, j etc.*

If K is reducible $K = I_1 K_1$ where I_1 is an invariant factor composed of powers of a_α^2, b_β^2 and K_1 is not reducible.

Therefore, by considering all possible products M, composed of the four factor types, mentioned in the theorem above, we will obtain a system in terms of which every concomitant K of the two quadratics, if multiplied by an invariant factor, can be expressed. But any such product $M = \sum I_i G_i$, where each G_i is a concomitant involving the variables properly convolved and each I_i is an invariant composed of powers of Θ, Φ, Ψ. Hence, if we determine the complete system of the G_i, we have a system of concomitants in terms of which every concomitant K, if multiplied by a suitable invariant factor, can be expressed.

Let us consider the concomitants G_i in the following order:

1. Consider G_1 before G_2 if G_1 is of less degree in the coefficients of the two quadratics than G_2.

2. If 1 fails to distinguish G_1 and G_2, consider G_1 before G_2 if G_1 has more symbols α (or β) than G_2 and not less symbols β (or α).

3. If both 1 and 2 fail, then consider G_1 before G_2 if G_1 has more symbols A (or B) than G_2 and not less symbols B (or A). That this is a legitimate procedure follows from the fact, that each G_i is also a concomitant K and can therefore always be obtained from symbols i_x etc., in which the symbols α, β, A, B appearing in G_i also appear in the i_x.

It follows immediately from this scheme and the manner in which the formulae (8) to (11) were obtained, that we are at liberty to use these formulae in determining the G_i from the products M. Moreover,

$$\begin{aligned}(23\xi\eta) &= a_\beta(B\dot{a}_2\dot{a}_3)(a'\dot{a}_1 \mid \xi\eta) - a'_\beta(B\dot{a}_2\dot{a}_3)(a\dot{a}_1 \mid \xi\eta) \qquad \alpha = a_1a_2a_3,\\ &\equiv \dot{a}_{3\beta}(B\dot{a}_2a)(a'\dot{a}_1 \mid \xi\eta) - \dot{a}_{3\beta}(B\dot{a}_2a')(a\dot{a}_1 \mid \xi\eta) \qquad \text{mod } b_\beta.\\ &\equiv (AB)(\alpha \mid \beta\xi\eta) \qquad \text{mod } a_\alpha.\end{aligned}$$

We next proceed to show that of the factors of M we may neglect those in which equivalent symbols $1\,1'$, $2\,2'$, $3\,3'$, $4\,4'$ occur convolved.

Let us first consider the case in which $1\,1'$ is convolved. Since an invariant factor a_β may be introduced when 12 is convolved in one bracket factor, we have to consider the separate cases:

$$\begin{aligned}&(11' \mid \xi\eta) = (aa' \mid \xi\eta) = (A \mid \xi\eta),\\ &(121' \mid \xi\eta\zeta) \equiv a_\beta(Aa' \mid \xi\eta\zeta) \equiv a_\beta(\alpha \mid \xi\eta\zeta),\\ &(121'2' \mid xyzt) \equiv a_\beta a'_{\beta'}(AA' \mid xyzt) \equiv 0 \text{ mod } a_\alpha,\\ &(1231' \mid xyzt) \equiv a_\beta(AB)(\alpha a' \mid xyzt) \equiv 0 \text{ mod } a_\alpha.\end{aligned}$$

In every case the resulting G_i have been reduced to G's of a simpler type and so the factors above may be neglected. Similarly we do not need to consider factors in which $4\,4'$ appears convolved. The cases to be considered in which $2\,2'$ occurs convolved are

$$22', \quad 122', \quad 232', \quad 2342', \quad 232'3', \quad 1232'.$$

Of these the last three are reducible mod a_α:

$$\begin{aligned}(22' \mid \xi\eta) &= (\dot{a}_1\beta)(a'_1\beta')(\dot{a}_2a'_2 \mid \xi\eta) - (\dot{a}_1\beta)(a'_2\beta')(\dot{a}_2a'_1 \mid \xi\eta),\\ &\qquad\qquad A = a_1a_2, \quad A' = a'_1a'_2,\\ &= (\dot{a}_1\beta)(\dot{a}_2\beta')(A' \mid \xi\eta) - (A'\dot{a}_2 \mid \xi\eta\beta')(\dot{a}_1\beta),\\ &\equiv (A'\dot{a}_2 \mid \beta'\xi\eta)(\dot{a}_1\beta) \text{ mod } b_\beta:\\ (122' \mid \xi\eta\zeta) &\equiv (a\beta)(\dot{a}_1\beta')(A\dot{a}_2 \mid \xi\eta\zeta) \equiv (a\beta)(\dot{a}_1\beta')(\alpha \mid \xi\eta\zeta):\\ (232' \mid \xi\eta\zeta) &\equiv (AB)\,[(\dot{a}\beta)(a'_1\beta')(\dot{a}_1\dot{a}_2a'_2 \mid \xi\eta\zeta) - (\dot{a}\beta)(a'_2\beta')(\dot{a}_1\dot{a}_2a'_1 \mid \xi\eta\zeta)],\\ &\equiv 0 \text{ mod } a_\alpha, \quad b_\beta, \quad \alpha = aa_1a_2, \quad A' = a'_1a'_2.\end{aligned}$$

Once again the resulting G_i have been reduced to G's of a simpler type. Similarly we do not need to consider bracket factors in which $3\,3'$ appears convolved. Furthermore, since

$$i_\xi - i'_\eta = (ii' \mid \xi\eta)$$

we can interchange equivalent symbols i in any product and so do not require to use distinguishing marks on the equivalent symbols i.
But by (7)

$$\Delta^{-1}(ij \mid xy) = (ijuv) = (ijp),$$
$$\Delta^{-2}(ijk \mid xyz) = (ijku),$$
$$\Delta^{-3}(1234 \mid xyzt) = (1234),$$

and, since originally a power of Δ was introduced with each variable y, z, t, we can pass back to the original variables u and uv. If we write (ij) for (ijp) and (ijk) for $(ijku)$ we may state the final result in the theorem;

THEOREM II. *Every concomitant of the two quadratics, if multiplied by a suitable invariant factor, is a sum of terms, where each term is a product of symbolic factors of the four types* (1234), (ijk), (ij), i_x, *together with the invariants* a_α^2 *and* b_β^2 *and each symbol* i, j, k *must occur an even number of times in every product. Moreover no equivalent symbols* i, j, *etc. may occur in the same bracket factor.*

The values of the factors (ij), (ijk) are given in list A § 2. On considering this list we notice that corresponding to convolutions of the symbols 12, 23 and 34 factors a_β, (AB), b_α appear. Since we wish to determine all concomitants (not merely all the concomitants multiplied by an invariant factor), we must see if, in forming the factors (ij), (ijk), we have broken up 12, 23 or 34. This may have happened. Before convolving the variables x, y, z back to their original forms i. e. to xy, xyz, we might have the product $(12 \mid xy)(34 \mid xz)$. In convolving the variables x, y, z together we permute x, y and z to obtain

$$(123 \mid xyz)4_x - (124 \mid xyz)3_x \quad \text{or}$$
$$(324 \mid xyz)1_x - (314 \mid xyz)2_x.$$

In either case we have broken up a factor 12 or 34, and so must introduce the new factor

$$(12, 43) \equiv (123)4_x - (124)3_x,$$
$$\equiv (143)2_x - (243)1_x.$$

This is the only new factor that appears for (ij, kn) is always expressible in terms of the earlier factors, if ij or kn is 13 or 24. Further

$$(12, 23) = -(122)3_x + (123)2_x$$

where (122) has the equivalent symbols 2 2 convolved and so may be neglected and (123) has both 12 and 23 convolved. Similarly no factors of the type *(ijk, lm)* or *(ijk, lmn)* need to be introduced. Hence with the introduction of (12, 34) we are now assured that, since we have found all factors in which 12, 23, 34 are convolved, we may, from our list of concomitants multiplied by invariant factors, remove the invariant factors and obtain the actual concomitants.

6. *Identities.* All the identities in Turnbull's paper are obtained from the following:

$$(12)\quad \begin{cases} (ij)k_x = (ik)j_x + (jk)i_x, \\ (12)(34) = (13)(24) + (32)(14), \\ (123)4_x = (124)3_x - (134)2_x - (324)1_x, \\ (ijk)(im) = (ijm)(ik) - (ikm)(ij), \\ (12, 43) = (123)4_x - (124)3_x, \\ \qquad\qquad = (143)2_x - (243)1_x. \end{cases}$$

The first four of these identities are analogous to those for binary and ternary types and the others are a direct result of the definition of (12, 43).
We now deduce two other important identities.

$$\begin{aligned} (13)\quad & (14)(12, 43) = [(123)4_x - (124)3_x]\,(14), \\ & = (124)(13)4_x + (143)(12)4_x - (124)(13)4_x - (124)(34)1_x, \\ & = (143(12)4_x - (124)(34)1_x. \end{aligned}$$

$$\begin{aligned} (14)\quad & (23)(12, 43) = [(123)4_x - (124)3_x]\,(23), \\ & = (123)(43)2_x + (123)(24)3_x - (123)(24)3_x - (324)(21)3_x, \\ & = (123)(43)2_x - (324)(21)3_x. \end{aligned}$$

By means of the above identities all products of factors of the type indicated in theorem II can be expressed in terms of the list given § 2 B. These identities cannot however be applied blindly. Consider for example the identity, $(12)3_x = (13)2_x + (32)1_x$. This identity involves on the left hand side the factor a_β, because (12) is convolved. Accordingly, if 12 is convolved an even number of times in a concomitant, since eventually we shall discard the invariant factors, we are not at liberty to use this identity as it stands.

It may however always be used in the form $(13)2_x = (12)3_x + (23)1_x$ since (13) does not introduce an extraneous factor.

7. *The determination of the concomitants.* It is not our intention here to reproduce the work whereby the list B was obtained. We shall however give a few typical examples.

Complexes. The only factors that may occur are the six (ij) factors and (1234). If the factor (1234) does not occur, the problem of determining the irreducible complexes is strictly analogous to that of binary forms, and accordingly the result is,

The six quadratic complexes $(ij)^2$,
The four cubic complexes $(ij)\ (jk)\ (ki)$.

If the factor (1234) occurs and the form is a quadratic complex there are three possible cases,

$$(1234)(12)(34), \qquad (1234)(23)(14), \qquad (1234)(13)(24),$$

and of these the third is expressible in terms of the other two. If the form is a cubic complex it must be one of the four,

$$(1234)(ij)(ik)(im) \qquad\qquad (i, j, k, m, = 1, 2, 3, 4).$$

The only other possibility is

$$(1234)(12)(23)(34)(41)(13)(24)$$

and this is obviously reducible.

Concomitants involving the factor $(12, 43)$. If we consider a concomitant M, involving the factor $(12, 43)$, to be reducible when $(12, 43)$ can be expressed in terms of factors $(ijk)m_x$, any irreducible concomitant involving $(12, 43)$ must have 12 and 34 convolved an even number of times. [Last two of (12)].

Let M denote a concomitant containing $(12, 43)$. Then obviously M may be

$$(12, 43)\ (1234),$$
$$\text{or}\ (12, 43)^2.$$

If $M \neq (12, 43)(1234)$, 23 must be convolved an even number of times in M, for otherwise M would have the factor $(AB)(ABux)$. It follows immediately that, if M does not contain the variable p nor the factor (1234), there are only two possibilities for M:

$$\begin{aligned} M &= (12, 43)(123)(234)3_x 2_x, \\ \text{or} \qquad &= (12, 43)(124)(134)1_x 4_x \end{aligned}$$

where in the first 12, 23, 34 are all convolved twice and in the second 12 and 34 are convolved twice.

If $M = (1234)(12, 43)N$, 23 must be convolved once in N, for it cannot be convolved three times. Hence N contains (123) or (234). Therefore

$$N = (123)(124)3_x 4_x \qquad \text{or} \qquad (234)(431)2_x 1_x.$$

But $(123)(124)3_x 4_x = (123)\,[(123)4_x{}^2 - (12, 43)4_x]$ by (12) with a similar identity in the other case. Hence M is reducible.

If M contains the variable p, M cannot have the factor (14) or (23) (by (13) and (14)). It is now a simple matter to obtain the complete list of irreducible concomitants involving the factor (12, 43). The complete list is,

$$\begin{array}{ll}
(12, 43)^2 & \\
(12, 43)(1234) & \\
(12, 43)(123)(234)3_x 2_x & R \\
(12, 43)(124)(134)1_x 4_x & \\
(12, 43)(12)(34) & \\
(12, 43)(124)(34)4_x & \\
(12, 43)(134)(12)1_x & \\
(12, 43)(1234)(231)(12)3_x & R \\
(12, 43)(1234)(234)(34)2_x & R
\end{array}$$

8. *Three special reductions.* The concomitants above, which are marked R, appear in Turnbull's list but are reducible. We proceed to show this.

After removing the invariant factor, $(12, 43)(123)(234)3_x 2_x$ is equivalent to

$$\begin{aligned} &(A\beta x)\,(B\alpha x)(ABux)u_\alpha u_\beta \\ \equiv\, &2(aa'\beta x)(B\alpha x)(Bau)u_\alpha u_\beta a'_x \\ \equiv\, &2\,[a_\beta a'_x - a_x a'_\beta]\,(B\alpha x)(Bau)u_\alpha u_\beta a'_x. \end{aligned}$$

Each term is factorable, the first having the factor a_x^2 and the second $u_\beta a'_\beta a'_x$. Hence the concomitant is reducible.

$$(12, 43)(1234)(231)(12)3_x$$

is equivalent to

$$\begin{aligned} &(ABux)(B\alpha x)(Ap)u_\alpha \\ \equiv\ &2(Abu)(b_\alpha b'_x - b_x b'_\alpha)(Ap)b'_x u_\alpha \end{aligned}$$

and again each term has a concomitant factor. An exactly similar proof shows that $(12, 43)(1234)(234)(34)2_x$ is reducible. We are therefore left with 122 concomitants which form the complete system for the two quadratics.

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Concerning Continuous Curves in Metric Space.[*]

By W. L. Ayres.

It is the purpose of this paper to prove some results concerning continuous curves lying in a metric space. Most of these results are known for Euclidean space of two dimensions. The proofs for two dimensions do not extend readily since they depend, in general, upon two theorems, neither of which is true in space of more than two dimensions.[†] Also much of the complexity of the proofs of the theorems of the present paper is due to the failure of these two theorems. Wherever a theorem is known for the two-dimensional case, or for some other special case, we shall refer to this in a footnote. Throughout this paper we assume that all point sets mentioned lie in a metric space which is locally compact in the sense that for each point p and each number $\epsilon > 0$, the set of all points $[x]$ such that $\rho(p, x) \leqq \epsilon$ is a compact point set.

Notations. If K is any point set, $\bar{K}$ denotes the point set consisting of the points of K together with all limit points of K that do not belong to K. If x and y are points, the symbol $\rho(x, y)$ denotes the distance from x to y. If X and Y are point sets, the symbol $\rho(X, Y)$ denotes the greatest lower bound of the set of numbers $\rho(x, y)$ where x is a point of X and y is a point of Y. If xyz denotes an arc with end-points x and y, the symbols $<xzy$, $xzy>$, and $<xzy>$ denote $xzy - x$, $xzy - y$ and $xzy - x - y$, respectively. If x and y are points of an arc α, then the subset xy of α, or the subarc xy of α, denotes the arc with end-points x and y which is a subset of α. If X and Y are point sets, the symbol $X \cdot Y$ denotes the set of all points common to the sets X and Y.

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[†] The two theorems to which we refer are the accessibility theorem of R. L. Wilder and a theorem of the author that there remain only a finite number of maximal connected subsets of diameter greater than $\epsilon > 0$ when a bounded continuous curve is removed from a given continuous curve. See (A) R. L. Wilder, "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7 (1925), pp. 340-377, theorem 1, and (B) W. L. Ayres, "Note on a Theorem Concerning Continuous Curves," *Annals of Mathematics*, Vol. 28 (1927), pp. 501-502. An example to show that both theorems are false in three dimensions may be found at the end of Theorem 1 of the present paper. To see that the theorem of the author is false, we take $OP + \alpha$ as the bounded continuous curve. Then for any positive number $\epsilon < 2$ there are infinitely many maximal connected subsets of diameter greater than ϵ remaining after the set $OP + \alpha$ is removed.

THEOREM 1. *In order that an M-boundary point P of an M-domain D*[*] *of a continuous curve M be accessible*[†] *from D it is sufficient that if C_1 is any hypersphere*[‡] *with center at P there should exist a concentric hypersphere C_2 with radius smaller than that of C_1 such that only a finite number of the components*[§] *of $D \cdot I(C_1)$ contain points of $I(C_2)$.*

Proof. Let C_1 be a hypersphere of radius 1 with center P. By hypothesis there exists a hypersphere C' with center P and radius less than 1 such that only a finite number of the components of $D \cdot I(C_1)$ have points in $I(C')$. But if C'' is any hypersphere with center P and radius less than that of C', then the above property is satisfied with respect to C''. Hence there exists a hypersphere C_2 with center P and radius less than $\frac{1}{2}$ such that only a finite number of the components of $D \cdot I(C_1)$ have points in $I(C_2)$. Let these be $D_{11}, D_{12}, D_{13}, \cdots, D_{1n_1}$. Since P is a limit point of D there is some integer $k_1 (1 \leqq k_1 \leqq n_1)$ such that P is a limit point of D_{1k_1}. There exists a hypersphere C_3 with center P and radius less than $\frac{1}{3}$ and less than the radius of C_2 such that only a finite number of the components of $D \cdot I(C_2)$ have points in $I(C_3)$. Then there are only a finite number of components of $D_{1k_1} \cdot I(C_2)$ that have points in $I(C_3)$. Let these be $D_{21}, D_{22}, D_{23}, \cdots, D_{2n_2}$. As P is a limit point of D_{1k_1} there is an integer $k_2 (1 \leqq k_2 \leqq n_2)$ such that P is a limit point of D_{2k_2}. In general there exists a hypersphere C_{i+1} with center P and radius less than both $1/i + 1$ and the radius of C_i such that only a finite number of the components of $D \cdot I(C_i)$ have points in $I(C_{i+1})$. Hence there are only a finite number, $D_{i1}, D_{i2}, \cdots, D_{in_i}$, of components of $D_{i-1,\, k_{i-1}} \cdot I(C_i)$ which have points in $I(C_{i+1})$. Hence there is one of these which has P as a limit point. Let D_{ik_i} denote this one.

Let Q be any point of D and let P_1 be any point of D_{1k_1} distinct from Q. There is an arc α_1 of D whose end-points are Q and P_1.[¶] Let D_{m_1} be the first set in the series $D_{1k_1}, D_{2k_2}, \cdots$, which contains no point of α_1. Let P_2 be a point of D_{m_1}. There is an arc α_2 of D_{1k_1} from P_1 to P_2. Let D_{m_2} be the first set in the series $D_{1k_1}, D_{2k_2}, \cdots$, which contains no point of $\alpha_1 + \alpha_2$, and let P_3 be a point of D_{m_2}. There exists an arc α_3 of D_{m_1} with end-points P_2 and P_3. In general let D_{m_i} be the first set of the series $D_{1k_1}, D_{2k_2}, D_{3k_3}, \cdots$ which contains no point of $\sum_{j=1}^{i} \alpha_j$, and let P_{i+1} be a point D_{m_i}. There exists an arc α_{i+1} of $D_{m_{i-1}}$ with end-points P_i and P_{i+1}.

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[*] A connected subset D of a continuum M is said to be an *M-domain* if $M - D$ is closed. The set of all limit points of D that do not belong to D is the M-boundary of D. See reference (A), page 341.

[†] A boundary point P of a domain D is said to be *accessible* from D if for any point Q of D there exists an arc with end-points P and Q and which lies in D except for the point P.

[‡] If p is a point and r is a positive number, the *hypersphere* with center p and radius r is the set of all points $[x]$ of the space such that $\rho(x, p) = r$. If S is a hypersphere, $I(S)$ and $E(S)$ denote the interior and exterior, respectively, of S, i. e. the set of all points $[x]$ of the space such that $\rho(x, p) < r$ and $\rho(x, p) > r$, respectively.

[§] A connected subset H of a point set K is said to be a *component* of K if H is not a proper subset of any connected subset of K.

[¶] (C) R. L. Moore, "Concerning Continuous Curves in the Plane," *Mathematische Zeitschrift*, Vol. 15 (1922), pp. 254-260, Theorem 1. While Professor Moore proves the theorem for two dimensions only, it is obvious that the same proof holds for our space.

Let p_1 be the first point of α_2 on the arc α_1 in the order from Q to P_1. Let p_2 be the first point of α_3 on the arc α_2 in the order from p_1 to P_2. In general let p_i be the first point of α_{i+1} on the arc α_i in the order from p_{i-1} to P_i. It is easy to see that

$$P + \text{ subarc } Qp_1 \text{ of } \alpha_1 + \sum_{i=2}^{\infty} \text{ subarc } p_{i-1}p_i \text{ of } \alpha_i$$

is an arc from Q to P lying in D except for the point P.

The condition of Theorem 1 is by no means necessary and it is probable that much weaker conditions may be found. It is not the purpose of this paper to study accessibility except in so far as may be needed for the remaining theorems. However, it may be of interest to observe that neither of the conditions given by R. L. Wilder [*] is sufficient in Euclidean space of three dimensions. Consider the following example:

In a Euclidean three-space with a rectangular coördinate system, for every positive integer n let R_n be the set of all points (x, y, z) such that

$$y = nx,\ 0 \leqq y \leqq 1/n,\ 0 \leqq z \leqq 2 - 1/n.$$

In each rectangular set R_n let A_n be the point $(1/n^2, 1/n, 0)$, and let α be the arc consisting of the point $O = (0, 0, 0)$ and the straight line intervals A_1A_2, A_2A_3, A_3A_4, $\cdots$. Let P be the point $(0, 0, 2)$. Let

$$M = P + \alpha + \sum_{i=1}^{\infty} R_i,$$

and let the M-domain D be $M - OP$. Both conditions given by R. L. Wilder are satisfied but the point P of the M-boundary of D is not accessible from D.[†]

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[*] See reference (A).

[†] The accessibility theorem of R. L. Wilder places conditions upon the boundary of the domain only. In this example we have an M-domain whose M-boundary is simply an arc. As an arc is a very simple type of boundary, it seems evident that any general accessibility theorem for M-boundaries of M-domains in space of more than two dimensions must place some conditions on the M-domain itself.

DEFINITIONS. If M is a continuous curve, α is an arc of M, A and B are distinct points of α and D is a subset of $M-\alpha$, the set of arcs K will be said to be a *chain of D from A to B* if (a) K consists of a set of arcs x_1y_1, x_2y_2, $x_3y_3, \cdots, x_my_m$ such that (1) the arc α contains every point x_i and every point y_i, (2) for each i, x_{i+1} lies between x_i and y_i on α but not between y_{i-1} and A, and y_{i+1} lies on the subset $<y_iB$ of α, (3) D contains $<x_iy_i>$ for every i, (4) if $i \neq j$, $<x_iy_i>$ and $<x_jy_j>$ have no points in common, (5) $x_1=A$ and $y_n=B$; or (b) K consists of a set of arcs x_1y_1, x_2y_2, x_3y_3, $\cdots$ satisfying (1) to (4) above and such that $x_1=A$ and B is the sequential limit point of $[y_i]$; or (c) K consists of a set of arcs x_0y_0, $x_{-1}y_{-1}$, $x_{-2}y_{-2}, \cdots$ satisfying (1) to (4) above and such that $y_0=B$ and A is the sequential limit point of $[x_{-i}]$; or (d) K consists of a set of arcs $\cdots, x_{-2}y_{-2}$, $x_{-1}y_{-1}$, x_0y_0, x_1y_1, $x_2y_2, \cdots$ satisfying (1) to (4) above and such that A is the sequential limit point of $[x_i]$ for i negative and B is the sequential limit point of $[x_i]$ for i positive. The chain will be said to be of *type* (a), (b), (c) or (d) according as (a), (b), (c) or (d) is true. A chain will be said to be *convergent* if for any positive number ϵ it contains only a finite number of arcs of diameter greater than ϵ.

THEOREM 2. *If M is a continuous curve, α is an arc of M one of whose end-points is B, D is a component of $M-\alpha$ having B and a point A of α distinct from B as limit points, and ϵ is a positive number, then there is a point A' of α whose distance from A is less than ϵ such that there is a convergent chain of D of type (a) or (b) from A' to B.*

Proof. Let ϵ_1 be any positive number less than both ϵ and $\frac{1}{2}\rho(A,B)$. As M is connected im kleinen at A there exists a positive number δ_{ϵ_1} such that any point of M whose distance from A is less than δ_{ϵ_1} can be joined to A by an arc of M every point of which is at a distance from M less than ϵ_1. There is a point P of D such that $\rho(A,P)<\delta_{\epsilon_1}$. Let α_1 be an arc of M with end-points A and P, every point of which is within a distance ϵ_1 of A. Let A' be the first point of α on α_1 in the order from P to A. Then $\rho(A, A')<\epsilon_1<\epsilon$.

If B is accessible from D, there is an arc α_2 with end-points P and B and lying in D except for the point B. Let P' be the first point of the subarc PA' of α_1 on α_2 in the order B to P. Then the subarc $P'A'$ of α_1 plus the subarc $P'B$ of α_2 is a chain of D from A' to B.

If B is not accessible from D, by Theorem 1 there exists a hypersphere S with center at B and radius r such that if S' is any hypersphere with center B and radius less than r, then infinitely many of the components of $D\cdot I(S)$

have points in $I(S')$. It is evident that if S'' and S''' are any two hyperspheres with center B such that the radius of S''' is less than the radius of S'', which is less than or equal to r, then infinitely many of the components of $D \cdot I(S'')$ have points in $I(S''')$. Let $[x]$ be any infinite set of points of $D \cdot S'''$ such that no two belong to the same component of $D \cdot I(S'')$. Since $[x]$ is an infinite bounded set, it has at least one limit point y. The point y belongs to $D + \alpha$ since D and M-D-α are mutually separated.[*] Furthermore, if y belongs to D, it belongs to a component D_y of $D \cdot I(S'')$. But D_y and $D \cdot I(S'') - D_y$ are mutually separated.[†] Hence y belongs to α. Then every limit point of such a set $[x]$ belongs to α.

Let $S_1, S_2, S_3, \cdots$ be a sequence of hyperspheres with center at B and radii $r_1, r_2, r_3, \cdots$, such that (1) r_1 is the smaller of r and ϵ_1, (2) $r_{i+1} < \frac{1}{2} r_i$ for each i, (3) if x is any point of $\alpha \cdot [S_i + I(S_i)]$ then x lies on the subset $By>$ of α where y is any point of $\alpha \cdot S_{i-1}$. There exists a sequence $D_1, D_2, D_3, \cdots$ of distinct components of $D \cdot I(S_1)$ such that for each i, D_i contains a point z_i in $I(S_i)$. For each i and each $j < i$ let D_{ji} be the component of $D_i \cdot I(S_j)$ which contains z_i.

For each $i > 4$ let x_{5i} be a point of $D_i \cdot S_5$. Let v_1 be a limit point of $[x_{5i}]$. We have shown that α contains v_1. Let $\epsilon_{12} = \frac{1}{2} \rho(v_1, S_4 + S_6)$. There exists a positive number δ_{12} such that every point of M whose distance from v_1 is less than δ_{12} can be joined to v_1 by an arc of M every point of which is within a distance ϵ_{12} of v_1. There is a point x_{5k_1} such that $\rho(v_1, x_{5k_1}) < \delta_{12}$. Let α_4 denote an arc of M with end-points v_1 and x_{5k_1} every point of which is within a distance ϵ_{12} of v_1, and let t_1 be the first point of α on α_4 in the order from x_{5k_1} to v_1. Since D_{k_1} is a component of $D \cdot I(S_1)$ and α_4 is a subset of $I(S_1)$, every point of the subset $x_{5k_1}t_1>$ of α_4 belongs to D_{k_1}. There exists an arc α_3 of D with end-points P and x_{5k_1}. The set consisting of the arc α_3 plus the subarc PA' of α_1 plus the subarc $x_{5k_1}t_1$ of α_4 contains an arc β_1 with end-points A' and t_1 such that D contains $<\beta_1>$.

Only a finite number of the sets $D_{29}, D_{210}, D_{211}, \cdots$, contain points of β_1.[‡] For each $i \geqq 9$ let x_{3i} be a point of $D_{2i} \cdot S_3$ and let x_{9i} be a point of $D_{2i} \cdot S_9$. There exists a sequence of integers $i_1, i_2, i_3, \cdots$, such that

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[*] (D) C. Kuratowski, "Une définition topologique de la ligne de Jordan," *Fundamenta Mathematicae*, Vol. 1 (1920), pp. 40-43; and H. Hahn, "Ueber die Komponenten offener Mengen," *Fundamenta Mathematicae*, Vol. 2 (1921), pp. 189-192.

[†] See reference (D).

[‡] Since β_1 contains only a finite number of mutually exclusive subarcs containing a point of S_1 and a point of S_2.

(1) $[x_{3i_n}]$ has a sequential limit point u_2 on $\alpha \cdot S_3$,
(2) $[x_{9i_n}]$ has a sequential limit point v_2 on $\alpha \cdot S_9$,
(3) no set D_{2i_n} contains a point of β_1.

Let $\epsilon_{21} = \frac{1}{2}\rho(u_2, S_2 + S_4)$ and $\epsilon_{22} = \frac{1}{2}\rho(v_2, S_8 + S_{10})$. There exist positive numbers δ_{21} and δ_{22} such that any point of M whose distance from $\left\{\begin{matrix} u_2 \\ v_2 \end{matrix}\right\}$ is less than $\left\{\begin{matrix} \delta_{21} \\ \delta_{22} \end{matrix}\right\}$ can be joined to $\left\{\begin{matrix} u_2 \\ v_2 \end{matrix}\right\}$ by an arc of M every point of which is within a distance $\left\{\begin{matrix} \epsilon_{21} \\ \epsilon_{22} \end{matrix}\right\}$ of $\left\{\begin{matrix} u_2 \\ v_2 \end{matrix}\right\}$. Since u_2 and v_2 are sequential limit points of $[x_{3i_n}]$ and $[x_{9i_n}]$, there is a number k_2 (which is one of the numbers i_n) such that $\rho(u_2, x_{3k_2}) < \delta_{21}$ and $\rho(v_2, x_{9k_2}) < \delta_{22}$. Let α_5 and α_6 be arcs of M with end-points u_2 and x_{3k_2} and v_2 and x_{9k_2} respectively such that every point of α_5 is within a distance ϵ_{21} of u_2 and every point of α_6 is within a distance ϵ_{22} of v_2. Let s_2 and t_2 be the first points of α on α_5 and α_6 in the orders from x_{3k_2} to u_2 and from x_{9k_2} to v_2 respectively. Since D_{2k_2} is a component of $D \cdot I(S_2)$ the subsets $x_{3k_2}s_2 >$ of α_5 and $x_{9k_2}t_2 >$ of α_6 belong to D_{2k_2}. Let α_7 be an arc of D_{2k_2} with end-points x_{3k_2} and x_{9k_2}. Then the set composed of the arc α_7 plus the subarc $x_{3k_2}s_2$ of α_5 plus the subarc $x_{9k_2}t_2$ of α_6 contains an arc β_2 with end-points s_2 and t_2 and such that D_{2k_2} contains $< \beta_2 >$. Since D_{2k_2} contains no point of β_1, the arcs β_1 and β_2 have no common points. From the manner in which the hyperspheres were chosen we have the order $A's_2t_1t_2B$ on α.

Repeating the above process with the sets $D_{6,13}, D_{6,14}, D_{6,15}, \cdots$, we obtain an arc β_3 whose end-points are s_3 and t_3 such that (1) β_3 is a subset of $I(S_6)$, (2) s_3 is a point of α whose distance from S_7 is less than $\rho(S_7, S_6 + S_8)$ and t_3 is a point of α whose distance from S_{13} is less than $\rho(S_{13}, S_{12} + S_{14})$, (3) β_3 has no point in common with β_1 or β_2 and D contains $< \beta_3 >$, (4) on the arc α we have the order $A's_2t_1s_3t_2t_3B$.

Let us continue this process indefinitely. We obtain an infinite sequence of arcs $\beta_1, \beta_2, \beta_3, \beta_4, \cdots$, such that (1) β_i has end-points s_i and t_i * on α in the order $s_1s_2t_1s_3t_2s_4t_3s_5 \cdots B$, (2) for each $i > 1$, β_i is a subset of $I(S_{4i-6})$ and hence the diameter of β_i approaches zero as i increases indefinitely, (3) the point B is the sequential limit point of $[t_i]$, (4) for every i, D contains $< \beta_i >$ and if $i \neq j$ then $\beta_i \cdot \beta_j = 0$, (5) $\rho(A, A') < \epsilon$. Then $\beta_1, \beta_2, \beta_3, \cdots$, is the desired convergent chain of D of type (b) from A' to B.

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* Let s_1 be the point A'.

THEOREM 3.[*] *In order that a point P be an end-point*[†] *of a continuous curve M it is necessary and sufficient that P be a non-cut-point*[‡] *of M which lies on no simple closed curve of M.*

Proof. The proof given by the author[§] for the necessity of the condition in two dimensions holds equally well in the space considered here.

The condition is sufficient. Let P be a point of M satisfying the condition and suppose that P is not an end-point of M. Then there exists an arc α of M one of whose end-points is P, such that $M - (\alpha - P)$ contains a connected set N containing more than one point and containing P. Let

$$M - P = (\alpha - P) + M_1.$$

Then M_1 contains $N - P$. Let q be any point of $N - P$ and let D_q be the component of $M - \alpha$ containing q. As N is connected, one of the sets $N \cdot D_q$ and $N - N \cdot D_q$ contains a limit point of the other. The set $N \cdot D_q$ cannot contain a limit point of $N - N \cdot D_q$ since D_q is an open subset of M.[¶] Since D_q is a component of $M - \alpha$, no point of $N - N \cdot D_q - P$ is a limit point of $N \cdot D_q$. Hence P is a limit point of $N \cdot D_q$ and consequently of D_q. Since P is a non-cut-point of M, one of the sets $(\alpha - P + M_1 - D_q)$ and D_q must contain a limit point of the other. As $M - D_q$ is closed, the set $(\alpha - P + M_1 - D_q)$ must contain a limit point of D_q. But every limit point of D_q which does not belong to D_q belongs to α. Hence $\alpha - P$ contains a limit point A of D_q.

Let ϵ be a positive number less than $\rho(A, P)$. By Theorem 2 there is a point A' of α such that $\rho(A, A') < \epsilon$ and a convergent chain of $M - \alpha$ from A' to P. This chain contains just one arc or is of type (b). If the chain contains just one arc β_1, then β_1 plus the subarc $A'P$ of α is a simple closed curve of M containing P. But this is a contradiction as P lies on no simple closed curve of M.

If the chain is of type (b), let $\beta_1, \beta_2, \beta_3, \cdots,$ be the arcs of the chain.

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[*] For the theorem in two dimensions, see (E) W. L. Ayres, "Concerning continuous curves and correspondences," *Annals of Mathematics*, Vol. 28 (1927), pp. 396-418, Theorem 3.

[†] A point P of a continuous curve M is said to be an *end-point* of M if for any arc α of M one of whose end-points is P, the set $M - (\alpha - P)$ contains no connected subset containing more than one point and containing P. See reference (A), page 358.

[‡] A point P of a connected set M is said to be a *non-cut-point* or a *cut-point* of M according as $M - P$ is connected or not.

[§] See reference (E).

[¶] See reference (D).

Let

$$\alpha_1 = P + \sum_{i=1}^{\infty} \beta_{2i-1} + \sum_{i=1}^{\infty} \text{subarc } t_{2i-1}s_{2i+1} \text{ of } \alpha,$$
$$\alpha_2 = P + \text{subarc } A's_2 \text{ of } \alpha + \sum_{i=1}^{\infty} \beta_{2i} + \sum_{i=1}^{\infty} \text{subarc } t_{2i}s_{2i+2} \text{ of } \alpha.$$

Then α_1 and α_2 are arcs of M from A' to P which have only these two points in common. Hence $\alpha_1 + \alpha_2$ is a simple closed curve of M containing P. But by hypothesis P belongs to no simple closed curve of M.

As a corollary of Theorem 3 we have a theorem which was proved by G. T. Whyburn * for two dimensions.

THEOREM 4. *Every continuous curve is the sum of its end-points, cut-points and simple closed curves.*

THEOREM 5.† *In order that a point P of a continuous curve M should be an end-point of M it is necessary and sufficient that no arc of M should have P as an interior point.*

Proof. The proof of the necessity of the condition given for two dimensions holds without regard to the containing space.

The condition is sufficient. Suppose P is a point of M that is interior to no arc of M and that P is not an end-point of M. Then, by Theorem 3, the point P is either a cut-point of M or lies on some simple closed curve of M. If P lies on a simple closed curve J of M, let A and B be two points of J which are distinct and distinct from P. Then there is an arc of J with end-points A and B that contains P as an interior point, contrary to the assumed condition. If P is a cut-point of M, let $M - P = M_1 + M_2$, where M_1 and M_2 are non-vacuous mutually separated sets. Let A be a point of M_1 and let B be a point of M_2. Any arc of M with end-points A and B must have P as an interior point. Hence the assumption that P is not an end-point of M leads to a contradiction with the given condition.

Theorems 5 and 6 of my paper, "Concerning Continuous Curves and Correspondences", ‡ hold in our general space without change in proof since the proofs given depend only on Theorem 5 of the present paper. Theorems 7 to 15 inclusive of my paper, "Concerning Continuous Curves

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* (F) G. T. Whyburn, "Concerning Continua in the Plane," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 369-400, Theorem 22.

† For the theorem in two dimensions, see reference (F), Theorem 12. See also reference (E), Theorem 4.

‡ See reference (E). See also reference (F), Theorem 21. However, the proof given by Whyburn is strictly two-dimensional.

[illegible] Correspondences", extend to the space considered here with the [illegible] since they concern internal properties of continuous curves contai[illegible] most one simple closed curve, and every such curve is in continuous (1. [illegible] spondence with some curve of the same type in the plane.*

THEOREM 6.† *If K_1 and K_2 are mutually exclusive closed subsets of* [illegible] *continuous curve M and no point of M separates*‡ *K_1 and K_2 in M,* [illegible] *are two arcs α_1 and α_2 of M such that (1) for each i ($i = 1, 2$),* [illegible] *a point of K_1 as one end-point and a point of K_2 as the other end-p*[illegible] *and $M - K_1 - K_2$ contains $< \alpha_i >$, (2) $< \alpha_1 >$ and $< \alpha_2 >$ are mutu*[illegible] [illegible]*sive.*

Proof. Let u be a point of K_1 and v be a point of K_2, and let α' be [illegible] arc of M with end-points u and v. In the order from u to v let x be t[illegible] point of K_1 on α', and let y be the first point of K_2 on the subarc [illegible] α'. Let α denote the subarc of α' with end-points x and y. On α [illegible] order as being from x to y. If any component of $M - \alpha$ conta[illegible] [illegible]s of both K_1 and K_2, then this component contains an arc havi[illegible] [illegible] of K_1 as one end-point and a point of K_2 as the other end-point a[illegible] [illegible]g no other point in common with $K_1 + K_2$. Then this arc toget[illegible] with the arc α forms a pair of arcs having all the properties of our theor[illegible]. [illegible] component of $M - \alpha$ contains points of both K_1 and K_2, then for [illegible] component d of $M - \alpha$ let P_{xd} and P_{yd} denote the first and last limit points of d on α. Let H_1 denote the set of all points $P_{yd'}$, where d' is a component of $M - \alpha$ containing a point of K_1, and let H_2 denote the set of all points $P_{xd''}$, where d'' is a component of $M - \alpha$ containing a point of K_2. Let x' denote the last point of $\bar{H}_1 + x$ on α, and let y' denote the first point of $\bar{H}_2 + y$ on α.

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* For the case of a continuous curve containing no simple closed curve, see [illegible] Ważewski, "Sur les courbes de Jordan ne renfermant aucune courbe fermée de [illegible]" *Annales de la Société Polonaise de Mathématique*, Vol. 2 (1923), pp. 49-170, [illegible] Menger, "Über reguläre Baumkurven," *Mathematische Annalen*, Vol. 96 (1926), [illegible] 572-582, and H. M. Gehman, "Concerning Acyclic Continuous Curves," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 553-568. The case where [illegible] continuous curve contains just one simple closed curve is not essentially differ[illegible]

† For the case where M lies in a plane and K_1 and K_2 are single points, this th[illegible] has been proved by G. T. Whyburn, "Some Properties of Continuous Curv[illegible]" *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 305-308. The [illegible] The same special case was announced to the American Mathematical So[illegible] February 26, 1927, by the author, but it has never been published. See an abstrac[illegible] "On the Separation of Points of a Continuous Curve by Arcs and Simple Closed Curves," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 26[illegible] Theorems closely related to Theorem 6 have been stated by K. Menger, "Zur allgemeinen Kurventheorie," *Fundamenta Mathematicae*, Vol. 10 (1926), pp. 96-115, Statz [illegible] and N. E. Rutt, "Concerning the cut-points of a continuous curve when the arc-curve [illegible] contains exactly n independent arcs," *American Journal of Mathematics*, Vol. [illegible] (1929), pp. 217-246.

‡ If K_1 and K_2 are mutually exclusive subsets of a connected set M, a subset H of [illegible] said to separate K_1 and K_2 in M if $M - H$ is the sum of two mutually separate[illegible] sets, one containing K_1 and the other containing K_2.

If $x' \neq x$ and ϵ is any positive number, there is a point $P_{yd'}$ of H_1 such that $\rho(x', P_{yd'}) < \epsilon/2$. There exists a positive number δ_ϵ such that $\delta_\epsilon < \rho(P_{yd'}, K_1)$ and any point of M whose distance from $P_{yd'}$ is less than δ_ϵ can be joined to $P_{yd'}$ by an arc of M every point of which is within a distance $\epsilon/2$ of $P_{yd'}$. There is a point z of d' such that $\rho(z, P_{yd'}) < \delta_\epsilon$. Let γ_1 be an arc of M with end-points z and $P_{yd'}$, every point of which is within a distance $\epsilon/2$ of $P_{yd'}$. In the order from z to $P_{yd'}$ let x'' be the first point of α on γ_1. The component d' contains a point w of K_1, and let γ_2 be an arc of d' with end-points w and z. In the order from w to z let w' be the last point of K_1 and let z' be the first point of the subarc zx'' of γ_1 on the subarc $w'z$ of γ_2. The subarc $w'z'$ of γ_2 plus the subarc $z'x''$ of γ_1 is an arc of M having one end-point in K_1 and one end-point on α within a distance ϵ of x' and lying except for its end-points in $M - (K_1 + K_2 + \alpha)$. Similarly if $y' \neq y$, there is an arc having the same properties with respect to y' and K_2.

Now suppose y' precedes x' on α. There exists a positive number ϵ such that if y'' and x'' are points of α so that $\rho(x', x'') < \epsilon$ and $\rho(y', y'') < \epsilon$, then y'' precedes x'' on α. Since $x' \neq x$ and $y' \neq y$, there exist two mutually exclusive arcs γ_3 and γ_4 such that (1) γ_3 has a point of K_2 as one end-point and a point y'' of α such that $\rho(y', y'') < \epsilon$ as the other end-point, (2) γ_4 has a point of K_1 as one end-point and a point x'' of α such that $\rho(x', x'') < \epsilon$ as the other end-point, (3) $M - (K_1 + K_2 + \alpha)$ contains $< \gamma_3 >$ and $< \gamma_4 >$. Then the arc composed of γ_3 plus the subarc xy'' of α together with the arc composed of γ_4 plus the subarc yx'' of α form a pair of arcs satisfying the conditions of our theorem. This completes the proof for the case where y' precedes x' on α and for the remainder of the argument we will suppose that y' does not precede x' on α.

If $x' = y$, then either (1) y is a point of H_1 or (2) y belongs to $\bar{H}_1 - H_1$. In the first case there is a component d' of $M - \alpha$, containing a point q of K_1 such that $P_{yd'} = y$. If y is accessible from d' there exists an arc β whose end-points are q and y and which lies in d' except for y. In the order from q to y let q' be the last point of K_1 on β. Then α together with the subarc $q'y$ of β forms a set of two arcs satisfying the conditions of the theorem. If y is not accessible from d', by Theorem 2, there exists a convergent chain of

d' of type (b) from some point A' of α to y. Let $\beta_1, \beta_2, \beta_3, \cdots$, be the arcs of this chain. Let s_i be the first end-point of β_i on the arc α, and let t_i be the other end-point of β_i. There exists a smallest positive integer k such that for $i \geqq k$, the arc β_i contains no point of the set K_1. There exists a positive number ϵ such that if z is any point of α so that $\rho(z, t_k) < \epsilon$, then z lies on the subset $< s_2 y$ of α. There is a positive number δ_ϵ such that any point of M whose distance from t_k is less than δ_ϵ can be joined to t_k by an arc of M every point of which is within a distance ϵ of t_k. If $k > 1$, let $m = k$ and let h be the first point of K_1 on the arc β_{k-1} in the order from t_{k-1} to s_{k-1}. Let ζ denote the subarc $h t_{k-1}$ of β_{k-1}, and let t_{k-1} be denoted by g. If $k = 1$, let c be a point of d' such that $\rho(c, t_1) < \delta_\epsilon$. Let ζ_1 be an arc of d' with end-points c and q, and let ζ_2 be an arc of M with end-points c and t_1 such that every point of ζ_2 is within a distance ϵ of t_1. The set $\zeta_1 + \zeta_2$ contains an arc λ which has a point h of K_1 as one end-point and a point e of α as the other end-point and no other point in common with $K_1 + \alpha$. If λ has no point in common with $\sum_{i=1}^{\infty} < \beta_i >$, let $g = e$, $\zeta = \lambda$, and let m be the smallest integer such that e lies between s_m and t_m on α. If λ has a point in common with $\sum_{i=1}^{\infty} < \beta_i >$, in the order from h to e let b be the first point of $\sum_{i=1}^{\infty} < \beta_i >$ on the arc λ. It is easy to prove the existence of a first point. Let β_{m-1} be the arc containing b, and let $t_{m-1} = g$ and ζ be the arc composed of the subarc hb of λ plus the subarc $b t_{m-1}$ of β_{m-1}. Now in any of the preceding possibilities of case (1) let

$$\alpha_1 = \text{subarc } x s_m \text{ of } \alpha + \sum_{i=0}^{\infty} \beta_{m+2i} + \sum_{i=0}^{\infty} \text{subarc } t_{m+2i} s_{m+2i+2} \text{ of } \alpha,$$

$$\alpha_2 = \zeta + \text{subarc } g s_{m+1} \text{ of } \alpha + \sum_{i=0}^{\infty} \beta_{m+2i+1} + \sum_{i=0}^{\infty} \text{subarc } t_{m+2i+1} s_{m+2i+3} \text{ of } \alpha,$$

Then α_1 and α_2 are two arcs satisfying the conditions of our theorem.

In case (2) there exists a set of components $d_1, d_2, \cdots$, of $M - \alpha$ such that each component contains a point of K_1 and each contains a point p_i such that $\rho(p_i, y) < 1/i$. Let S be a hypersphere with center y such that every point of K_1 lies in $E(S)$. For each i let d'_i be the component of $I(S) \cdot d_i$ that contains p_i. By methods almost identical with those of Theorem 2 we may show that there exists a convergent chain of $[d'_i]$ of type (b) from some point of $< \alpha >$ to y. With the use of this chain we may select two arcs satisfying the hypothesis of our theorem as in the above paragraph.

This completes the argument for the case where $x' = y$. Similarly we may complete the proof if $y' = x$. For the remainder of our proof we shall consider $x' \neq y$ and $y' \neq x$.

For each point P of $<\alpha>$ let G_P denote the set of all points P_{xd} for all components d of $M-\alpha$ such that P lies between P_{xd} and P_{yd} on α. If $x'=y'$, let $Q=x'$. If $x'\neq y'$ let Q be some point of the subset $<x'y'>$ of α. Let T denote the point x if $x'=x$ and if $x'\neq x$ let it denote the subset $xx'>$ of α. Let Q_1 be the first point of $\bar{G}_Q$. If Q_1 belongs to T, then either (1) there is a component D of $M-\alpha$ which has a point of T and a point of α between Q and y as limit points, or (2) there is no such component. If we have case (1), by Theorem 2 there is a convergent chain of D from some point of α between y and Q to some point of T. If we have case (2), we must have $x'=x$; and if S_1 is a hypersphere with center x and radius sufficiently small that the subarc yQ of α lies in $E(S_1)$ and if S_2 is any hypersphere concentric with and interior to S_1, there are infinitely many components of $M-\alpha$ which have points in $I(S_2)$ and $E(S_1)$. In exactly the same manner as in the proof of Theorem 2 we may obtain a convergent chain in this situation.

If Q_1 does not belong to T, let ϵ_1 be a positive number less than 1 such that if z is any point of the subarc yQ_1 of α so that $\rho(z, Q_1)<\epsilon_1$, then there is a component d of $M-\alpha$ such that both z and Q_1 lie between P_{xd} and P_{yd}. There is a point s_1 of G_Q such that $\rho(s_1, Q_1)<\epsilon_1/2$. Let d_1 be a component of $M-\alpha$ such that $P_{xd_1}=s_1$ and P_{yd_1} lies on the subset $yQ>$ of α. There exists a number $\delta_1>0$ such that any point of M whose distance from s_1 is less than δ_1 can be joined to s_1 by an arc of M every point of which is within a distance $\epsilon_1/2$ of s_1. Let p_1 be a point of d_1 so that $\rho(p_1, s_1)<\delta_1$, and let α_1 denote an arc of M from p_1 to s_1 every point of which is within a distance $\epsilon_1/2$ of s_1. Let u_1 be the first point of α on α_1 in the order from p_1 to s_1. Then $\rho(u_1, Q_1)<\epsilon_1$.

The component d_1 has a limit point t_1 on the subset $yQ>$ of α. There is a point p_2 of d_1 and an arc α_2 of M from p_2 to t_1 such that every point of α_2 is at a distance from t_1 less than $\rho(t_1$, subarc xQ of $\alpha)$. Let v_1 be the first point of α on the arc α_2 in the order p_2 to t_1. There is an arc α_3 of d_1 with end-points p_1 and p_2. The set $\alpha_1+\alpha_2+\alpha_3$ contains an arc β_1 with end-points u_1 and v_1 and such that d_1 contains $<\beta_1>$. It is easy to see that on α we have the order $xQ_1u_1Qv_1y$.

Let Q_2 be the first point of the set $\bar{G}_{u_1}$ on α. If Q_2 belongs to T, our argument is completed as above. If Q_2 does not belong to T, let ϵ_2 be a positive number less than ½ such that if z is any point of the subset $<Q_2y$ of α such that $\rho(z, Q_2)<\epsilon_2$, then there is a component d of $M-\alpha$ such that both z and Q_2 lie between P_{xd} and P_{yd} on α. Just as above we may show that there is a component d_2 of $M-\alpha$ different from d_1 and an arc β_2 with end-points u_2 and v_2 of α such that $\rho(u_2, Q_2)<\epsilon_2$, d_2 contains $<\beta_2>$, and on α we have the order $xQ_2u_2Q_1u_1v_2v_1$.

Let Q_3 be the first point of the set $\bar{G}_{u_2}$ on α. If Q_3 belongs to T, our argument is completed as above. If Q_3 does not belong to T, let ϵ_3 be a positive number less than 1/3 such that if z is any point of the subarc [illegible] of α so that $\rho(z, Q_3) < \epsilon_3$, then there is a component d of $M-\alpha$ such that both z and Q_3 lie between P_{xd} and P_{yd}. There exists a component d_3 of $M-\alpha$ different from both d_1 and d_2 and an arc β_3 with end-points u_3 and v_3 of α such that $\rho(u_3, Q_3) < \epsilon_3$, d_3 contains $<\beta_3>$, and we have the order $xQ_3u_3v_3u_1v_2v_1y$ on α.

Continue this process indefinitely unless for some n, the point Q_n belongs to T. In this case we may complete the argument as above showing that there is a convergent chain of $M-\alpha$ joining v_1 and some point of T. If the process continues indefinitely, the points $u_1, u_2, u_3, \cdots$, approach a point x_1 as a sequential limit point. If $x_1 \neq x$, there is a component d of $M-\alpha$ such that x_1 lies between P_{xd} and P_{yd}. Let j be the smallest integer such that u_j lies on the subset $x_1P_{yd}>$ of α. Then the point Q_{j+1} lies [illegible] P_{xd} of α. And u_{j+2} precedes Q_{j+1} on the arc α. But this is absurd, for x_1 precedes every point u_i and P_{xd} precedes x_1. Therefore $x_1 = x$ and the set $\beta_1, \beta_2, \beta_3, \cdots$, is a chain of $M-\alpha$ from v_1 to x.

We shall now show that the chain is convergent. Suppose there exists a positive number η and an increasing sequence of integers $n_1, n_2, n_3, \cdots$, such that β_{n_i} is of diameter greater than η for every positive integer i. Let S be a hypersphere of radius $\eta/2$ with x as a center, and for each value of i let z_i be a point of $S \cdot \beta_{n_i}$. There exists a point z' and a subsequence $k_1, k_2, k_3, \cdots$ of the sequence $n_1, n_2, n_3, \cdots$, such that z' is the sequential limit point of $[z_{k_i}]$. Since no two of the sets $<\beta_{k_i}>$ lie in the same component of $M-\alpha$, it is easy to see that the arc α contains z'. On the subarc xz' of α let u_j be the last point of the set $[u_i]$ in the order from x to z'. Let γ be the smaller of the numbers $\rho(z'$, subarc xu_j of $\alpha)$ and $\rho(x$, subarc [illegible] of $\alpha)$. There exists a number $\delta_\gamma > 0$ such that any point of M whose distance from z' or x is less than δ_γ can be joined to z' or x, as the case may be, by an arc of M every point of which is within a distance γ of z' or x. There is an arc β_{k_n} containing points z_{k_n} and w_{k_n} such that $\rho(z', z_{k_n}) < \delta_\gamma$ and $\rho(x, w_{k_n}) < \delta_\gamma$. Then the component of $M-\alpha$ containing $<\beta_{k_n}>$ has limit points on α preceding Q_{j+1} and following u_j. But this is contrary to the definition of Q_{j+1}. Thus $\beta_1, \beta_2, \beta_3, \cdots$, is a convergent chain of $M-\alpha$ of type (b) from v_1 to x.

If $y' = y$, let U be the point y. If $y' \neq y$, let U be the subset $<y'y$ of α. In exactly the same manner as above we determine a convergent chain of $M-\alpha$ of type (a) or (b) from a point of α preceding Q to a point of the set U. From the two chains we may select a convergent chain of $M-\alpha$ from

a point of T to a point of U by omitting a finite number of arcs. From this convergent chain, the arc α, and, if $x' \neq x$, an arc of M one of whose end-points belongs to K_1 and the other is a point of α sufficiently close to x' and which has no other point in common with $K_1 + K_2 + \alpha$, and, if $y' \neq y$, an arc of M one of whose end-points is a point of K_2 and the other a point of α sufficiently close to y' and having no other point in common with $K_1 + K_2 + \alpha$, we may select two arcs which satisfy the conditions of the theorem as we did in case (1) of $x' = y$.

THEOREM 7. *If x and y are distinct points of a continuous curve M, then either there is a point of M which separates x and y in M or there is a simple closed curve of M containing both x and y.*

This theorem is a corollary of Theorem 6.

THEOREM 8. *If K_1 and K_2 are two mutually exclusive closed subsets of the continuous curve M and for every point P of M, the sets $(K_1 + P) - P$ and $(K_2 + P) - P$ are non-vacuous and are not separated in M by the point P, then there exist two mutually exclusive arcs α_1 and α_2 of M such that each arc has a point of K_1 as one end-point and a point of K_2 as the other and, except for these points, lies entirely in $M - K_1 - K_2$.*

THEOREM 9. *If K_1 and K_2 are mutually exclusive closed subsets of the continuous curve M and for every point P of $M - K_2$, the sets $(K_1 + P) - P$ and K_2 are non-vacuous and are not separated in M by the point P, then there exist two arcs α_1 and α_2 such that* (1) *each arc has a point of K_1 as one end-point and a point of K_2 as the other and lies, except for these points, entirely in $M - K_1 - K_2$,* (2) *the arcs α_1 and α_2 have no point in common except possibly their end-points that belong to K_2.*

The proof given for Theorem 6 is sufficient to prove Theorems 8 and 9 since the two arcs constructed in proving Theorem 6 have in common their end-points that belong to K_1 only if $x' = x$, and the end-points belonging to K_2 are identical only if $y' = y$.

THEOREM 10.* *In order that a continuous curve be cyclicly connected it is necessary and sufficient that it contain no cut-point.*

Proof. The proof of the necessity of the condition given for two dimensions holds equally well in the space considered here.

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* For the result in two dimensions, see (G) G. T. Whyburn, "Cyclicly Connected Continuous Curves," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 31-38, Theorem 1. A continuous curve M is said to be cyclicly connected if every two points of M lie together on some simple closed curve belonging to M.

The condition is sufficient. Let x and y be any two distinct points of M. Since M contains no cut-point, there is no point of M which separates x and y in M. Then, from Theorem 7, there is a simple closed curve of M containing x and y. Hence M is cyclicly connected.

Theorems 2-7 and 9 of G. T. Whyburn's paper, "Cyclicly Connected Continuous Curves,"[*] hold in n dimensions as the proofs given by Whyburn depend only on our theorem 10 (Whyburn's theorem 1), two theorems of G. T. Whyburn,[†] and a theorem of R. L. Moore,[‡] all of which are true in the space considered here. Also theorem 8 of Whyburn's paper[*] is true in our space but a different proof of part (2) is necessary to avoid the Wilder accessibility theorem.

THEOREM 11.[§] *If the point x of a continuous curve M lies on no simple closed curve of M and α is any arc of M whose end-points are x and any other point z of M, then x is a limit point of the points of M which lie on α and separate x and z in M.*

Proof. Suppose the theorem is not true. Then there is an arc α of M whose end-points are x and some other point z of M, and a positive number ϵ such that no point of α whose distance from x is less than ϵ separates x and z in M. There is a point y of α such that every point of the subarc xy of α is at a distance from x less than ϵ. Hence no point of the subarc xy of α separates x and y in M. Moreover if there is any point of M which separates x and y in M, it must belong to every arc of M with end-points x and y, and thus to the subarc xy of α. Then by Theorem 7 there is a simple closed curve of M containing both x and y. But by hypothesis x belongs to no simple closed curve of M.

THEOREM 12.[¶] *If K is any subset of the continuous curve M and $M(K)$ denotes the arc-curve of K with respect to M,[‖] then the point set $\overline{M(K)} - M(K)$ is a subset of the point set $\bar{K} - K$.*

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[*] See reference (G).

[†] See reference (F), theorems 15 and parts 1) and 3) of 24. Theorems 24 and theorems 10 and 23 of (F), on which theorems 15 and 24 are based, require different proofs in our more general space.

[‡] See reference (C), Theorem 1.

[§] For the two-dimensional case, see (H) W. L. Ayres, "On the Structure of a Plane Continuous Curve," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 749-754.

[¶] For the theorem in two dimensions, see (I) W. L. Ayres, "Concerning the Arc Curves and Basic Sets of a Continuous Curve," *Transactions of the American Mathematical Society*, Vol. 30 (1928), pp. 567-578, Theorem 2.

[‖] The set of all points $[P]$ such that P lies on some arc of M whose end-points belong to K is called the *arc-curve of K with respect to M*. If K contains just one point, we define the arc-curve of K with respect to M as being this single point.

Proof. Suppose $\overline{M(K)} - M(K)$ contains a point x which does not belong to $\bar{K} - K$. Since $M(K)$ contains K, the point x belongs to $M - \bar{K}$. Let z be a point of K and let xz be an arc of M with end-points x and z. In the order from x to z let y be the first point of $\bar{K}$ on xz. Then $x \neq y$. Let α be the subarc of xz with end-points x and y. Since x belongs to $\overline{M(K)} - M(K)$, the set K contains more than one point. For each component d of $M - \alpha$ containing a point of K let P_{xd} denote the first limit point of d on α in the order from x to y. Let H denote the set of all such points P_{xd} for all such components d of $M - \alpha$. Let w be the first point of $\bar{H} + y$ on the arc α in the order x to y. If $w = y$, then

$$M - y = M_1 + M_2,$$

where M_1 and M_2 are non-vacuous mutually separated sets containing x and $\bar{K} - y$ respectively. Then $M(K)$, and thus $\overline{M(K)}$, is a subset of $M_2 + y$. Then x cannot be a point of $\overline{M(K)}$, contrary to hypothesis. Hence $w \neq y$.

In much the same way we may show that there is no point P which separates x and $\bar{K}$ in M. Now let P be a point of $\bar{K}$. If $P \neq y$, then $(\bar{K} + P) - P$ is non-vacuous since it contains y, and P does not separate x and $(\bar{K} + P) - P$ in M since the arc α does not contain P and joins x and a point y of $\bar{K}$. If $P = y$, then $\bar{K} - P$ is non-vacuous and P does not separate x and $\bar{K} - P$ in M since there is a component d of $M - \alpha$ containing a point of $\bar{K}$ which has a limit point u on the subset $wy >$ of α, and the subarc xu of α plus the set d is a connected subset of $M - P$ containing x and a point of K. Thus the hypothesis of Theorem 9 is satisfied and there exist two arcs α_1 and α_2 of M having only the point x in common such that each arc has x as one end-point and a point of $\bar{K}$ as the other end-point and lies in $M - \bar{K} - x$ except for these points. Let y_1 and y_2 be the end-points of α_1 and α_2 belonging to $\bar{K}$. If y_1 belongs to K, let t_1, y_1' and α_3 all be the point y_1. If y_1 belongs to $\bar{K} - K$, let $\epsilon_1 = \rho(y_1, \alpha_2)/2$. There is a positive number δ_1 such that any point of M whose distance from y_1 is less than δ_1 can be joined to y_1 by an arc of M of diameter less than ϵ_1. There is a y_1' of K such that $\rho(y_1', y_1) < \delta_1$ and let α_3 be an arc of M with end-points y_1 and y_1' and of diameter less than ϵ_1. In the order from y_1' to y_1 let t_1 be the first point of α_1 on α_3. In a similar manner we define y_2', t_2 and α_4. Then the sets composed of the subarc $y_1't_1$ of α_3 plus the subarc xt_1 of α_1 plus the subarc xt_2 of α_2 plus the subarc t_2y_2' of α_4 is an arc of M containing x and with the points y_1' and y_2' of K as end-points. Then, by definition, the point x belongs to $M(K)$. But we assumed that $\overline{M(K)} - M(K)$ contains the point x.

Theorems 1, 3 to 9 and 11 to 13 inclusive, and part (1) of 10 of my [illegible]. "Concerning the Arc-Curves and Basic Sets of a Continuous Curve"* [illegible] our present space without change in proof since they depend only on theorems 3, 5 and 12 of the present paper, theorems 2 and 3 of G. T. Whyburn's paper, "Cyclicly Connected Continuous Curves",† and a theorem of my paper, "Continuous Curves which are Cyclicly Connected",‡ all of which hold in this space. Part (2) of theorem 10 holds also in this space but the proof must be modified slightly to avoid using the accessibility theorem of [illegible].

THEOREM 13.§ *If x is a point of a continuous curve M and C_x is the set consisting of x and all points $[y]$ such that M contains a simple closed curve containing both x and y, then C_x is a continuous curve.*

Proof. Obviously C_x is connected. Let z be any limit point of C_x. If there is any point P which separates x and z in M, then

$$M - P = M_x + M_z,$$

where M_x and M_z are mutually separated sets containing x and z respectively. Since z is a limit point of C_x, the set M_z contains a point y of C_x. There exists a simple closed curve J of M containing x and y. Both of the arcs of J with end points x and y cannot contain the point P. But every connected set containing a point of M_x and a point of M_z must contain the point P. Hence there is no point of M which separates x and z in M. Then by Theorem 7 there is a simple closed curve of M containing both x and z. Therefore z belongs to C_x by definition. Then every limit point of C_x belongs to C_x. We may prove that C_x is connected im kleinen as in the two-dimensional case.

THEOREM 14.¶ *Under the hypothesis of Theorem 13, if K is a component of $M - C_x$, then K has just one limit point in C_x.*

Suppose K has two limit points p and q in C_x. Let $\epsilon = \frac{1}{2}\rho(p, q)$. There exists a positive number δ_ϵ such that any point of M whose distance from p or q is less than δ_ϵ can be joined to p or q, as the case may be, by an arc of M of diameter less than ϵ. The set K contains two points u and v such that

$$\rho(u, p) < \delta_\epsilon \quad \text{and} \quad \rho(v, q) < \delta_\epsilon.$$

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[*] See reference (I).

[†] See reference (G).

[‡] *Bulletin de l'Académie Polonaise des Sciences et des Lettres* (1928), pp. 127-1[illegible]. The theorem referred to is: If x, y and z are distinct points of a cyclicly connected continuous curve, there exists an arc belonging to the continuous curve, containing z and having x and y as end-points.

[§] For the theorem in two dimensions, see reference (H), Theorem 2.

[¶] For the result in two dimensions see reference (H), Theorem 3.

Let α_1 and α_2 denote arcs of M of diameter less than ϵ with end-points u and p and v and q respectively. Let s and t be the first points of C_x on the arcs α_1 and α_2 in the orders from u to p and from v to q respectively. The set K contains an arc α_3 with end points u and v. The set composed of α_3 plus the subarc su of α_1 plus the subarc tv of α_2 contains an arc α_4 with end-points s and t and such that K contains $< \alpha_4 >$. By an argument following that used in the two-dimensional case of Theorem 13, we may show that every point of an arc whose end-points belong to C_x is a point of C_x. But $M - C_x$ contains the set K and $< \alpha_4 >$ is a subset of K.

THEOREM 15.[*] *If the continuous curve M contains a cut-point and a simple closed curve, it contains a cut-point which lies on a simple closed curve of M.*

Proof.[†] Let P be a cut-point of M and let J be a simple closed curve of M. There is a maximal cyclic curve C of M containing J.[‡] If C contains the point P, there is a simple closed curve of C containing the point P. If C does not contain P, let R be the component of $M - C$ containing the point P. The component R has just one limit point Q in C.[§] Hence Q must be a cut-point of M. There is a simple closed curve containing Q and lying in C. Then Q is a cut-point of M which lies on a simple closed curve of M.

Theorems 4 and 6 to 13 inclusive of my paper, "On the Structure of a Plane Continuous Curve",[¶] hold in our more general space with the same proofs as given for two dimensions as these proofs depend only on theorems 10, 11, 13, 14 and 15 of the present paper and theorems 7 and 10, part (2) of my paper, "Concerning the Arc-Curves and Basic Sets of a Continuous Curve",[‖] all of which hold in our more general space.

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[*] For the two-dimensional case of this theorem, see reference (H), Theorem 5.

[†] This very simple proof of this theorem was suggested by G. T. Whyburn. It is somewhat shorter than my original proof.

[‡] See reference (G), Theorem 3, and a statement following Theorem 10 of the present paper on the validity of this result for the more general space considered here. A cyclicly connected continuous curve C which is a subset of a continuous curve M is said to be a *maximal cyclic curve* of M if C is not a proper subset of any cyclicly connected continuous curve which is a subset of M.

[§] See reference (G), Theorem 2, and a statement on this theorem in the present space following Theorem 10 of the present paper.

[¶] See reference (H).

[‖] See reference (I) and a statement following Theorem 12 of the present paper on the validity of these theorems in our more general space.

On A Certain Function of the Masses in the Problem of Three Bodies.

By H. E. Buchanan.

Introduction. In his discussion of the stability of an infinitesimal body near the equilateral triangle positions Professor Moulton[*] found a certain function of the two finite masses whose sign determined the character of the motion of the infinitesimal body. He chose the two finite masses to be μ and $1-\mu$ and found the infinitesimal body to be in stable equilibrium if

$$1-27\mu(1-\mu)\geqq 0,$$

and this inequality is satisfied by $\mu \leqq .0385$.

It is the purpose of this paper to discuss the corresponding function when all the masses are finite.

The foundation for the discussion has already been laid in the author's recent paper on Periodic Orbits[†] near the Equilateral Triangle Solutions. We make use of the equations (5), (6) and (7) of that paper, together with the results obtained from them. The function which we wish to discuss is $(m_1+m_2+m_3)^2-27(m_1 m_2+m_1 m_3+m_2 m_3) = f(m_1, m_2, m_3)$. If $f(m_1, m_2, m_3) > 0$, the characteristic exponents are pure imaginaries, in pairs, except for the double root zero. If $f(m_1, m_2, m_3) = 0$ there appears another pair of double roots, $+i\omega/2^{1/2}$ and $-i\omega/2^{1/2}$. If $f(m_1, m_2, m_3) < 0$ there appears, in addition to the double root zero, a pair of conjugate complex roots. It follows that the equilateral triangle solutions are always unstable for every arrangement of the masses. We designate the three cases above:

Case I. $f(m_1, m_2, m_3) > 0$.
Case II. $f(m_1, m_2, m_3) = 0$.
Case III. $f(m_1, m_2, m_3) < 0$.

The cone $f(m_1, m_2, m_3)=0$. Since $f(m_1, m_2, m_3)$ is homogeneous and of the second degree in m_1, m_2, and m_3, $f(m_1, m_2, m_3)=0$ may be regarded as the equation of a cone, m_1, m_2, m_3 being the rectangular coordinates of any point. This cone has its vertex at the origin and, from the symmetry, the line $m_1=m_2=m_3$ is its axis. We wish to define precisely those regions in which all the masses are positive and $f(m_1, m_2, m_3) \gtreqless 0$. Further, if one of the masses is regarded as given we shall find the limiting values of the other two in each of the three cases. Finally, if $m_3 = 0$, we shall get Moulton's results and obtain a geometrical interpretation for them.

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[*] Moulton, *Celestial Mechanics*, revised ed., p. 307.

[†] *American Journal of Mathematics*, Vol. 50, No. 4 (1928).

Rotating the $m_1\, m_3$ plane clockwise through 45° changes the equation of the cone to

$$27x^2 + 2m_2^2 - 23z^2 - 50\,(2)^{1/2}\,m_2 z = 0.$$

Rotating the $m_2 z$ plane counter clockwise through the proper angle to remove the $m_2 z$ term we find

$$3x^2 + 3y^2 - (16/3)z^2 = 0,$$

which is a circular cone. The generating angle is approximately 53° 7′.

The Traces on the reference planes. Returning to the original variables we find the traces on the $m_1\, m_2$ plane by making $m_3 = 0$. This gives

$$m_1^2 + m_2^2 - 25m_1\, m_2 = 0,$$

or

$$m_1/m_2 = .04005,\quad m_1/m_2 = 24.9599.$$

These represent two straight lines whose inclinations are, approximately 2° 17′ and 87° 43′. Since m_1, m_2 and m_3 are interchangeable it follows that the traces on the other two reference planes are similarly situated.

Let us agree that $m_3 \geqq m_2 \geqq m_1$ and let us cut the cone by the plane

$$m_1 + m_2 + m_3 = C$$

where c is the largest value m_3 may have. Since the masses are always positive we confine our attention to that portion of space bounded by the coordinate planes and the plane

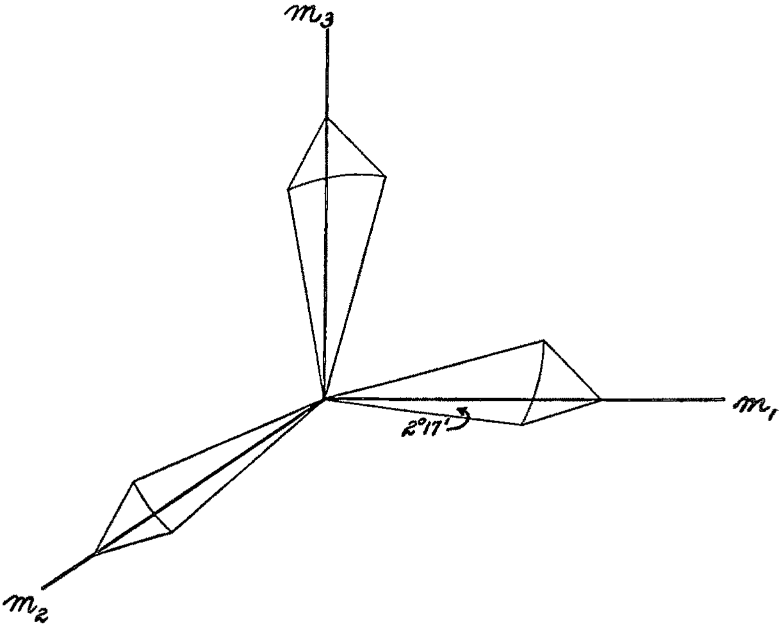
$$m_1 + m_2 + m_3 = C.$$

It is evident that $f(m_1, m_2, m_3) < 0$ for $m_1 = m_2 = m_3$, that is, for any point on the axis of the cone. The function can change sign only as (m_1, m_2, m_3) passes through the surface of the cone. Hence the only values of the masses which give rise to case I are determined by points inside the slender, rapier-like volumes extending out each axis, bounded by the coordinate planes and the cone. These are shown in the adjoining figure. For the purposes of drawing we were obliged to make them more blunt than they are in fact. If m_3 may be regarded as given then the largest value m_2 may have is $.04005m_3$ and for these values $m_1 = 0$. But if $m_2 < .04005m_3$ then m_1 may have any value from zero up to

$$\tfrac{1}{2}\left[25(m_2+m_3)-\{621(m_2^2+m_3^2)+1350\,m_2\,m_3\}^{1/2}\right],$$

which in turn must be less than $.04005m_3$.

The Infinitesimal Case. If $m_3=0$ and if we take the sum of the masses as unity we may find the limiting values of m_1 and m_2 by solving $m_1+m_2=1$ simultaneously with $m_1=.04005\,m_2$. The result is $m_1=.0385$, $m_2=.9615$. If we had solved simultaneously with the line near the m_2 axis the values of m_1 and m_2 would have been interchanged. This result checks exactly with Moulton's.



An Application. The mass of Jupiter is $954.8/10^6$ of the mass of the sun. If we let $m_3=1$ represent the sun, and $m_2=.0009548$ represent Jupiter, then the largest value m_1 can have, in order that Case I may hold, is $m_1=.039$. For these values, and for all smaller positive values of m_1 the equilateral triangle position is stable except for the presence, in the X_t and Y_t equations, of terms which are linear in t. A similar computation made for the earth and sun shows that Case I will hold for a body at the equilateral triangle position 13,000 times the mass of the earth.

If Gylden [*] and Moulton [†] are correct in their explanation of the Gegenschein, then it seems that there ought to be a larger accumulation near the equilateral triangle positions than near the straight line positions, for the latter give rise to terms of the type e^{at} in addition to linear terms in t.* This accumulation, however, would not be as readily visible as the Gegenschein because of its greater distance from the earth and because less than half the surfaces of any bodies there would appear illuminated, as seen from the earth.

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[*] *Bulletin Astronomique*, Vol. I.

[†] Moulton, *Astronomical Journal*, N. 483.

Cases II and III. If m_1, m_2 and m_3 are such that $f(m_1, m_2, m_3) = 0$, that is for any point on the cone, the characteristic exponents are

$$\pm i\omega,\ +i\omega/2^{1/2},\ +i\omega/2^{1/2},\ -i\omega/2^{1/2},\ -i\omega/2^{1/2},\ 0,\ 0.$$

Hence in this case also the positions are unstable, due to the presence of terms $K_4 t$ and $K_8 t e^{i\omega t/2^{1/2}}$.

Finally, for any point inside the cone the positions are unstable, due to the presence of terms of the type $K_4 t$ and of the type $K_6 e^{at} e^{i\beta t}$.

Thus in any system consisting of a sun and a planet, if the mass of the planet is less than 4% of its principal, Case I holds for any mass at the triangle point which is less than 1/10 of 1% of the principal. The planets of our system all come under Case I but the binary star systems undoubtedly come under Case III.

Conclusion. If we assume the initial conditions, $X_i = \alpha_i$, $Y_i = \beta_i$, $X' = \gamma_i$, $Y'_i = \delta_i$, $i = 1, 3$ at $t = 0$ and apply these conditions to equations (7)† it appears that the $K_{ij} (j = 1 \cdots 8)$ are linear homogeneous functions of the initial constants. Therefore K_{14} may be made zero if the initial constants are chosen so that a linear relation is satisfied. In this event seven of the initial constants are completely arbitrary and small oscillations of the three finite bodies will contain only periodic terms in Case I.

If the masses are such that Case II holds then three linear relations must be satisfied by the initial constants in order that only periodic terms may be present in equations (7). If Case III holds then five linear relations must hold between the initial constants in order that only periodic terms appear in the solutions.

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* Buchanan, "Periodic Orbits," *American Journal of Mathematics*, Vol. 45, No. 2, (1923).

† *American Journal of Mathematics*, Vol. 50, Oct. (1928).

Invariants of Sets of Points under Inversion.

By Joseph William Peters.

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1. *Introduction.* Two points, P and P', on a line with a point O are called inverse points if $OP \cdot OP' = \kappa$, where κ is a real constant. The point O is the center of the inversion. The transformation applies not only to points in a plane but to points in flat spaces of any number dimensions. For $\kappa > 0$, there is in the plane a circle of fixed points, in three dimensions a sphere of fixed points, and so on. To every point P different from O, there corresponds a unique finite point. In order to have a one-to-one correspondence in all cases, we close our space by a single point at infinity which is the inverse of the center, O. As a consequence of this assumption, circles and lines in the plane are inversely equivalent; spheres and planes in three dimensions are inversely equivalent; and so on. A further important property of inversions concerns the relationship between the distance from P to Q and the distance between the inverse points P' and Q'. If O is the center of inversion and κ the constant of inversion, the relation is

$$PQ = (\kappa / OP' \cdot OQ') \cdot P'Q'.$$

With these general remarks concerning inversions in mind, let us now proceed to the discussion of invariants under inversion.

2. *Inversive Properties of Sets of Points.* We shall first take the case of three points. Let (ij) represent the distance between the points i and j and let $(ij) = (ji)$. If 0 be a variable point in the plane of the three fixed points 1, 2, 3, consider the equations

$$(01)(23) = (02)(31) = (03)(12). \tag{2.1}$$

Under an inversion with any other point in the plane of 1, 2, 3 as a center, these equations are reproduced in the primed numbers. Hence we will say that they are invariant under inversions, or simply inversive.

The equations (2.1) represent three circles which intersect in two real points. For if we let the point 3 approach infinity, the equations of the three circles become

$$(01) = (12) \qquad (02) = (12) \qquad (01) = (02).$$

The first circle has its center at 1 and radius (12), the second has its center at 2 and radius (12), hence they intersect on the perpendicular bisector of (12) which is the third circle.

Since the circles (2.1) intersect in two real points, let us move one of these points to infinity by using it as the center of an inversion. The circles (2.1) now become straight lines with the equations

$$(01) = (02) = (03),$$

and the triangle 1, 2, 3 becomes equilateral since

$$(12) = (23) = (31).$$

It is now evident that each circle passes through one vertex of the triangle 1, 2, 3 and has the other two vertices as inverse points. Hence the equations $(01) = (02) = (03)$ represent the Apollonian Circles of the equilateral triangle 1, 2, 3.

In general, the equations (2.1) represent the Apollonian Circles of the triangle 1, 2, 3. Their two points of intersection are called the Hessian points of the triangle.

Consider now the equations

$$(01)(23) \pm (02)(13) \pm (03)(12) = 0 \tag{2.2}$$

using all combinations of signs. The equations are inversive. If we invert with one of the Hessian points as a center, the triangle 1, 2, 3 becomes equilateral as before, and the equations (2.2) become

$$(01) \pm (02) \pm (03) = 0.$$

The case where the signs are all positive is an inversion in the circumcircle of the triangle 1, 2, 3.

$(01) + (02) - (03) = 0$ represents the arc of the circumcircle from the point 1 to the point 2.

$(01) - (02) + (03) = 0$ represents the arc of the circumcircle from the point 3 to the point 1.

$(01) - (02) - (03) = 0$ represents the arc of the circumcircle from the point 2 to the point 3.

The point simultaneously satisfying $(01) = (02) + (03)$ and $(02) = (03)$ is on the circumcircle and is called the counterpoint of 1. Similarly $(02) = (01) + (03)$ and $(01) = (03)$ give the counterpoint of 2 and $(03) = (01) + (02)$ and $(01) = (02)$ give the counterpoint of 3.

To summarize, we may say the invariant relations of three points under inversion give the Hessian points; the Apollonian Circles; the circumcircle of the three points, not as a whole but in three arcs which make it complete;

an inversion in the circumcircle; and lastly the counterpoints of the three given points.

Take now the case of four points. Let 1, 2, 3, 4 be four points in a three-dimensional space with 0 as a variable point in that space and consider the equations homogeneous in the points 0, 1, 2, 3, 4

$$\begin{aligned}(01)^2(23)(34)(42) &= (02)^2(34)(41)(13)\\ = (03)^2(41)(12)(24) &= (04)^2(12)(23)(31).\end{aligned} \tag{2.3}$$

These six equations are inversive. They represent six spheres which intersect in two points, for if the point 4 approaches infinity the equations (2.3) become

$$(01)^2(23) = (02)^2(13) = (03)^2(12) = (12)(23)(13).$$

The spheres $(01)^2 = (12)(13)$ and $(02)^2 = (12)(23)$ will intersect if

$$(01) + (02) > (12)$$

or

$$(13)^{1/2} + (23)^{1/2} > (12)^{1/2}.$$

We assume that none of the four points coincide, so $(13) + (23) > (12)$. This inequality implies $(13)^{1/2} + (23)^{1/2} > (12)^{1/2}$. A similar proof shows that $(01)^2 = (12)(13)$ and $(03)^2 = (23)(13)$ intersect and likewise $(02)^2 = (12)(13)$ and $(03)^2 = (23)(13)$ intersect. These three spheres will intersect in two points and from the original conditions imposed by the equations the other three spheres must also pass through these two points, called the canonizant pair.

Let us put one of these canonizant points at infinity by an inversion with that point as a center. Then the six spheres become six planes whose equations are given by

$$(01)^2 = (02)^2 = (03)^2 = (04)^2.$$

The distances between the four points will then satisfy the following equations:

$$(23)(34)(42) = (34)(41)(13) = (14)(12)(24) = (12)(23)(13),$$

which reduce to

$$(12)^2 = (34)^2, \quad (13)^2 = (24)^2; \quad (14)^2 = (23)^2.$$

Four points in a three-dimensional space are always on a sphere, and these equations arrange the four points so that, if the center of the sphere is the origin, the points may be given coordinates (a, b, c), $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$. In this same reference frame, the Cartesian equations of the six planes are

$$\begin{array}{ll}
(01)^2 = (02)^2 & by + cz = 0, \\
(01)^2 = (03)^2 & ax + cz = 0, \\
(01)^2 = (04)^2 & ax + by = 0, \\
(02)^2 = (03)^2 & ax - by = 0, \\
(02)^2 = (04)^2 & ax - cz = 0, \\
(03)^2 = (04)^2 & by - cz = 0.
\end{array}$$

Thus when one of the canonizant points is at infinity, the other is at the center of the sphere on the four points.

Consider now the following equations, using all combinations of signs:

$$(2.4) \qquad (01)^2(23)(34)(42) \pm (02)^2(34)(41)(13) \\ \pm (03)^2(41)(12)(24) \pm (04)^2(12)(23)(31) = 0.$$

The eight equations are inversive and represent spheres or inversions. To study these equations, it will be convenient to put one of the canonizant points at infinity, as was done before. Then the equations (2.4) become

$$(01)^2 \pm (02)^2 \pm (03)^2 \pm (04)^2 = 0,$$

and the points $1, 2, 3, 4$ may be given coordinates (a, b, c), $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$, respectively. To determine the nature of these spheres or inversions, we will find their equations in the Cartesian coordinate system determined by the naming of the four points.

$$(01)^2 + (02)^2 + (03)^2 + (04)^2 = 0$$

becomes

$$x^2 + y^2 + z^2 = -(a^2 + b^2 + c^2).$$

This represents an inversion with $(0, 0, 0)$ as the center. The sphere of fixed points does not exist. Under this inversion

$$\begin{array}{lcl}
(a, b, c) & \rightarrow & (-a, -b, -c), \\
(a, -b, -c) & \rightarrow & (-a, b, c), \\
(-a, b, -c) & \rightarrow & (a, -b, c), \\
(-a, -b, c) & \rightarrow & (a, b, -c).
\end{array}$$

The right-hand column gives four points known as the counterpoints of the first set. The equation

$$(01)^2 + (02)^2 + (03)^2 - (04)^2 = 0$$

becomes in the Cartesian reference frame

$$(x-a)^2+(y-b)^2+(z+c)^2=0.$$

It evidently represents the point $(a, b, -c)$. Likewise

$$(01)^2+(02)^2-(03)^2+(04)^2=0$$

represents the point $(a, -b, c)$,

$$(01)^2-(02)^2+(03)^2+(04)^2=0$$

represents the point $(-a, b, c)$,

and
$$(01)^2-(02)^2-(03)^2-(04)^2=0$$

represents the point $(-a, -b, -c)$. Thus for an odd number of minus signs in the equations (2. 4) the counterpoints of the four given points appear. The relation of the four points to the four counterpoints is mutual. The sphere
$$(01)^2+(02)^2-(03)^2-(04)^2=0$$

represents the coordinate plane $x=0$, in the Cartesian system chosen. Likewise
$$(01)^2-(02)^2+(03)^2-(04)^2=0$$

represents the coordinate plane $y=0$, and

$$(01)^2-(02)^2-(03)^2+(04)^2=0$$

represents the coordinate plane $z=0$. These three spheres (or planes) are mutually perpendicular and intersect on the canonizant points, which in this case are $(0, 0, 0)$ and ∞. The sphere containing the four points $1, 2, 3, 4$ and the planes $x=0$ and $y=0$ intersect in the points $(0, 0, \pm 1)$ if the sphere is regarded as being of unit radius. Likewise the sphere and the planes $x=0$ and $z=0$ intersect at $(0, \pm 1, 0)$, and the sphere and the planes $y=0$ and $z=0$ intersect at $(\pm 1, 0, 0)$. These six points are called the Jacobian points of $1, 2, 3, 4$ and the three spheres (or planes) $x=0$, $y=0$, $z=0$ are called the Jacobian spheres.

The inversive theory of four points gives, in short, six spheres which intersect in two points called the canonizant pair. It also gives the four counterpoints and an inversion connecting them with the original four points. Likewise the three Jacobian spheres and the six Jacobian points appear.

Since four points in three-dimensional space are always on a sphere, we may consider the special case when the sphere is a plane. The counterpoints will also be in the plane and the inversion sending them into the original four points will be determined. Since the three Jacobian spheres intersect the sphere containing the four points, they will intersect the plane in three circles called the Jacobian circles. These circles will be perpendicular to each other and intersect in the six Jacobian points.

Let us now attempt to extend this theory to five points in a four-dimensional space. We will first consider the equations

$$\begin{aligned}\frac{(01)^3}{(12)(13)(14)(15)} &= \frac{(02)^3}{(12)(23)(24)(25)} = \frac{(03)^3}{(13)(23)(34)(35)} \\ &= \frac{(04)^3}{(14)(24)(34)(45)} = \frac{(05)^3}{(15)(25)(35)(45)}\end{aligned} \tag{2.5}$$

These equations are inversive and represent sixteen hyperspheres. However, if we try to prove that the spheres intersect as in the case of four points, we find that they may not. Hence, if they do not intersect, we do not have a canonizant pair of points as we had in the other cases. We have, however, a canonizant circle.

Let us now consider the equations

$$\begin{aligned}\frac{(01)^3}{(12)(13)(14)(15)} &\pm \frac{(02)^3}{(12)(23)(24)(25)} \pm \frac{(03)^3}{(13)(23)(34)(35)} \\ &\pm \frac{(04)^3}{(14)(24)(34)(45)} \pm \frac{(05)^3}{(15)(25)(35)(45)} = 0.\end{aligned} \tag{2.6}$$

We have here sixteen equations representing hyperspheres or inversions. Let us suppose the five points so chosen that the equations (2. 6) become

$$(01)^3 \pm (02)^3 \pm (03)^3 \pm (04)^3 \pm (05)^3 = 0.$$

The conditions imposed on the five points in that case are

$$\begin{aligned}(12)(13)(14)(15) &= (12)(23)(24)(25) = (13)(23)(34)(35) \\ &= (14)(24)(34)(45) = (15)(25)(35)(45).\end{aligned}$$

They do not seem to place the points in any symmetrical arrangement.

If we calculate the "power" or bilinear invariant of each hypersphere of the set

$$(01)^3 \pm (02)^3 \pm (03)^3 \pm (04)^3 \pm (05)^3 = 0$$

with itself, we find there are six inversions and ten real hyperspheres. The six inversions are given by the one equation with all plus signs and the five equations with one minus sign. The ten hyperspheres are given by the ten equations with two minus signs. The geometric connection between these hyperspheres and inversions and the five points is not evident, and in this paper we will leave the theory here.

3. *The Cayley Determinants.* Cayley[*] has shown that for four points to be on a circle, the following determinant is zero,

$$C_4 = \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & 0 & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & 0 & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 0 \end{vmatrix} = 0$$

where λ_{ij} is the square of the distance between the points i and j and $\lambda_{ij} = \lambda_{ji}$. Five points on a sphere satisfy a similar relation, $C_5 = 0$, where C_5 is a five-rowed determinant similar to C_4.

In the same paper, Cayley gave the condition for three points to be on a line, namely

$$D_3 = \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & 1 \\ \lambda_{21} & 0 & \lambda_{23} & 1 \\ \lambda_{31} & \lambda_{32} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

Four points in a plane satisfy the relation $D_4 = 0$ where D_4 is the determinant C_4 with a bordering row and column of units. Similarly five points in three-dimensional space satisfy $D_5 = 0$ where D_5 is the determinant C_5 with a bordering row and column of units.

The relations $C_4 = 0$ and $C_5 = 0$ are inversive, for if we replace λ_{ij} by $\kappa^2 \lambda'_{ij}/\lambda'_{ci}\lambda'_{cj}$, where c is the center of the inversion and not one of the points, $C_4 = 0$ is reproduced in the primed letters, and likewise $C_5 = 0$ becomes $C_5' = 0$. If the center of the inversion, c, is one of the given points, then $C_4 = 0$ becomes $D_3 = 0$, the fourth point having gone into the point at infinity. Likewise if we invert with the point 5 as a center, $C_5 = 0$ becomes $D_4 = 0$. The relations $D_3 = 0$, $D_4 = 0$, $D_5 = 0$ are not inversive. If we invert with a point c, not in the line of 1, 2, 3 as a center, the point c goes to infinity, the point at infinity becomes a point 4, and $D_3 = 0$ becomes $C_4 = 0$. Likewise we may derive the condition that five points be on a sphere, $C_5 = 0$, by an inversion applied to $D_4 = 0$, where the center of the inversion must not be in the plane of the four points satisfying $D_4 = 0$.

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[*] Cayley, *Collected Mathematical Papers*, Vol. I, p. 1; also Salmon, *Conic Sections*, 6th ed. (1879), p. 134.

It is evident that determinants such as these may be set down for any number of points. $D_n = 0$ is the relation connecting any n points in S_{n-2}. $C_n = 0$ is the condition that n points be on a "round" or hypersphere in S_{n-2}. The relations $C_n = 0$ are all inversive.

The question arises as to what geometric interpretation we may give these determinants when they are not zero. The following two theorems will answer the question for three points and then extensions will be made to any number of points.

THEOREM 1. *If* Δ *be the area of the triangle formed by any three points, then*

$$D_3 = -16\Delta^2.$$

Proof: Three points on a line satisfy the relation $D_3 = 0$. If the point 3 is at a distance p from the line joining 1 and 2, then

$$\begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} - p^2 & 1 \\ \lambda_{21} & 0 & \lambda_{23} - p^2 & 1 \\ \lambda_{31} - p^2 & \lambda_{32} - p^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

On splitting this determinant up into sums, we find that $D_3 + 2p^2D_2 = 0$. But $D_2 = 2\lambda_{12}$ and since $p^2\lambda_{12}$ is four times the square of the area of the triangle 1, 2, 3, then $D_3 = -16\Delta^2$.

THEOREM 2. *If* r *be the radius of the circumcircle of any three points not in a straight line, then* $C_3 = -2r^2D_3$.

Proof: For three points not in a straight line, we have

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) \\ \cos(21) & 1 & \cos(23) \\ \cos(31) & \cos(32) & 1 \end{vmatrix} = 0$$

where $\cos(ij)$ is the cosine of the angle subtended at the circumcenter by the arc of the circumcircle joining the points i and j and $\cos(ij) = \cos(ji)$. If we replace $\cos(ij)$ by $1 - \lambda_{ij}/2r^2$, where r is the radius of the circumcircle and λ_{ij} is the square of the distance between the points i and j, the determinant becomes

$$\begin{vmatrix} 1 & 1 - \lambda_{12}/2r^2 & 1 - \lambda_{13}/2r^2 \\ 1 - \lambda_{21}/2r^2 & 1 & 1 - \lambda_{23}/2r^2 \\ 1 - \lambda_{31}/2r^2 & 1 - \lambda_{32}/2r^2 & 1 \end{vmatrix} = 0.$$

After splitting this determinant up into a sum of determinants, we find $C_3 + 2r^2D_3 = 0$. Since from theorem 1, $D_3 = -16\Delta^2$, we have $C_3 = 32r^2\Delta^2$.

The following are the corresponding theorems for four points not in a plane, with a sketch of the proofs:

THEOREM 1′: *If V be the volume of the tetrahedron formed by the four points* 1, 2, 3, 4, then $D_4 = 288V^2$.*

Proof: If p be the distance from the plane of the points 1, 2, 3 to the point 4, we may replace in the relation $D_4 = 0$, λ_{i4} by $\lambda_{i4} - p^2$ $(i = 1, 2, 3)$. Splitting the resulting determinant up into sums, we find $D_4 + 2p^2D_3 = 0$.

Since
$$D_3 = -16\Delta^2, \quad D_4 = 32p^2\Delta^2.$$

But since $p\Delta = 3V$, we have $D_4 = 288V^2$.

THEOREM 2′: *If r be the radius of the sphere on the four points, then*

$$C_4 = -2r^2D_4.$$

Proof: Four points in a three-dimensional space always satisfy the relation †

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) & \cos(14) \\ \cos(21) & 1 & \cos(23) & \cos(24) \\ \cos(31) & \cos(32) & 1 & \cos(34) \\ \cos(41) & \cos(42) & \cos(43) & 1 \end{vmatrix} = 0.$$

where (ij) is the arc of the great circle joining the points i and j on the sphere on the four points. Replacing $\cos(ij)$ in this determinant by $1 - \lambda_{ij}/2r^2$ and splitting up the resulting determinant into sums, we find

$$C_4 + 2r^2D_4 = 0.$$

Since $D_4 = 288V^2$, we have

$$C_4 = -576r^2V^2.$$

These theorems can be extended to any number of points. Thus we have

THEOREM I. *If V be the content of the solid formed by n points in S_{n-1}, then $D_n = kV^2$, where $k = (-1)^n \cdot (2)^{n-1}\,[(n-1)\,!]^2$.*

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* Salmon, *Analytic Geometry of Three Dimensions*, 3d ed. (1874), p. 34.

† Salmon, *loc. cit.*, p. 35.

THEOREM II. *If r be the radius of the hypersphere in S_{n-1} on which the n points lie, then $C_n = -2r^2D_n$, or $C_n = 2kr^2V^2$, where k is defined as above.*

4. *The Doubly Bordered Cayley Determinant.* Four points in a plane satisfy the relation $D_4 = 0$. If about the point 4, a circle of radius ρ is drawn, then

$$\lambda_{14} = (\delta_1 + \rho)^2, \quad \lambda_{24} = (\delta_2 + \rho)^2, \quad \lambda_{34} = (\delta_3 + \rho)^2,$$

where $\delta_1, \delta_2, \delta_3$ represent perpendicular distances from the points $1, 2, 3$, respectively, to the circumference of the circle. The relation, a quadratic in ρ, is now written

$$\begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & (\delta_1+\rho)^2 & 1 \\ \lambda_{21} & 0 & \lambda_{23} & (\delta_2+\rho)^2 & 1 \\ \lambda_{31} & \lambda_{32} & 0 & (\delta_3+\rho)^2 & 1 \\ (\delta_1+\rho)^2 & (\delta_2+\rho)^2 & (\delta_3+\rho)^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

If now the point 4 moves off to infinity, ρ becomes infinite, the circle becomes a straight line, and the coefficient of ρ^2 in the above relation is zero. Therefore

$$2\begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & \delta_1 & 1 \\ \lambda_{21} & 0 & \lambda_{23} & \delta_2 & 1 \\ \lambda_{31} & \lambda_{32} & 0 & \delta_3 & 1 \\ \delta_1 & \delta_2 & \delta_3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & 1 \\ \lambda_{21} & 0 & \lambda_{23} & 1 \\ \lambda_{31} & \lambda_{32} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

The determinant on the left is Cayley's C_3 doubly bordered, first with δ's, then with units. The determinant on the right is D_3 which is equal to $-16\Delta^2$ where Δ is the area of the triangle formed by the three points $1, 2, 3$. Let $\lambda_{23} = a^2$, $\lambda_{13} = b^2$, $\lambda_{12} = c^2$ and let A, B, C be the interior angles of the triangle $1, 2, 3$ at the points $1, 2, 3$ respectively. Then if the left-hand determinant is expanded, we have

$$(4.1) \qquad a^2\delta_1^2 + b^2\delta_2^2 + c^2\delta_3^2 - 2ab\cos C\,\delta_1\delta_2 - 2ac\cos B\,\delta_1\delta_3 - 2bc\cos A\,\delta_2\delta_3 = 4\Delta^2.$$

The $\delta_1, \delta_2, \delta_3$ now represent the perpendicular distances from the points $1, 2, 3$ to a straight line. For a given δ_2 and δ_3 there are ordinarily two straight lines satisfying the equation. They are common tangents to the es with centers at 2 and 3 and radii δ_2 and δ_3 respectively. They on external pairs according as 2 and 3 are on the opposit side of the lines. The internal tangents coincide when $\delta_2 - -\delta$ Likewise the external tangents coincide when $\delta_2 - \delta_3 = \pm a$. ally we may therefore say that the discriminant of equation (4.1) to δ, when set equal to zero, must reduce to $\delta_2 - \delta$ minant set equal to zero gives

$$b^2 \sin^2 C\,\delta_2^2 \quad c^2 \sin^2 B\,\delta_3^2 - 2bc\,(\cos A + \cos B \cos C)\,\delta_2\delta_3 = 4\Delta.$$

equation should reduce to $(\delta_2 - \delta_3)^2 = a^2$. Therefore the following must be satisfied.

$$b^2 \sin^2 C = bc\,(\cos A + \cos B \cos C),$$
$$b/c = \sin B/\sin C, \quad ab \sin C = 2\Delta.$$

The first relation is a form of the law of cosines for a triangle. the law of sines. The third gives an expression for the area. relation gives

$$b^2 \sin^2 C = bc \cos A + bc \cos (B + C) + bc \sin B \sin C,$$

upon using the second relation becomes $\cos A = -\cos (B + C)$, whence $A + B + C = \pi$. Here we have sufficient information about to determine any of its properties. Can we do the same for in S_3?

To find the properties of the tetrahedron, or four points in S_3, start points satisfying the relation $D_5 = 0$. About the point 5 draw a sphere ρ, and let $\delta_1, \delta_2, \delta_3, \delta_4$ be perpendicular distances from the poi to the sphere. Then in the relation $D_5 = 0$, set $\lambda_{i5} = (\delta$ ρ 1, 2, 3, 4. If now the point 5 moves off to infinity in a given tion, ρ becomes infinite, the sphere becomes a plane, and the co the determinant $D_5 = 0$ is zero. Therefore

$$\begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} & \delta_1 & 1 \\ \lambda_{21} & 0 & \lambda_{23} & \lambda_{24} & \delta_2 & 1 \\ \lambda_{31} & \lambda_{32} & 0 & \lambda_{34} & \delta_3 & 1 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 0 & \delta_4 & 1 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{vmatrix} = 144V^2,$$

where V is the volume of the tetrahedron 1, 2, 3, 4. The above determinant is C_4, doubly bordered, first with δ's, then with units. Remembering that $D_3 = -16\Delta^2$, we have on expanding the above determinant

$$
\begin{aligned}
(4.2)\qquad & \Delta_1^2\delta_1^2 + \Delta_2^2\delta_2^2 + \Delta_3^2\delta_3^2 + \Delta_4^2\delta_4^2 \\
& - 2\Delta_1\Delta_2\delta_1\delta_2 \cos(12) - 2\Delta_1\Delta_3\delta_1\delta_3 \cos(13) \\
& - 2\Delta_1\Delta_4\delta_1\delta_4 \cos(14) - 2\Delta_2\Delta_3\delta_2\delta_3 \cos(23) \\
& - 2\Delta_2\Delta_4\delta_2\delta_4 \cos(24) - 2\Delta_3\Delta_4\delta_3\delta_4 \cos(34) = 9V^2,
\end{aligned}
$$

where $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ are the areas of the faces opposite the points 1, 2, 3, 4 respectively, and cos (12) is the cosine of the angle between the faces Δ_1 and Δ_2.

NOTE: In the expansion of the above determinant, we used the fact that

$$
\begin{vmatrix}
\lambda_{12} & \lambda_{23} & \lambda_{24} & 1 \\
\lambda_{13} & 0 & \lambda_{34} & 1 \\
\lambda_{14} & \lambda_{43} & 0 & 1 \\
1 & 1 & 1 & 0
\end{vmatrix} = -16\Delta_1\Delta_2 \cos(12).
$$

To prove this we choose a rectangular reference frame in the plane of the points 2, 3, 4, giving the points coordinates (x_2, y_2), (x_3, y_3), (x_4, y_4) respectively. The projection of the point 1 on this plane is given coordinates (x_1', y_1'). Then

$$
\begin{vmatrix}
x_2^2 + y_2^2 & -2x_2 & -2y_2 & 1 \\
x_3^2 + y_3^2 & -2x_3 & -2y_3 & 1 \\
x_4^2 + y_4^2 & -2x_4 & -2y_4 & 1 \\
1 & 0 & 0 & 0
\end{vmatrix}
\begin{vmatrix}
1 & x_1' & y_1' & x_1'^2 + y_1'^2 \\
1 & x_3 & y_3 & x_3^2 + y_3^2 \\
1 & x_4 & y_4 & x_4^2 + y_4^2 \\
0 & 0 & 0 & 1
\end{vmatrix} = -16\,\Delta_1\Delta_2 \cos(12).
$$

$$
\text{Whence}\quad
\begin{vmatrix}
\lambda_{1'2} & \lambda_{23} & \lambda_{24} & 1 \\
\lambda_{1'3} & 0 & \lambda_{34} & 1 \\
\lambda_{1'4} & \lambda_{43} & 0 & 1 \\
1 & 1 & 1 & 0
\end{vmatrix} = -16\Delta_1\Delta_2 \cos(12).
$$

Replacing in this determinant $\lambda_{1'i}$ by $\lambda_{1i} - p^2$, $(i = 2, 3, 4)$, where p is the distance from the point 1 to the plane of 2, 3, 4, and then splitting the resulting determinant into sums of determinants, we have the result.

For given $\delta_2, \delta_3, \delta_4$ there are two values of δ_1, or two planes, satisfying the equation (4.2). When will these planes coincide? Take three balls, no two of the same size, and place them on a table. Their centers will represent the points 2, 3, 4 and their radii $\delta_2, \delta_3, \delta_4$. The plane of the table will be one tangent plane and there will be also an upper external tangent plane, for which a piece of cardboard may be used. If we now bring the three points of contact of the spheres with the table into a line, the upper tangent plane will swing around and finally coincide with the table. Hence the three points

2, 3, 4 will be in a plane through the line of contact and perpendicular to the table. Then

$$(4.3)\qquad a^2\delta_2^2+b^2\delta_3^2+c^2\delta_4^2-2ab\cos C\,\delta_2\delta_3 \\ -2ac\cos B\,\delta_2\delta_4-2bc\cos A\,\delta_3\delta_4=4\Delta_1^2,$$

where a, b, c are the sides of the face Δ_1 opposite the points 2, 3, 4 respectively, and A, B, C are the interior angles at the points 2, 3, 4.

The same relation is true when the internal tangent planes coincide. Hence the discriminant of equation (4. 2) with respect to δ_1 must reduce to equation (4. 3) when the tangent planes coincide. The discriminant when set equal to zero, gives

$$\begin{aligned}&\Delta_2^2\sin^2(12)\,\delta_2^2+\Delta_3^2\sin^2(13)\,\delta_3^2+\Delta_4^2\sin^2(14)\,\delta_4^2\\ &-2\Delta_2\Delta_3\delta_2\delta_3\,[\cos(23)+\cos(12)\cos(13)]\\ &-2\Delta_2\Delta_4\delta_2\delta_4\,[\cos(24)+\cos(12)\cos(14)]\\ &-2\Delta_3\Delta_4\delta_3\delta_4\,[\cos(34)+\cos(13)\cos(14)]=9V^2.\end{aligned}$$

This equation must reduce to equation (4. 3). Therefore the following relations must hold true:

$$\begin{gathered}\Delta_2\sin(12)/a=\Delta_3\sin(13)/b=\Delta_4\sin(14)/c,\\ \sin(12)\sin(13)\cos C=\cos(23)+\cos(12)\cos(13),\\ \sin(12)\sin(14)\cos B=\cos(24)+\cos(12)\cos(14),\\ \sin(13)\sin(14)\cos A=\cos(34)+\cos(13)\cos(14),\\ 2\Delta_1\Delta_2\sin(12)=3aV.^*\end{gathered}$$

Here we have information about the tetrahedron. The first set of equations gives the law of sines for the dihedral angles, the second set gives the law of cosines. The last equation is a formula for the volume.

This theory can be extended to any number of points. The doubly-bordered Cayley determinant for n points can always be set down. On expanding the determinant, we will always have a quadratic expression in the δ's, such that the discriminant with respect to δ_1, when set equal to zero, must reduce to the relation on the δ's for $(n-1)$ points. The algebraic conditions imposed by this fact give information about the n point.

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[*] Casey, *Spherical Trigonometry*, p. 133.

Independent Event Histories.*

By ARTHUR H. COPELAND.

It has been proved that there exist N *independent numbers* † associated with a given probability p, N being arbitrarily large. The condition that a set of numbers be independent is expressed by a certain set of equations which must be satisfied by these numbers. Thus in a sense, independence is the exception and not the rule. However, another point of view is possible with respect to this question. It was proved that there exists a set of *admissible numbers* having the power of the continuum. We are led to inquire whether a set of independent numbers can have the power of the continuum, or whether such a set is necessarily finite or denumerable. We shall see that it is possible for a set of independent numbers to have the power of the continuum, and hence from this point of view we cannot say that independence is exceptional.‡

It is important to notice that a set of numbers can be independent when these numbers are not admissible. As admissible numbers are the ones which are chiefly of interest in the theory of probability, we shall restrict ourselves in this paper to the discussion of numbers which are both independent and admissible. We shall introduce the following definition:

DEFINITION: A set of numbers, $x_1, x_2, x_3, \cdots x_N$, is an admissibly independent set if and only if there exists a corresponding set of numbers, $p_1, p_2, p_3, \cdots p_N$, such that every sub-set of numbers, $x_{\nu_1}, x_{\nu_2}, x_{\nu_3}, \cdots x_{\nu_n}$, of the set $x_1, x_2, x_3, \cdots x_N$, satisfies the condition that $x_{\nu_1} \cdot x_{\nu_2} \cdot x_{\nu_3} \cdots x_{\nu_n}$ belongs to to $A(p_{\nu_1} p_{\nu_2} p_{\nu_3} \cdots p_{\nu_n})$.

Let us observe that it is possible for two numbers, x and y, to be independent and admissible when the product, $x \cdot y$ is not admissible. That is, x and y can be admissible and independent without being admissibly independent. For example, we can take the numbers, x and $(2/1)x$, where x belongs to $A(p)$. Then $(2/1)x$ belongs to $A(p)$.§ We have the equations

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* This paper was presented to the Society September 7, 1928. It is based on a paper by the author entitled "Admissible Numbers in the Theory of Probability," *American Journal of Mathematics*, Vol. 50 (1928), pp. 535-552.

† The reader is referred, for notation and for the definition of independent and admissible numbers, to the memoir just cited, §§ I, III.

‡ See Theorems 2 and 3 below, also see definition of the independence of infinite sets, p. 613.

§ See the memoir cited, Theorem 13.

$$p\,[(1/3)\{x\cdot(2/1)x\}] = p\,[(1/3)x\cdot(2/3)x] = p^2 \text{ and}$$
$$p\,[(2/3)\{x\cdot(2/1)x\}] = p\,[(2/3)x\cdot(3/3)x] = p^2, \text{ but}$$
$$p\,[(1/3)\{x\cdot(2/1)x\}\cdot(2/3)\{x\cdot(2/1)x\}] =$$
$$p\,[(1/3)x\cdot(2/3)x\cdot(3/3)x] = p^3 \neq p^4.$$

Hence $x\cdot(2/1)x$ is not admissible. To establish the independence of x and $(2/1)x$ we have to prove that $p\,[x\cdot(2/1)x] = p^2$. We see that $p\,[(r/n)\{x\cdot(2/1)x\}] = p\,[(r/n)x\cdot(r+1/n)x] = p^2$, if $r = 1, 2, 3 \cdots, n-1$. Hence the independence of these numbers is a simple consequence of the following theorem.

THEOREM 1.[*] *If x is such that $p\{(r/n)x\} = p$ for every n and every r such that $0 < r \leqq n - q$, where q is a fixed integer, then $p(x) = p$.*

By hypothesis, if we have given a positive number, ϵ, then there exists a number, μ_0 such that

$$(a)\quad p - \epsilon/3 < p_\mu\{(r/n)x\} < p + \epsilon/3 \text{ if } \mu \geqq \mu_0 \text{ and } 0 < r \leqq n - q.$$

Let $\nu = \mu n + m$ where $0 \leqq m < n$, then

$$p_\nu(x) = \sum_{r=1}^{n-q} \mu\, p_\mu\{(r/n)x\}/\nu + \sum_{r=n-q+1}^{n} \mu\, p_\mu\{(r/n)x\}/\nu + \sum_{k=n\mu+1}^{n\mu+m} x^k/\nu.$$

But $0 \leqq p_\mu\{(r/n)x\} \leqq 1$ if $n - q < r \leqq n$, and $0 \leqq \sum_{k=n\mu+1}^{n\mu+m} x^k/\nu \leqq m/\nu$. Hence

$$(p - \epsilon/3)(n - q)\mu/\nu \leqq p_\nu(x) \leqq (p + \epsilon/3)(n - q)\mu/\nu + q\mu/\nu + m/\nu.$$

Since $(1 - q/n)\mu/(\mu + 1) \leqq (n - q)\mu/\nu \leqq 1$, we have the inequalities

$$p\mu/(\mu + 1) - \epsilon/3 - q/n \leqq p_\nu(x) \leqq p + \epsilon/3 + (q\mu + m)/(n\mu + m).$$

These inequalities can be replaced by the stronger inequalities

$$p - (\epsilon/3 + q/n + 1/\mu) \leqq p_\nu(x) \leqq p + (\epsilon/3 + q/n + 1/\mu).$$

Let us first choose n so that $q/n < \epsilon/3$, and next choose μ_0 so that condition (a) is satisfied and so that $1/\mu_0 < \epsilon/3$. Then $|\,p_\nu(x) - p\,| < \epsilon$ if $\nu \geqq n\mu_0$. Therefore $p(x) = p$.

We shall now consider infinite sets of admissible independent numbers, such sets being defined as follows:

DEFINITION. An infinite set, E, is an admissibly independent set if and only if every finite sub-set of E is admissibly independent.

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[*] Special cases of this theorem were used in proving Theorems 11 and 13 in the memoir cited.

THEOREM 2. *There exists an admissibly independent set, E, having the power of the continuum.*

In order to prove this theorem we shall make use of the following lemma:

LEMMA. For every number, a, such that $0 < a < 1$, there exists a sequence of integers, $q_s(a)$ $(s = 1, 2, 3, \cdots)$ such that $q_s(a) < s$ and such that if $a_1 \neq a_2$, then the equation, $q_s(a_1) = q_s(a_2)$, has only a finite number of solutions.[*]

Let $q_s(a)$ be such that $(q_s(a) + 1)/s > a \geqq q_s(a)/s$. Then $| q_s(a)/s - a | < 1/s$, and therefore $\lim_{s \to \infty} q_s(a)/s = a$. Let a_1 and a_2 be two numbers such that $0 < a_1, a_2 < 1$ and let us assume that $a_1 < a_2$. Let ϵ be such that $a_2 - \epsilon > a_1$. Then we can choose s_0 such that $a_2 - q_s(a_2)/s < \epsilon$ if $s \geqq s_0$. Then $q_s(a_2)/s > a_2 - \epsilon > a_1 > q_s(a_1)/s$ if $s \geqq s_0$, and hence the equation $q_s(a_1) = q_s(a_2)$, has at most s_0 solutions. This completes the proof of the lemma.

Corresponding to any number, a, such that $0 < a < 1$, we can define a number, $x(a)$ such that the digits $\nu_s + 1$ to ν_{s+1} of $x(a)$ are the same as the digits 1 to N_s of $(q_s(a)/s)y$, where $\nu_s = N_1 + N_2 + N_3 \cdots + N_{s-1}$ and where y belongs to $A(p)$. We wish to show that the integers, $N_1, N_2, N_3, \cdots N_s, \cdots$, can be so selected that the set of numbers, $x(a)$ is admissibly independent. We have to prove that if $a_1, a_2, a_3, \cdots a_\alpha$ is any set of distinct numbers such that $0 < a_1, a_2, a_3, \cdots a_\alpha < 1$, then $x(a_1) \cdot x(a_2) \cdots x(a_\alpha)$ belongs to $A(p^\alpha)$. Let

$$Y_{n,s} = (r_1/n)\{(\rho_1/s)y \cdot (\rho_2/s)y \cdots (\rho_\beta/s)y\} \cdot (r_2/n)\{(\rho_1/s)y \cdots (\rho_\beta/s)y\} \cdots (r_k/n)\{(\rho_1/s)y \cdots (\rho_\beta/s)y\}$$

where $n \leqq s$ and $r_1, r_2, r_3, \cdots r_k$ is any set of distinct positive integers such that $r_i \leqq n$, and where $\rho_1, \rho_2, \rho_3 \cdots \rho_\beta$ is any set of positive integers such that $\rho_j \leqq s$. Then if $\rho_1, \rho_2 \cdots \rho_\beta$ are distinct, $p(Y_{n,s}) = p^{k\beta}$.

Let $\epsilon_1 > \epsilon_2 > \cdots \epsilon_s > \cdots$ be such that $\lim_{s \to \infty} \epsilon_s = 0$. We can define two sets of integers, $M_1, M_2, M_3, \cdots M_s, \cdots$, and $N_1, N_2, N_3, \cdots N_s, \cdots$, such that

(a) $\quad | p_N(Y_{n,s+1}) - p^{k\beta} | < \epsilon_s/3$ if $N \geqq M_s/n$

(b) $\quad | p_N(Y_{n,s}) - p^{k\beta} | + (\nu_s + M_s)/N_s < \epsilon_s/3$ if $N \geqq N_s/n$

where $n \leqq s$ and where $\rho_1, \rho_2 \cdots \rho_\beta$ are distinct and $\rho_j \leqq s$. The numbers,

---

[*] It is interesting to notice that if the condition $q_s(a) < s$ were replaced by the condition $q_s(a) < N$, N being fixed, then the lemma would no longer be true.

N_s and M_s, can be chosen so that conditions (a) and (b) hold for all possible numbers, $Y_{n,s}$ and at the same time, the numbers, N_s, can be chosen so that ν_s/n is an integer if $n \leqq s$. Then N_s/n is an integer. Let

$$X_n = (r_1/n)\{x(a_1) \cdot x(a_2) \cdots x(a_\alpha)\} \cdot (r_2/n)\{x(a_1) \cdots x(a_\alpha)\} \cdots (r_k/n)\{x(a_1) \cdots x(a_\alpha)\}.$$

Then digits $\nu_s/n + 1$ to ν_{s+1}/n of X_n are the same as digits 1 to N_s/n of $Y_{n,s}$, if $\rho_j = q_s(a_j)$ and $\beta = \alpha$. In order that conditions (a) and (b) can be made to hold for this choice of the number, $Y_{n,s}$, the integers, $q_s(a_1)$, $q_s(a_2)$, $q_s(a_3), \cdots q_s(a_\alpha)$, must be distinct. However, it follows from the lemma that a number, s_0, can be chosen so that these integers are distinct if $s \geqq s_0$.

In the work which follows it will be assumed that $s \geqq s_0$ and $s \geqq n$. Then

$$\text{(c)} \quad | p_N(X_n) - \frac{N_s}{nN} p_{(N_s/n)}(Y_{n,s}) | \leqq (nN - N_s)/nN < (\nu_s + M_s)/N_s < \epsilon_s$$

if $\nu_{s+1}/n \leqq N \leqq (\nu_{s+1} + M_s)/n.$

Combining (b) and (c) we get

$$\text{(d)} \qquad | p_N(X_n) - \frac{N_s}{nN} p^{k\alpha} | < (1 + N_s/nN)\epsilon_s/3 < 2\epsilon_s/3.$$

Combining (d) with the inequality

$$\text{(e)} \qquad | p^{k\alpha} \cdot N_s/nN - p^{k\alpha} | < 1 - N_s/nN < \epsilon_s/3$$

we get

$$\text{(f)} \qquad | p_N(X_n) - p^{k\alpha} | < \epsilon_s \text{ if } \nu_{s+1}/n \leqq N \leqq (\nu_{s+1} + M_s)/n.$$

If $(\nu_{s+1} + M_s)/n \leqq N \leqq \nu_{s+2}/n$, then

$$\text{(g)} \qquad | p_N(X_n) - \frac{N_s}{nN} p_{(N_s/n)}(Y_{n,s}) - \frac{N - \nu_{s+1}/n}{N} p_{(N-\nu_{s+1}/n)}(Y_{n,s+1}) | \leqq (\nu_{s+1} - N_s)/nN < \epsilon_s/3.$$

Combining (a), (b), and (g) we get

$$\text{(h)} \quad | p_N(X_n) - \frac{nN - \nu_s}{nN} p^{k\alpha} | < (1 + \frac{nN - \nu_s}{nN}) \epsilon_s/3 < 2\epsilon_s/3.$$

Combining (h) with the inequality

$$\text{(i)} \qquad | p^{k\alpha} - \frac{nN - \nu_s}{Nn} p^{k\alpha} | < \epsilon_s/3$$

we get

(j) $$| p_N(X_n) - p^{ka} | < \epsilon_s \text{ if } (\nu_{s+1} + M_s)/n \leqq N \leqq (\nu_{s+2})/n.$$

From (f) and (j) it follows that $P(X_n) = p^{ka}$ and since this equation holds for all possible numbers of the form, X_n, it follows that $x(a_1) \cdot x(a_2) \cdot x(a_3) \cdots x(a_a)$ belongs to $A(p^a)$. Hence the numbers, $x(a)$ (where $0<a<1$) constitute an admissibly independent set having the power of the continuum.

THEOREM 3. *There exists an admissibly independent set, I, such that every set, $I \cdot A(p)$, where $0 < p < 1$, has the power of the continuum.*

The set, I, will consist of the numbers $x(a, p)$, where $0 < a < 1$ and $0 < p < 1$, and where $x(a, p)$ belongs to the set $A(p)$. We have to show that if $a_1, a_2, a_3, \cdots, a_\gamma, p_1, p_2, \cdots, p_\gamma$ is any set of numbers such that $0 < a_1, a_2, \cdots, a_\gamma, p_1, \cdots, p_\gamma < 1$ and such that the equations, $a_i = a_j$ and $p_i = p_j$ are not both satisfied for any two distinct indices i and j, then $x(a_1, p_1) \cdot x(a_2, p_2) \cdots x(a_\gamma, p_\gamma)$ belongs to $A(p_1 \cdot p_2 \cdots p_\gamma)$.

We shall make use of an auxiliary set of numbers, $y^s{}_{a,\beta}$, where $y^s{}_{a,\beta}$ belongs to $A(\beta/2^s)$ and where $\alpha = 1, 2, 3, \cdots, s$ and $\beta = 1, 2, 3, \cdots, (2^s - 1)$. We shall define these numbers by means of the following equations:

$$y^s{}_{a,\beta} = \sum_{i=1}^{\beta} \{ \prod_{j=1}^{r_i} (s(\alpha - 1) + s^2(\beta - 1) + r_{ij}/2^s s^2) y.$$

$$\prod_{j=r_i+1}^{s} [1 - (s(\alpha - 1) + s^2(\beta - 1) + r_{ij}/2^s s^2) y] \},$$

where y belongs to $A(\frac{1}{2})$, $0 < r_{ij} \leqq s$, $r_{ij} \neq r_{ij'}$ if $j \neq j'$, and where the terms of the summation represent mutually exclusive event histories. It will be observed that $s(\alpha - 1) + s^2(\beta - 1) + r_{ij} < 2^s s^2$ and hence $y^s{}_{a,\beta}$ belongs to $A(\beta/2^s)$.* Next let us consider the product, $y^s{}_{a_1,\beta_1} \cdot y^s{}_{a_2,\beta_2} \cdot y^s{}_{a_3,\beta_3} \cdots y^s{}_{a_m,\beta_m}$ where $m \leqq s$ and where the equations $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$ are not both satisfied for any two distinct indices, i and j. This product can be expanded by means of the laws of ordinary algebra and we obtain an expression of the form

$$\sum_{i=1}^{\beta_1 \cdot \beta_2 \cdots \beta_m} \{ \prod_{j=1}^{R_i} (R_{i,j}/2^s s^2) y \cdot \prod_{j=r_i+1}^{sm} [1 - (R_{i,j}/2^s s^2) y] \},$$

where $0 < R_{i,j} < 2^s s^2$, $R_{i,j} \neq R_{i,j''}$ if $j \neq j'$ and where the terms of the summation represent mutually exclusive event histories. Hence $y^s{}_{a_1,\beta_1} \cdot y^s{}_{a_2,\beta_2} \cdots y^s{}_{a_m \beta_m}$ belongs to $A(\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})$. Let $Y_{n,s} = (r_1/n) \{ y^s{}_{a_1,\beta_1} \cdots y^s{}_{a_m,\beta_m} \} \cdot (r_2/n) \{ y^s{}_{a_1,\beta_1} \cdots \} \cdots (r_k/n) \{ y^s{}_{a_1,\beta_1} \cdots \}$ where $r_i \neq r_j$ if $i \neq j$ and where $0 < r_i \leqq n \leqq s$. Then $p(Y_{n,s}) = (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})^k$.

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* See theorem 16 of the memoir cited.

Let $b_1, b_2, b_3, \cdots, b_m$ be a set of numbers such that $0 < b_i < 1$, and let β_i be the integer such that $(\beta_i + 1)/2^s > b_i \geqq \beta_i/2^s$. (In particular we could set $m = \gamma$, and $b_i = p_i$.) Then

$$
\begin{aligned}
(b_1 \cdot b_2 \cdots b_m)^k - (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})^k \\
< ((\beta_1 + 1)(\beta_2 + 1) \cdots (\beta_m + 1)/2^{sm})^k \\
- (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})^k.
\end{aligned}
$$

If $0 < \beta_i \leqq 2^s - 1$, $0 < k \leqq s$, and $0 < m \leqq s$, then the maximum of $((\beta_1 + 1) \cdots (\beta_m + 1/2^{sm})^k - (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})^k$ occurs when $\beta_1 = \beta_2 = \cdots = \beta_m = 2^s - 1$ and $k = m = s$ (s being fixed). Therefore

$$(b_1 \cdot b_2 \cdots b_m)^k - \beta_1 \cdot \beta_2 \cdots \beta_m)^k < 1 - (1 - 2^{-s})^{s^2}.$$

Let $\epsilon_s = 6[1 - (1 - 2^{-s})^{s^2}]$ then $\lim\limits_{s \to \infty} \epsilon_s = 0.$

We can define two sets of integers, $M_1, M_2, \cdots, M_s, \cdots$, and $N_1, N_2, \cdots, N_s, \cdots$, such that

$$
\begin{aligned}
&\text{(a)} \quad | p_N(Y_{n,s+1}) - (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{(s+1)m})^k | < \epsilon_s/6 && \text{if } N \geqq M_s/n \\
&\text{(b)} \quad | p_N(Y_{n,s}) - (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})^k | + \nu_s + M_s/N_s < \epsilon_s/6 && \text{if } N \geqq N_s/n
\end{aligned}
$$

where $\nu_s = N_1 + N_2 + \cdots + N_{s-1}$. The numbers M_s and N_s are such that conditions (a) and (b) hold simultaneously for all numbers, $Y_{n,s}$, for which $0 < \beta_i \leqq 2^s - 1$, $0 < \alpha_i$, $m \leqq s$ and $0 < k \leqq n \leqq s$. At the same time the numbers, N_s, are chosen so that ν_s/n is an integer if $n \leqq s$.

We can now define the numbers, $x(a, p)$, as follows: *The digits* $\nu_s + 1$ *to* $\nu_s + 1$ *of* $x(a, p)$ *are the same as digits* 1 *to* N_s of $y^s{}_{\alpha^s, \beta^s}$, *where* $\alpha^s = q_s(a)$ *and* $\beta^s = q_{2^s}(p)$.

Let us consider the number, $Y_{n,s}$, where $m = \gamma$, $\alpha_i{}^s = q_s(a_i)$, and $\beta_i{}^s = q_{2^s}(p_i)$, and let s_0 be such that the equations, $\alpha_i{}^s = \alpha_j{}^s$ and $\beta_i{}^s = \beta_j{}^s$ are not both satisfied if $s \geqq s_0$, and $i \neq j$. Combining (a) and (b) with the inequality

$$| (\beta_1{}^s \cdot \beta_2{}^s \cdots \beta_m{}^s/2^{\gamma s})^k - (p_1 p_2 p_3 \cdots p_\gamma)^k | < \epsilon_s/6$$

we get

$$
\begin{aligned}
&\text{(c)} \quad | p_N(Y_{n,s+1}) - (p_1 p_2 p_3 \cdots p_\gamma)^k | < \epsilon_s/3 && \text{if } N \geqq M_s/n \\
&\text{(d)} \quad | p_N(Y_{n,s}) - (p_1 p_2 \cdots p_\gamma)^k | + (\nu_s + M_s)/N_s < \epsilon_s/3 && \text{if } N \geqq N_s/n
\end{aligned}
$$

whenever $s \geqq s_0$.

Let X_n be defined by the equation

$$
\begin{aligned}
X_n = (r_1/n)\{x(a_1, p_1) \cdot x(a_2, p_2) \cdots x(a_\gamma, p_\gamma)\} \cdot \{r_2/n) x(a_1, p_1) \cdots \} \\
\cdots (r_k/n)\{x(a_1, p_1) \cdots \}.
\end{aligned}
$$

Then it is easily seen that $|\, p_N(X_n) - (p_1 p_2 p_3 \cdots p_\gamma)^k \,| < \epsilon_s$, if $s \geqq s_0$ and if $\nu_{s+1}/n \leqq N \leqq \nu_{s+2}/n$. Therefore $p(X_n) = (p_1 p_2 \cdots p_\gamma)^k$. It follows that $x(a_1, p_1) \cdot x(a_2, p_2) \cdots x(a_\gamma, p_\gamma)$ belongs to $A(p_1 p_2 p_3 \cdots p_\gamma)$ and hence the numbers, $x(a, p)$, constitute an admissibly independent set.

We have now established the existence of non-denumerably infinite sets of admissibly independent numbers. One might ask whether the added condition of admissibility enables us to lighten the independence restriction. That is, can we say that n numbers are *admissibly* independent provided each is *admissibly* independent of each of the other $n - 1$? We have to answer this question in the negative as the following example shows. Let x belong to $A(\frac{1}{2})$ and let $x_1 = (1/2)x$, $x_2 = (2/2)x$ and $x_3 = (1/2)x \cdot (2/2)x + [1 - (1/2)x] \cdot [1 - (2/2)x]$. Then $x_1 \cdot x_2 = x_2 \cdot x_3 = x_3 \cdot x_1 = (1/2)x \cdot (2/2)x$ and $x_1 \cdot x_2 \cdot x_3 = (1/2)x \cdot (2/2)x$. Hence each of the numbers, x_1, x_2, x_3, is admissibly independent of each of the other two, but the set, x_1, x_2, x_3, is not admissibly independent.

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The Primitive Groups of Class Fourteen.

BY W. A. MANNING.

1. No primitive group of class 14 contains a substitution of degree 14 and order 7.[*] Then the primitive groups of class 14 can be studied to the best advantage under two heads: (a) those in which the positive subgroup is of class > 15, to be determined in §§ 3-36 of this paper, (b) all others, deferred to §§ 37-48. It will be shown that there are but four groups of the first sort, and one of the second:

(1) *The simply transitive primitive group according to which the symmetric group of degree 9 permutes its 36 transpositions.*

(2) *The simply transitive primitive group of degree 49 and order* $2(7!)^2$ *isomorphic to* $\{ab, bcdefg, \alpha\beta, \beta\gamma\delta\epsilon\zeta\eta, a\alpha \cdot b\beta \cdot c\gamma \cdot d\delta \cdot e\epsilon \cdot f\zeta \cdot g\eta\}$.

(3) *The triply transitive group of degree 22 and order* $22 \cdot 21 \cdot 20 \cdot 96$ *which occurs in Mathieu's quintuply transitive group of degree 24 as a transitive constituent of the largest subgroup in which the subgroup that leaves two letters fixed is invariant.*

(4) *The doubly transitive subgroup of (3) of degree 21 and order* $21 \cdot 20 \cdot 96$.

(5) *The doubly transitive group of degree 21 and order* $21 \cdot 20 \cdot 288$ *which occurs in the quintuply transitive group of degree 24 as a transitive constituent of the largest subgroup in which the subgroup that leaves three letters fixed is invariant.*

The positive subgroup of (5) is a doubly transitive group of class 15, which was omitted from the author's published list of the primitive groups of class 15.[†] The error in that paper is corrected in §§ 39-48. A direct proof of the existence of the quintuply transitive group of degree 24 is given in § 46.

2. In this investigation the following theorem is of constant use. It is, however, only a special case of a more general theory.[‡]

Let certain substitutions of prime order generate an invariant subgroup of a primitive group G. If H is an intransitive subgroup generated by some

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[*] *Transactions of the American Mathematical Society,* Vol. 4 (1903), p. 351.

[†] *American Journal of Mathematics,* Vol. 39 (1917), p. 281.

[‡] *Transactions of the American Mathematical Society,* Vol. 12 (1911), p. 378.

of these substitutions, there is in G a substitution similar to one of the given generators which replaces a letter of an arbitrarily chosen transitive constituent of H by a letter of another constituent of H.

Occasional use will be made of one of Jordan's theorems, which may be stated as follows:[*]

Let G be a transitive group generated by certain substitutions of prime order p, some of which generate an intransitive subgroup H. Let there be in G a substitution similar to one of the given generators of G which replaces a letter of an arbitrarily chosen transitive constituent A of H by a letter of some other constituent of H. Of such substitutions let $t, t_1, \cdots$ be those that displace a minimum number of letters new to H. Then if every power of some one of them, say t, replaces a letter of A by a letter of A, there is at most one letter new to H in any cycle of t.

The last condition of this theorem is certainly satisfied if the generating substitutions are of order 2 and if the degree of A exceeds the number of cycles in t.

The primitive groups of degree less than 21 are known and among them there are no primitive groups of class 14.[†]

3. If the positive subgroup of the primitive group G is of class > 15, a substitution s_2 of order 2 and degree 14 will with

$$s_1 = a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2$$

generate an Abelian group or one of the following non-Abelian diedral groups:

$$\begin{aligned}
D_0 &= \{s_1,\ s_2 = a_1b_1 \cdot c_1\alpha_1 \cdot c_2\alpha_2 \cdot d_1\beta_1 \cdot d_2\beta_2 \cdot e_1\gamma_1 \cdot e_2\gamma_2\},\\
D'_0 &= \{s_1,\ s_2 = a_1b_1 \cdot c_1d_1 \cdot e_1f_1 \cdot g_1\alpha_1 \cdot g_2\alpha_2 \cdot \beta_1\beta_2 \cdot \gamma_1\gamma_2\},\\
D_1 &= \{s_1,\ s_2 = a_1a_2 \cdot b_1c_1 \cdot d_1e_1 \cdot f_1\alpha_1 \cdot f_2\alpha_2 \cdot g_1\beta_1 \cdot g_2\beta_2\},\\
D_2 &= \{s_1,\ s_2 = a_1b_1 \cdot c_1d_1 \cdot e_1\alpha_1 \cdot e_2\alpha_2 \cdot f_1\beta_1 \cdot f_2\beta_2 \cdot \gamma_1\gamma_2\},\\
D_3 &= \{s_1,\ s_2 = a_1a_2 \cdot b_1b_3 \cdot c_1c_3 \cdot d_1d_3 \cdot e_1e_3 \cdot f_1f_3 \cdot g_1g_3\},\\
D_4 &= \{s_1,\ s_2 = a_1a_3 \cdot b_1b_3 \cdot c_1c_3 \cdot d_1d_3 \cdot e_1e_3 \cdot f_1f_3 \cdot g_1g_3\},\\
D_5 &= \{s_1,\ s_2 = a_1b_1 \cdot a_2c_1 \cdot b_2c_2 \cdot d_1d_3 \cdot e_1e_3 \cdot f_1f_3 \cdot g_1g_3\},\\
D_6 &= \{s_1,\ s_2 = a_1b_1 \cdot a_2c_1 \cdot b_2d_1 \cdot c_2d_2 \cdot e_1f_1 \cdot g_1\alpha_1 \cdot g_2\alpha_2\},\\
D_7 &= \{s_1,\ s_2 = a_1b_1 \cdot a_2c_1 \cdot b_2d_1 \cdot e_1f_1 \cdot e_2g_1 \cdot f_2\alpha_1 \cdot g_2\alpha_2\},
\end{aligned}$$

The accuracy of this list can be checked by the construction of the positive substitutions of degree > 15 and < 22[‡] which can occur in G. D'_0 is the same group as D_0, with s_1 and s_2 transposed.

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[*] C. Jordan, *Crelle's Journal*, Vol. 79 (1874), p. 249.

[†] See references, *American Journal of Mathematics*, Vol. 35 (1913), p. 229.

[‡] *Transactions of the American Mathematical Society*, Vol. 18 (1917), p. 473.

Since no invariant subgroup of G can be Abelian unless it is regular, some two substitutions of a complete set of conjugate substitutions of degree 14 must generate one of these 8 diedral groups. Our immediate object is to show that G can not contain D_1, D_3, D_6, or D_7. They will be studied in the order D_7, D_3, D_6, D_1, D_5, D_0, D_2, D_4. A complete list of the possible Abelian groups generated by two substitutions of degree 14 might be given at this point, but there is nothing to be gained thereby. All such groups will be found in §§ 29-33.

4. The last group of this list, D_7, is of order 16 and has only two transitive constituents. It will be shown that it is not a subgroup of any primitive group of the sort we are studying. The following substitutions of D_7 will be needed:

$$\begin{aligned} s_1 &= a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2, \\ s_2 &= a_1b_1 \cdot a_2c_1 \cdot b_2d_1 \cdot e_1f_1 \cdot e_2g_1 \cdot f_2\alpha_1 \cdot g_2\alpha_2. \\ t_1 &= a_2d_1 \cdot b_2c_1 \cdot c_2d_2 \cdot e_1g_1 \cdot e_2f_1 \cdot f_2\alpha_2 \cdot g_2\alpha_1. \\ t_2 &= a_1b_2 \cdot a_2b_1 \cdot c_1d_2 \cdot c_2d_1 \cdot f_1g_2 \cdot f_2g_1 \cdot \alpha_1\alpha_2. \end{aligned}$$

There is a substitution s_3 of degree 14 that replaces a_1 by a letter of the second constituent and has at most one letter new to D_7 in any cycle. Under transformation by t_1 and $e_1e_2 \cdot f_1g_1 \cdot f_2g_2 \cdot \alpha_1\alpha_2$, the four transpositions (a_1e_1), (a_1e_2), (a_1f_1), and (a_1g_1) are conjugate. So also are (a_1f_2), (a_1g_2), $(a_1\alpha_1)$, and $(a_1\alpha_2)$. For s_3 we have to try only $(a_1e_1) \cdots$ and $(a_1f_2) \cdots$.

Since we are not interested in primitive groups of degree < 21,[*] it is well to know if a transitive group of degree 16 or 18 of which D_7 is a subgroup can be multiply imprimitive.[†] It is easy to verify the following statements:

The only possible system of imprimitivity of two letters including a_1 of any imprimitive group of degree 16 or 18 of which D_7 is a subgroup is a_1b_1.

The only possible system of imprimitivity of four letters including a_1 of any imprimitive group of degree 16 of which D_7 is a subgroup is $a_1b_1c_2d_2$.

No systems of three or nine letters are possible in an imprimitive group of degree 18 of which D_7 is a subgroup.

Then D_7 is a subgroup of a primitive group of degree < 21 unless s_3 displaces at least four letters new to D_7.

Neither $\{s_1, s_3\}$, $\{s_2, s_3\}$, $\{t_1, s_3\}$, or $\{t_2, s_3\}$ is Abelian or of the form D_6 or D_7. Nor can the first two be D_3 or D_4.

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[*] See references, *American Journal of Mathematics*, Vol. 35 (1913), p. 229.

[†] C. Jordan, *Liouville's Journal*, Ser. 2, Vol. 16 (1871), p. 383; Marggraff, Dissertation, *Ueber primitive Gruppen mit transitiven Untergruppen geringeren Grades*, Giessen, 1889; Manning, *Primitive Groups*, 1921, p. 92.

5. Consider first the substitution

$$s_3 = (a_1e_1)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —)\cdots.$$

Let both $\{s_1, s_3\}$ and $\{s_2, s_3\}$ be octic. Now $s_3 = (a_1e_1)(a_2)(b_1)\cdots$ so that $\{t_2, s_3\}$ is octic and $s_3 = (a_1e_1)(b_2\beta_1)(b_1)\cdots$, which is impossible when $\{s_1, s_3\}$ is octic.

Let $\{s_1, s_3\}$ be octic and $\{s_2, s_3\}$ such a group as D_5. Now $s_3 = (a_1e_1)(b_1x)(f_1y)(\beta_1 —)\cdots(a_2)(e_2)$, where (xy) is (b_2d_1), $(f_2\alpha_1)$, or $(g_2\alpha_2)$. If $\{s_1, s_3\}$ is D_1, $s_3 = (a_1e_1)(c_1\beta_1)(c_2\beta_2)(a_2)(e_2)\cdots$, which is impossible with s_2. Therefore $\{s_1, s_3\}$ is D_0 or D_2, and $s_3 = (a_1e_1)(b_1d_1)(f_1b_2)\cdots$ or $(a_1e_1)(f_1\alpha_1)(b_1f_2)\cdots$, both of which are impossible; or else $xy = g_2\alpha_2$, in which case e_2 and g_1 are fixed, impossible with s_2.

If $\{s_1, s_3\}$ is D_5 and $\{s_2, s_3\}$ is octic, transformation by $a_1e_1\cdot a_2f_1\cdot b_1e_2\cdot b_2g_1\cdot c_1f_2\cdot c_2\alpha_1\cdot d_1g_2\cdot d_2\alpha_2$ shifts this case back to the preceding.

Let $\{s_1, s_3\}$ and $\{s_2, s_3\}$ be D_5. It is not possible to have $s_3 = (a_1e_1)(a_2x)(e_2y)\cdots = (a_1e_1)(b_1u)(f_1v)\cdots$.

6. Consider second $s_3 = (a_1f_2)\cdots$.

Let $\{s_1, s_3\}$ and $\{s_2, s_3\}$ be octic. Since now s_2 fixes a_2 and b_1, $\{t_2, s_3\}$ is D_0 or D_2. Therefore $s_3 = (a_1f_2)(a_2)(b_1)(b_2)(f_1)(g_1)(\alpha_1)\cdots$, and $s_3 \neq (\alpha_2\beta_1)(\alpha_1)\cdots$. Thus s_3 has no cycle new to s_1, and in consequence $\{s_1, s_3\}$ is D_0. Then s_3 fixes g_2, so that $\{t_2, s_3\}$ is impossible.

Let $\{s_1, s_3\}$ be octic and $\{s_2, s_3\}$ of order 6. Now $s_3 = (a_1f_2)(b_1x)(\alpha_1y)(a_2)(f_1)\cdots$, where the cycle (xy) of s_2 is (b_2d_1), (e_2g_1), or $(g_2\alpha_2)$. Moreover, $x \neq g_1$, e_1, or e_2, and $y \neq d_1$, d_2, c_1, or c_2, for otherwise $(b_1x)(\alpha_1y)\cdots$ can be transformed into $(a_1e_1)\cdots$. We cannot have $s_3 = (a_1f_2)(b_1d_1)(\alpha_1b_2)\cdots$, and neither $(a_1f_2)(b_1g_2)(\alpha_1\alpha_2)\cdots$ nor $(a_1f_2)(b_1\alpha_2)(\alpha_1g_2)\cdots$ is possible.

Let $\{s_1, s_3\}$ be D_5 and $\{s_2, s_3\}$ octic. As before, $s_3 = (a_1f_2)(a_2x)(f_1y)(b_1)(\alpha_1)\cdots$, where (xy) is (c_1c_2), (d_1d_2), (e_1e_2), or (g_1g_2), and $x \neq e_1$, e_2, or g_2, $y \neq b_1$, c_1, or d_1. The three forms of s_3, $(a_1f_2)(a_2c_1)(f_1c_2)\cdots$, $(a_1f_2)(a_2d_1)(f_1d_2)\cdots$ and $(a_1f_2)(a_2g_1)(f_1g_2)\cdots$ are all impossible.

Finally, $\{s_1, s_3\}$ and $\{s_2, s_3\}$ cannot both be such groups as D_5.

Therefore D_7 can be struck from our list.

7. Consider a group G in which D_3 is a subgroup. If G exists there is in it a substitution $s_3 = (a_1b_1)\cdots$. Now s_3 is not $(a_1b_1)(a_2b_2)\cdots$. Nor is $s_3 = (a_1b_1)(a_2c_1)(b_2c_2)\cdots$ or $(a_1b_1)(a_2c_2)\cdots$. Then $\{s_1, s_3\}$ and $\{s_2, s_3\}$ must be octic groups. If s_3 fixes a_2, it fixes also b_2 and b_3, so that $\{s_2s_1s_2, s_3\}$ is an impossibility. If s_3 displaces a_2, $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are both D_6, $s_3 =$

$(a_1b_1)(a_2c_1)(b_2d)(c_2d)(b_3 —)\cdots$, and $\{s_1s_2s_1, s_3\}$ is D_0, D'_0, D_1, or D_2. Hence $s_3 = (a_1b_1)(a_2c_1)(b_2d_1)(c_2d_2)(b_3 —)(c_3)\cdots$. But if $\{s_2, s_3\}$ is D_6, s_3 should displace c_3. *This consideration removes D_3 from our list.*

8. We can also rid our list of D_6. Its substitutions of degree 14 are

$$\begin{aligned} s_1 &= a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2, \\ s_2 &= a_1b_1 \cdot a_2c_1 \cdot b_2d_1 \cdot c_2d_2 \cdot e_1f_1 \cdot g_1\alpha_1 \cdot g_2\alpha_2, \\ t_1 &= a_1d_1 \cdot a_2d_2 \cdot b_1c_1 \cdot b_2c_2 \cdot e_1f_2 \cdot e_2f_1 \cdot \alpha_1\alpha_2, \\ t_2 &= a_1c_2 \cdot a_2b_2 \cdot b_1d_2 \cdot c_1d_1 \cdot e_2f_2 \cdot g_1\alpha_2 \cdot g_2\alpha_1, \end{aligned}$$

All the substitutions of the group $\{\sigma = e_1f_1 \cdot e_2f_2,\ \sigma_1 = g_1g_2 \cdot \alpha_1\alpha_2,\ \sigma_2 = b_1c_2 \cdot b_2c_1 \cdot e_1e_2 \cdot f_1f_2 \cdot \alpha_1\alpha_2,\ \sigma_3 = a_2b_1 \cdot b_2c_1 \cdot c_2d_1 \cdot e_1g_1 \cdot e_2\alpha_1 \cdot f_1g_2 \cdot f_2\alpha_2\}$ transform D_6 into itself. There is then but one substitution s_3 to be investigated: $(a_1e_1)\cdots$. Upon this substitution may be imposed the condition that it introduces a minimum number of new letters to D_6, and therefore at most one new letter to any cycle.

If $\{D_6, s_3\}$ $(= H_3)$ is an imprimitive group of degree 16 or 18, the only system, including α_1, of two letters is $\alpha_1\alpha_2g_1g_2$, and there can be no system of three letters. Hence if H_3 is a transitive group of degree < 20, D_6 is a subgroup of a primitive group of degree < 21. That is, if H_3 is of degree < 20, it is intransitive.

9. The group $\{s_1, s_3\}$ can be Abelian, or of one of the types D_0, D'_0, D_2, D_3, or D_6.

Let $\{s_1, s_3\}$ be Abelian. If s_3 and t_1 are also commutative, $s_3 = (a_1e_1)(a_2e_2)(d_1f_2)(d_2f_1)\cdots$. By s_2, $s_3 = (a_1e_1)(f_1d_2)(b_1c_2)(a_2e_2)(c_1)\cdots$. Then $\{t_1, s_3\}$ is octic. If $\{t_1, s_3\}$ is D_6, there are just two letters in s_3 new to t_1, and they replace the letters of one cycle of t_1. These new letters are not g_1 or g_2. Therefore $s_3 = (a_1e_1)(a_2e_2)(\alpha_1\beta_1)(\alpha_2\beta_2)\cdots$. The only other letter new to s_2 in s_3 is now f_2, so that $\{s_2, s_3\}$ can only be D_1 or D_5, and neither case is possible. If $\{t_1, s_3\}$ is D'_0, D_1, or D_2, s_3 has four or six letters new to t_1. Now $\{s_2, s_3\}$ is octic because s_3 fixes f_1 and therefore $s_3 = (a_1e_1)(a_2e_2)(c_1 —)(b_1)(b_2)(d_1)(d_2)(f_1)(f_2)\cdots$, and this is impossible.

Let $\{s_1, s_3\}$ be D_0. The substitution $s_3 = (a_1e_1)(a_2)(e_2)\cdots$ has six letters new to s_1 in six cycles, of which at least four are β_1, β_2, β_3, β_4. If s_3 displaces α_1 and α_2, $\{t_1, s_3\}$ is D_0, D_1, D_2, or D_5. If it is D_5, $\{s_2, s_3\}$ is impossible; and if it is D_0, D_1, or D_2, s_3 fixes a_2, e_2, d_1, d_2, f_1, f_2, and therefore $s_3 = (a_1e_1)(b_1\beta_1)(b_2\beta_2)(c_1\beta_3)(c_2\beta_4)(\alpha_1g)(\alpha_2g)\cdots$, absurd with s_2. If s_3 displaces α_1 and fixes α_2, $\{t_1, s_3\}$ is D_2 and $s_3 = (a_1e_1)(g_1\beta_1)(g_2\beta_2)(\alpha_1x)\cdots$, where x is new to s_2 and to t_2 and is a letter of s_1, an impossibility. If s_3 fixes

α_1 and α_2, it displaces six letters β in six cycles, so that $\{t_1, s_3\}$ and $\{s_2, s_3\}$ are also D_0. This is impossible.

Let $\{s_1, s_3\}$ be D'_0. Now s_3 has six letters new to s_1 and four of these new letters are in two cycles. Therefore $s_3 = (a_1e_1)(\alpha_1\beta_1)(\alpha_2\beta_2)(\beta_3x)(\beta_4y) \cdots$, where (xy) is a cycle of s_1. We note that $\{t_1, s_3\}$ is octic and is not D_6. Therefore s_3 fixes a_2, e_2, d_1, f_2, and (xy) is (b_1b_2), (c_1c_2), or (g_1g_2). Because of s_2, (xy) is not (b_1b_2). If (xy) is (c_1c_2), t_1 requires that b_1 and b_2 be displaced, and this s_1 does not permit. If (xy) is (g_1g_2), $\{s_2, s_3\}$ is octic and therefore s_3 fixes f_1 and f_2, whereas s_1 requires that f_1 or f_2 be displaced.

Let $\{s_1, s_3\}$ be D_1. Here s_3 has a cycle in common with s_1 and has four letters new to s_1. If s_3 displaces α_1 it is of the form $(a_1e)(\alpha_1g)(\alpha_2g) \cdots$. For if H_3 is of degree < 20, it is intransitive. This form of s_3 is inconsistent with s_2 or t_2. Let s_3 replace g_1 by g_2. Thus $s_3 = (a_1e_1)(g_1g_2)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —) \cdots$. With s_2, t_1 and t_2, s_3 generates octic groups and therefore fixes f_1 and f_2, but should displace a letter from every cycle of s_1. Let $s_3 = (a_1e_1)(g_1\beta_1)(g_2\beta_2) \cdots$. This is commutative with t_1 and replaces d_1 by f_2. Hence if s_3 has one cycle in common with s_1, it has two.

Let $\{s_1, s_3\}$ be D_2. Now s_3 has six letters new to s_1, in five cycles. If $s_3 = (a_1e_1)(\alpha_1\beta_1)(\alpha_2 —)(\beta_2 —)(\beta_3 —)(\beta_4 —) \cdots$, the groups $\{t_1, s_3\}$, $\{s_2, s_3\}$ and $\{t_2, s_3\}$ are octic, so that $s_3 = (a_1e_1)(\alpha_1\beta_1)(g\alpha_2)(g\beta_2) \cdots$, impossible with s_2 and t_2. If $s_3 = (a_1e_1)(\alpha_1\beta_1)(\beta_2 —)(\beta_3 —)(\beta_4 —)(\beta_5 —) \cdots$, $\{t_1, s_3\}$ is D_5; but $(a_1e_1)(\alpha_1\beta_1)(d_1 —)(f_2 —) \cdots$ is absurd if the blanks are to be filled from t_1. It would require eight cycles in s_3. Then $s_3 = (a_1e_1)(\alpha_1\alpha_2)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —) \cdots$ and $s_3 \neq (g_1g_2) \cdots$. Therefore s_3 is not commutative with s_2 or t_2. Both $\{s_2, s_3\}$ and $\{t_1, s_3\}$ are octic. Hence s_3 fixes f_1, f_2, g_1 and g_2.

10. Let $\{s_1, s_3\}$ be D_5. Now $s_3 = (a_1e_1)(a_2x)(e_2y)(\beta_1 —)(\beta_2 —) \cdots$, where (xy) is a cycle of s_1. In s_3 there are four letters new to s_1. If one of them is α_1, H_3 is intransitive and therefore $s_3 = (\alpha_1g)(\alpha_2)(g) \cdots = (a_1e_1)(d_1x_1)(f_2y_1) \cdots$ [(x_1y_1) is a cycle of t_1], which is impossible. Hence $s_3 = (\alpha_1)(\alpha_2)(a_1e_1)(a_2x)(e_2y)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —)$. Since $\{t_1, s_3\}$ is D_0 or D_2, s_3 fixes d_1 and f_2, and then, because of s_1, must replace d_2 and f_1 by letters β. But $s_3 = (e_1a_1)(f_1\beta_1) \cdots$ is impossible with s_2.

Let $\{s_1, s_3\}$ be D_6. H_3 is intransitive. Then $s_3 = (a_1e_1)(g_1\beta_1)(g_2\beta_2) \cdots$ or $(g_1\alpha)(g_2x) \cdots$, where x is new to s_1. But $(g_1\alpha)(g_2\beta)(\alpha) \cdots$ is impossible, as is $(g_1\alpha)(g_2\alpha) \cdots$. Now $s_3 = (a_1e_1)(g_1\beta_1)(g_2\beta_2)(\alpha_1)(\alpha_2)(d_1f_2) \cdots$. Since $\{s_2, s_3\}$ and $\{t_2, s_3\}$ are of order 6, $s_3 = a_1e_1 \cdot a_2f_1 \cdot b_1c_1 \cdot d_1f_2 \cdot d_2e_2 \cdot g_1\beta_1 \cdot g_2\beta_2$.

The group $H_3 = \{D_6, s_3\}$ is unique. It is transformed into itself by $g_1g_2 \cdot \alpha_1\alpha_2 \cdot \beta_1\beta_2$ and by $t_3 = c_1e_2 \cdot c_2e_1 \cdot b_1f_2 \cdot b_2f_1 \cdot d_1d_2 \cdot \alpha_1\beta_2 \cdot \alpha_2\beta_1$, so that for s_4 two substitutions, $(a_1g_1) \cdots$ and $(a_1\alpha_1) \cdots$, are to be studied. Now $(a_1g_1) \cdots$ can only be the transform of s_3 above by the substitution $\sigma_3 = a_2b_1 \cdot b_2c_1 \cdot c_2d_1 \cdot e_1g_1 \cdot e_2\alpha_1 \cdot f_1g_2 \cdot f_2\alpha_2$ of the group whose substitutions transform D_6 into itself: $\sigma_3s_3\sigma_3 = a_1g_1 \cdot a_2b_2 \cdot b_1g_2 \cdot c_2\alpha_2 \cdot d_2\alpha_1 \cdot e_1x \cdot f_1y$, where x and y are letters new to D_6. This is impossible with s_3. The other substitution, $(a_1\alpha_1) \cdots$, is the transform of s_3 by $\sigma_2\sigma_3$: $\sigma_3\sigma_2s_3\sigma_2\sigma_3 = a_1\alpha_1 \cdot b_1\alpha_2 \cdot c_1d_1 \cdot c_2g_2 \cdot d_2g_1 \cdot e_1x \cdot f_1y$. This too is inconsistent with s_3.

This leaves in our list of diedral groups D_0, D'_0, D_1, D_2, D_4 and D_5.

11. It is also possible to reject the group D_1. The four negative substitutions of order 2 in D_1 are

$$\begin{aligned}
s_1 &= a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2,\\
s_2 &= a_1a_2 \cdot b_1c_1 \cdot d_1e_1 \cdot f_1\alpha_1 \cdot f_2\alpha_2 \cdot g_1\beta_1 \cdot g_2\beta_2,\\
t_1 &= a_1a_2 \cdot b_2c_2 \cdot d_2e_2 \cdot f_2\alpha_1 \cdot f_1\alpha_2 \cdot g_2\beta_1 \cdot g_1\beta_2,\\
t_2 &= a_1a_2 \cdot b_1c_2 \cdot b_2c_1 \cdot d_1e_2 \cdot d_2e_1 \cdot \alpha_1\alpha_2 \cdot \beta_1\beta_2.
\end{aligned}$$

All the substitutions of the group

$$\begin{gathered}
\{b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2, \quad b_1c_1 \cdot d_1e_1 \cdot f_1\alpha_1 \cdot f_2\alpha_2 \cdot g_1\beta_1 \cdot g_2\beta_2,\\
b_1d_1 \cdot b_2d_2 \cdot c_1e_1 \cdot c_2e_2, \quad b_1f_1 \cdot b_2\alpha_1 \cdot c_1f_2 \cdot c_2\alpha_2 \cdot d_1g_1 \cdot d_2\beta_1 \cdot e_1g_2 \cdot e_2\beta_2,\\
b_1c_1 \cdot b_2c_2,\ d_1e_1 \cdot d_2e_2,\ f_1f_2 \cdot \alpha_1\alpha_2,\ g_1g_2 \cdot \beta_1\beta_2,\ f_1g_1 \cdot f_2g_2 \cdot \alpha_1\beta_1 \cdot \alpha_2\beta_2\}
\end{gathered}$$

transform D_1 into itself. Then for s_3 only the substitution $(a_1b_1) \cdots$ need be studied.

If s_3 is commutative with s_1, $\{s_2, s_3\}$ is not one of our groups D_0, D'_0, D_1, D_2, D_4, or D_5. If $\{s_1, s_3\}$ is octic, s_3 fixes a_2, b_2, c_1 and c_2 and then $\{t_1, s_3\}$ is not a possible subgroup of G. Hence $\{s_1, s_3\}$ is D_5: $s_3 = (a_1b_1)(a_2 —)(b_2 —) \cdots$, and $(t_1s_3)^4 = (s_2s_3)^2 = 1$. Indeed, $\{t_1, s_3\}$ is D_1. Therefore

$$\begin{aligned}
s_3 &= a_1b_1 \cdot a_2c_1 \cdot b_2c_2 \cdot d_2\gamma_1 \cdot e_2\gamma_2 \cdot f_1\alpha_1 \cdot g_1\beta_1\\
\text{or} \qquad s'_3 &= a_1b_1 \cdot a_2c_1 \cdot b_2c_2 \cdot d_2\gamma_1 \cdot e_2\gamma_2 \cdot f_1\beta_1 \cdot g_1\alpha_1.
\end{aligned}$$

The only other substitutions of degree 14 which replace a_1 by b_1 are the transforms of s_3 and s'_3 by $f_1f_2 \cdot \alpha_1\alpha_2$, $g_1g_2 \cdot \beta_1\beta_2$ and $f_1f_2 \cdot g_1g_2 \cdot \alpha_1\alpha_2 \cdot \beta_1\beta_2$. The sets of transitivity of H_3 are $a_1a_2b_1b_2c_1c_2$, $d_1d_2e_1e_2\gamma_1\gamma_2$, $f_1f_2\alpha_1\alpha_2$, $g_1g_2\beta_1\beta_2$, and of H'_3 are $a_1a_2b_1b_2c_1c_2$, $d_1d_2e_1e_2\gamma_1\gamma_2$, $f_1f_2g_1g_2\alpha_1\alpha_2\beta_1\beta_2$. Both H_3 and H'_3 are invariant under the substitutions of the group $\{d_1e_1 \cdot d_2e_2 \cdot \gamma_1\gamma_2,\ b_1c_2 \cdot b_2c_1 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2,\ f_1g_1 \cdot f_2g_2 \cdot \alpha_1\beta_1 \cdot \alpha_2\beta_2,\ a_1a_2 \cdot b_1c_1 \cdot d_1e_1 \cdot f_1\alpha_1 \cdot f_2\alpha_2 \cdot g_1\beta_1 \cdot g_2\beta_2\}$. Hence for s_4 we have to discuss only the three substitutions $(a_1d_1) \cdots$, $(a_1\gamma_1) \cdots$ and $(a_1f_1) \cdots$.

It will be convenient to use w to represent s_3 and s'_3 indifferently.

12. Let $s_4=(a_1f_1)\cdots$. Take the eight substitutions $(a_1b_1)\cdots$ which are possible with $\{s_1, s_2\}$ and transform them by $b_1f_1\cdot b_2\alpha_1\cdot c_1f_2\cdot c_2\alpha_2\cdot d_1g_1\cdot d_2\beta_1\cdot e_1g_2\cdot e_2\beta_2$. Transform again by $d_1e_1\cdot d_2e_2\cdot\gamma_1\gamma_2$ and by $b_1c_2\cdot b_2c_1\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2\cdot g_1g_2$. There remain for further consideration two substitutions having the first cycles $a_1f_1\cdot a_2f_2\cdot\alpha_1\alpha_2\cdot\beta_1x_1\cdot\beta_2x_2$ and the last two cycles $b_1b_2\cdot d_1d_2$ or $b_1d_2\cdot b_2d_1$. Of the letters x_1 and x_2 we know as yet merely that they are new to D_1. Now $s_4=(f_1a_1)(\alpha_1\alpha_2)\cdots$ is not possible with s_3. The group $\{s'_3, s_4\}$ is D_5, so that $s_4=a_1f_1\cdot a_2f_2\cdot b_1d_2\cdot b_2d_1\cdot\alpha_1\alpha_2\cdot\beta_1\gamma_1\cdot\beta_2\gamma_2$. In the primitive group G there is no other substitution $(a_1f_1)\cdots$ similar to s_1. This means that no G containing this H_4 is doubly transitive, for if so there would be four substitutions with (a_1f_1) in common, conjugate to s_1, s_2, t_1 and t_2, respectively. Since $\{H'_3, s_4\}$ is transitive of degree 20, and since there is no simply transitive primitive group of degree 20, this $s_4=(a_1f_1)\cdots$ can be dropped.

Let $s_4=(a_1d_1)\cdots$. Now there are six substitutions s_4 with $a_1d_1\cdot a_2e_1\cdot d_2e_2\cdot b_2x_1\cdot c_2x_2$ in common and with the last two cycles $f_1\alpha_1\cdot g_1\beta_1$, $f_1\alpha_1\cdot g_2\beta_2$, $f_2\alpha_2\cdot g_2\beta_2$, $f_1\beta_1\cdot g_1\alpha_1$, $f_1\beta_2\cdot g_2\alpha_1$, or $f_2\beta_2\cdot g_2\alpha_2$. This s_4 is commutative with $s_1s'_3s_1=a_1c_2\cdot a_2b_2\cdot b_1c_1\cdot d_1\gamma_1\cdot e_1\gamma_2\cdot f_2\beta_1\cdot g_2\alpha_1$. Thus we have uniquely $s_4=a_1d_1\cdot a_2e_1\cdot d_2e_2\cdot b_2\gamma_2\cdot c_2\gamma_1\cdot f_1\beta_2\cdot g_2\alpha_1$. The group $\{H_3, s_4\}$ is impossible. Of $H_4=\{H'_3, s_4\}$ it can at least be said that it has two sets of transitivity. Transformation by substitutions of D_1 and by $s'_3t_1s'_3=b_1c_1\cdot b_2c_2\cdot f_1g_2\cdot f_2g_1\cdot\alpha_1\beta_2\cdot\alpha_2\beta_1\cdot\gamma_1\gamma_2$ shows that there is but one substitution to try for s_5: $(a_1f_1)\cdots$, and it has already been rejected.

Let $s_4=(a_1\gamma_1)\cdots$, with the strong condition that in no substitution of G, similar to s_1, is a_1 (or a_2) replaced by d_1, d_2, e_1, or e_2. Let s_4 fix a_2. If $ws_4=s_4w$, $s_4=(a_1\gamma_1)(b_1d_2)\cdots$, and the transform of $t_1wt_1=(a_2b_1)\cdots$ by s_4 is $(a_2d_2)\cdots$. If $\{w, s_4\}$ is octic, $s_4=(a_1\gamma_1)(a_2)(b_1)(b_2 —)(c_1 —)(c_2)\cdots$, and is inconsistent with t_2. If $\{w, s_4\}$ is of order 6, $s_4=(a_1\gamma_1)(b_1 —)(d_2 —)(c_1 —)(a_2)(b_2)(d_1)(c_2)\cdots$, clearly impossible with t_1,

Now the only letter by which s_4 can replace a_2 is γ_2: $s_4=(a_1\gamma_1)(a_2\gamma_2)\cdots$. If $ws_4=s_1w$, $s_4=(a_1\gamma_1)(a_2\gamma_2)(b_1d_2)(b_2)\cdots$. Then $s_4s_1ws_1s_4=(b_2\gamma_2)\cdots$ and the transform of this by s_1ws_1 is $(a_2e_1)\cdots$. If $\{w, s_4\}$ is octic, $s_4=(a_1\gamma_1)(a_2\gamma_2)(b_1)\cdots$, and $ws_4ws_4w=(a_1d_2)\cdots$.

Then the only non-Abelian diedral groups left are D_0, D'_0, D_2, D_4 and D_5.

13. Consider D_5, in which

$$\begin{aligned}
s_1&=a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2\cdot g_1g_2,\\
s_2&=a_1b_1\cdot a_2c_1\cdot b_2c_2\cdot d_1d_3\cdot e_1e_3\cdot f_1f_3\cdot g_1g_3,\\
t_1&=a_2b_2\cdot a_1c_2\cdot b_1c_1\cdot d_2d_3\cdot e_2e_3\cdot f_2f_3\cdot g_2g_2.
\end{aligned}$$

The group D_5 is invariant under $\{\sigma_1 = b_1c_2 \cdot b_2c_1 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2,\ \sigma_2 = a_2b_1 \cdot b_2c_1 \cdot d_2d_3 \cdot e_2e_3 \cdot f_2f_3 \cdot g_2g_3\}$ so that we have to study only the one type of substitution, $s_3 = (a_1d_1) \cdots$. If $\{s_1, s_3\}$ is an octic group, $\sigma_1s_3s_1s_3\sigma_1 = (a_1d_1)(a_2d_2) \cdots$, and if $\{s_2, s_3\}$ is octic, $\sigma_1\sigma_2s_3s_2s_3\sigma_2\sigma_1 = (a_1d_1)(a_2d_2) \cdots$. If $(s_2s_3)^2 = 1$, $\sigma_2s_3\sigma_2 = (a_1d_1)(a_2d_2) \cdots$. It is not possible to have all the groups $\{s_1, s_3\}$, $\{s_2, s_3\}$, $\{t_1, s_3\}$ of order 6, as is seen by considering one of the transitive constituents of degree 3. If $(s_1s_3)^3 = (s_2s_3)^3 = (t_1s_3)^4 = 1$, $s_3 = (a_1d_1)(c_2\alpha)(b_2)(a_2e_2)(e_3) \cdots$, and the transform of the cycles $(a_2b_2)(e_2e_3)$ of t_1 by $s_3s_1\sigma_2\sigma_1(d_1e_1)(d_2e_2)(d_3e_3)$ is $(a_1d_1)(a_2d_2)$. Then in all cases $s_3 = (a_1d_1)(a_2d_2) \cdots$. If $(s_2s_3)^4 = 1$, $s_3 = (a_1d_1)(a_2d_2)(b_1)(d_3) \cdots$, and $s_3\sigma_2\sigma_1$ transforms $s_2 = (a_1b_1)(d_1d_3)(a_2c_1) \cdots$ into $(d_2a_2)(a_1d_1)(d_3 —) \cdots$, which with s_2 generates such a group as D_5. Then

$$s_3 = a_1d_1 \cdot a_2d_2 \cdot b_1e_1 \cdot b_2e_2 \cdot d_3e_3 \cdot f_1f_2 \cdot g_3g_4$$

or

$$s'_3 = a_1d_1 \cdot a_2d_2 \cdot b_1e_1 \cdot b_2e_2 \cdot d_3e_3 \cdot f_3f_4 \cdot g_3g_4.$$

14. We take up the first of these two substitutions and study the group H_3. The substitutions of degree 14 in H_3 are

$$\begin{aligned}
s_1 &= a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2,\\
s_2 &= a_1b_1 \cdot a_2c_1 \cdot b_2c_2 \cdot d_1d_3 \cdot e_1e_3 \cdot f_1f_3 \cdot g_1g_3,\\
t_1 &= a_2b_2 \cdot a_1c_2 \cdot b_1c_1 \cdot d_2d_3 \cdot e_2e_3 \cdot f_2f_3 \cdot g_2g_3,\\
s_3 &= a_1d_1 \cdot a_2d_2 \cdot b_1e_1 \cdot b_2e_2 \cdot d_3e_3 \cdot f_1f_2 \cdot g_3g_4,\\
t_2 &= b_1d_3 \cdot c_1d_2 \cdot a_1e_3 \cdot c_2e_2 \cdot d_1e_1 \cdot f_2f_3 \cdot g_1g_4,\\
t_3 &= b_2d_3 \cdot c_2d_1 \cdot a_2e_3 \cdot c_1e_1 \cdot d_2e_2 \cdot f_1f_3 \cdot g_2g_4.
\end{aligned}$$

The sets of transitivity are $a_1a_2b_1b_2c_1c_2d_1d_2d_3e_1e_2e_3$, $f_1f_2f_3$, and $g_1g_2g_3g_4$. H_3 is invariant under $\sigma = a_2c_2 \cdot b_2c_1 \cdot d_2e_2 \cdot d_1e_3 \cdot d_3e_1 \cdot f_1f_3 \cdot g_1g_3$. There are then four types of substitutions that need be used for s_4: $(a_1f_1) \cdots$, $(a_1f_2) \cdots$, $(a_1g_1) \cdots$, and $(a_1g_2) \cdots$. If s_1s_4 is of order 4, $s_4s_1s_4$ is $(a_1f_2)(a_2f_1) \cdots$, $(a_1f_1)(a_2f_2) \cdots$, $(a_1g_2)(a_2g_1) \cdots$, or $(a_1g_1)(a_2g_2) \cdots$, and may be used for s_4. The products s_1s_4 and s_2s_4 are not both of order 2. Hence the substitution s_4 may be determined subject to the following conditions:

(a) $(s_1s_4)^2 = (s_2s_4)^3 = 1$,
(b) $(s_1s_4)^3 = (s_2s_4)^2 = 1$,
(c) $(s_1s_4)^3 = (s_2s_4)^3 = 1$,
(d) $(s_1s_4)^2 = (s_2s_4)^4 = 1$, $s_4 = (a_1f_2) \cdots$, or $(a_1g_2) \cdots$,
(e) $(s_1s_4)^3 = (s_2s_4)^4 = 1$, $s_4 = (a_1f_2) \cdots$, or $(a_1g_2) \cdots$.

To justify the limitation in (d) and (e), consider $s_4 = (a_1f_1)(b_1)(f_3) \cdots$. The transform $t = t_3s_4s_2s_4t_3 = (a_1f_1)(b_1f_3) \cdots$, and we may assume that ts_1 is not of order 4. For if it were of order 4, $ts_1t = (a_1f_2)(a_2f_1) \cdots$ could be used for s_4, under (a), or under (d) as stated. If $s_4 = (a_1 g_1)(b_1)(g_3) \cdots$, we use $t = \sigma s_4 s_2 s_4 \sigma = (a_1 g_1)(b_1 g_3) \cdots$ and again choose another substitution $(a_1 g_1)(a_2 g_2) \cdots$ for s_4 if $t s_1$ is of order 4. It will be seen that not all the substitutions s_4 determined subject to the above conditions have to be used for s_4, but these five cases certainly include all the necessary forms of s_4. The following 19 substitutions result immediately from this classification:

$$\begin{aligned}
S_{4\,1} &= a_1f_1 \cdot a_2f_2 \cdot b_1g_1 \cdot b_2g_2 \cdot d_1d_2 \cdot f_3g_3 \cdot e_3\alpha\,,\\
S_{4\,2} &= a_1f_2 \cdot a_2f_1 \cdot c_1e_1 \cdot c_2e_2 \cdot d_1d_2 \cdot e_3f_3 \cdot g_3\alpha\,,\\
S_{4\,3} &= a_1g_1 \cdot a_2g_2 \cdot b_1f_1 \cdot b_2f_2 \cdot e_1e_2 \cdot f_3g_3 \cdot d_3\alpha\,,\\
S_{4\,4} &= a_1g_2 \cdot a_2g_1 \cdot c_1d_1 \cdot c_2d_2 \cdot e_1e_2 \cdot d_3g_3 \cdot f_3\alpha\,,\\
S_{4\,5} &= a_1f_1 \cdot a_2d_1 \cdot b_1f_3 \cdot c_1d_3 \cdot d_2f_2 \cdot g_1g_3 \cdot e_2\alpha\,,\\
S_{4\,6} &= a_1g_1 \cdot a_2e_1 \cdot b_1g_3 \cdot c_1e_3 \cdot d_2g_4 \cdot e_2g_2 \cdot f_1f_3,\\
S_{4\,7} &= a_1g_1 \cdot a_2e_1 \cdot b_1g_3 \cdot c_1e_3 \cdot d_2g_4 \cdot e_2g_2 \cdot f_2\alpha\,,\\
S_{4\,8} &= a_1f_2 \cdot a_2d_1 \cdot b_2\alpha_1 \cdot c_1g_3 \cdot d_2f_1 \cdot d_3g_1 \cdot e_1\alpha_2,\\
S_{4\,9} &= a_1g_1 \cdot a_2e_2 \cdot b_1f_3 \cdot c_2\alpha_1 \cdot d_1\alpha_2 \cdot e_1g_2 \cdot f_1g_3,\\
S_{4\,10} &= a_1f_2 \cdot a_2f_1 \cdot b_1g_2 \cdot b_2g_1 \cdot d_1d_2 \cdot d_3\alpha_1 \cdot g_4\alpha_2,\\
S_{4\,11} &= a_1g_2 \cdot a_2g_1 \cdot b_1f_2 \cdot b_2f_1 \cdot e_1e_2 \cdot e_3\alpha_1 \cdot g_4\alpha_2,\\
S_{4\,12} &= a_1g_2 \cdot a_2e_2 \cdot b_1\alpha_1 \cdot c_1\alpha_2 \cdot d_1f_3 \cdot e_1g_1 \cdot f_2g_4,\\
S_{4\,13} &= a_1g_2 \cdot a_2e_1 \cdot b_1g_4 \cdot c_2d_3 \cdot d_2g_3 \cdot e_2g_1 \cdot f_2\alpha\,.
\end{aligned}$$

Cases (c) and (e) permit also

$$S_{4\,14} = a_1g_2 \cdot a_2e_1 \cdot b_2\alpha_1 \cdot c_1f_3 \cdot d_1\alpha_2 \cdot e_2g_1 \cdot e_3f_1,$$

transformed by σ into $a_1g_2 \cdot b_2f_1 \cdot c_1\alpha_1 \cdot c_2d_3 \cdot d_1f_3 \cdot d_2g_3 \cdot e_3\alpha_2$, which with s_1 generates an octic group; and

$$\begin{aligned}
S_{4\,15} &= \sigma t_3 & S_{4\,10}\, t_3\sigma &= a_1f_2 \cdot a_2d_2 \cdot b_1g_4 \cdot c_1\alpha_1 \cdot d_1f_1 \cdot e_1g_3 \cdot g_2\alpha_2,\\
S_{4\,16} &= \sigma t_3 & S_{4\,1}\, t_3\sigma &= a_1f_1 \cdot a_2d_2 \cdot b_1g_3 \cdot c_2\alpha \cdot d_1f_2 \cdot e_1g_4 \cdot f_3g_1,\\
S_{4\,17} &= \sigma & S_{4\,2}\, \sigma &= a_1f_2 \cdot a_2d_2 \cdot b_2d_3 \cdot c_2f_3 \cdot d_1f_1 \cdot e_2e_3 \cdot g_1\alpha\,,\\
S_{4\,18} &= \sigma, & S_{4\,4}\, \sigma &= a_1g_2 \cdot a_2e_2 \cdot b_2e_3 \cdot c_2g_3 \cdot d_2d_3 \cdot e_1g_1 \cdot f_1\alpha\,,\\
S_{4\,19} &= s_1s_3 & S_{4\,5}\, s_3s_1 &= a_1f_2 \cdot a_2d_1 \cdot b_1\alpha \cdot c_2e_3 \cdot d_2f_1 \cdot e_2f_3 \cdot g_2g_4.
\end{aligned}$$

The following relations reduce the number of the substitutions we shall have to test for s_4:

$$\begin{aligned}
&s_1s_3t_1S_{4\,1}\ t_2S_{4\,1}\ t_1s_3s_1 = \sigma S_{4\,4}\sigma,\\
&t_3S_{4\,8}\ t_2S_{4\,8}\ t_3 = S_{4\,1},\\
&t_3S_{4\,3}\ t_2S_{4\,3}\ t_3 = S_{4\,4},\\
&s_3s_1s_2S_{4\,9}\ t_3S_{4\,9}\ s_2S_{4\,9}\ t_3S_{4\,9}\ s_2s_1s_3 = S_{4\,6},\\
&t_1S_{4\,11}\ s_2S_{4\,11}\ t_1 = S_{4\,1},\\
&s_3S_{4\,12}\ t_3S_{4\,12}\ s_3 = \sigma S_{4\,2}\,\sigma,\\
&S_{4\,4}\ s_3S_{4\,4} = S_{4\,6},\\
&S_{4\,10}\ t_2S_{4\,10}\ s_2S_{4\,10}\ t_2S_{4\,10} = S_{4\,5},
\end{aligned}$$

Thus if $S_{4\,6}$ were fully discussed, $S_{4\,4}$, $S_{4\,8}$, $S_{4\,11}$, $S_{4\,1}$, $S_{4\,9}$, and $S_{4\,3}$ would require no further attention, as whatever primitive groups they might lead to would be revealed in following up completely the implications of $S_{4\,6}$. The substitutions then which we shall use for s_4 are $S_{4\,7}$, $S_{4\,6}$, $S_{4\,2}$, $S_{4\,5}$, and $S_{4\,13}$.

15. No primitive group of the sort under discussion contains $H_4 = \{H_3, S_{4\,7}\}$. For since H_4 is invariant under the substitutions of $\{t_3, t_4\}$, where $t_4 = t_1 S_{4\,7}\, t_1 = b_1e_2 \cdot b_2e_1 \cdot c_1g_2 \cdot c_2g_1 \cdot d_3g_4 \cdot e_3g_3 \cdot f_3f_4$, it is sufficient to put $s_5 = (a_1f_1) \cdots$ or $(a_1f_2) \cdots$. Now when in s_5 a_1 is replaced by f_1 or f_2, s_5 can be so taken as to come under one of the cases (a), (b), $\cdots$ of § 14. But none of the nine substitutions available are consistent with $S_{4\,7}$.

16. Let $S_{4\,6}$ be our s_4. To the six substitutions of order 2 and degree 14 in H_3 we add

$$
\begin{aligned}
s_4 &= a_1g_1 \cdot a_2e_1 \cdot b_1g_3 \cdot c_1e_3 \cdot d_2g_4 \cdot e_2g_2 \cdot f_1f_3, \\
t_4 &= b_1e_2 \cdot b_2e_1 \cdot c_1g_2 \cdot c_2g_1 \cdot d_3g_4 \cdot e_3g_3 \cdot f_1f_2, \\
t_5 &= a_1e_2 \cdot a_2g_2 \cdot b_2g_3 \cdot c_2e_3 \cdot d_1g_4 \cdot e_1g_1 \cdot f_2f_3, \\
t_6 &= a_1d_2 \cdot a_2d_1 \cdot c_1g_1 \cdot c_2g_2 \cdot d_3g_3 \cdot e_3g_4 \cdot f_1f_2, \\
t_7 &= a_2g_3 \cdot b_1d_2 \cdot b_2g_2 \cdot c_1d_3 \cdot d_1g_1 \cdot e_1g_4 \cdot f_2f_3, \\
t_8 &= a_1g_3 \cdot b_1g_1 \cdot b_2d_1 \cdot c_2d_3 \cdot d_2g_2 \cdot e_2g_4 \cdot f_1f_3.
\end{aligned}
$$

This group H_4 is transformed into itself by σ (of § 14) and by $\tau = a_1b_1 \cdot b_2c_1 \cdot d_1g_1 \cdot d_2g_3 \cdot d_3g_2 \cdot e_2e_3 \cdot f_2f_3$.

The group H_3 has the invariant subgroup

$$
\begin{aligned}
1,\ s_1s_3 &= a_1d_2 \cdot a_2d_1 \cdot b_1e_2 \cdot b_2e_1 \cdot c_1c_2 \cdot d_3e_3 \cdot g_1g_2 \cdot g_3g_4, \\
s_2t_3 &= a_1b_1 \cdot a_2e_1 \cdot b_2d_1 \cdot c_1e_3 \cdot c_2d_3 \cdot d_2e_2 \cdot g_1g_3 \cdot g_2g_4, \\
t_1t_2 &= a_1e_2 \cdot a_2b_2 \cdot b_1d_2 \cdot c_1d_3 \cdot c_2e_3 \cdot d_1e_1 \cdot g_1g_4 \cdot g_2g_3.
\end{aligned}
$$

This axial group is transformed into itself by

$$s_1s_2 = a_1c_1b_2 \cdot a_2b_1c_2 \cdot d_1d_2d_3 \cdot e_1e_2e_3 \cdot f_1f_2f_3 \cdot g_1g_2g_3,$$

and s_1 transforms the tetraedral group $\{s_1s_3, s_2t_3, s_1s_2\}$ into itself and thereby generates the group $\{s_1s_3, s_2t_3, s_1s_2, s_1\} = \{s_1, s_2, s_3\} = H_3$, of order 24. The 16 letters of the larger constituent of H_4 fall into four systems of imprimitively of four letters each in just three ways: $a_1a_2e_3g_3$, $d_1d_2d_3g_4$, $b_1c_1e_1g_1$, $b_2c_2e_2g_2$; $a_1b_1d_2e_2$, $a_2b_2d_1e_1$, $c_1c_2d_3e_3$, $g_1g_2g_3g_4$; or $a_1c_2d_1g_1$, $a_2c_1d_2g_2$, $b_1b_2d_3g_3$, $e_1e_2e_3g_4$. There are no other systems of imprimitivity in this constituent, not even systems of two letters each. Now

$$
\begin{aligned}
s_2s_4 &= a_1g_3 \cdot a_2e_3 \cdot b_1g_1 \cdot b_2c_2 \cdot c_1e_1 \cdot d_1d_3 \cdot e_2g_2 \cdot d_2g_4, \\
s_1t_4 &= a_1a_2 \cdot b_1e_1 \cdot b_2e_2 \cdot c_1g_1 \cdot c_2g_2 \cdot d_1d_2 \cdot d_3g_4 \cdot e_3g_3,
\end{aligned}
$$

and the transitive elementary group of order 16

$$\{s_1s_3, s_2t_3, s_2s_4, s_1t_4\}$$

is invariant in H_4. By the adjunction of s_1s_2, a group of order 48 is generated. The group $\{s_1s_3, s_2t_3, s_2s_4, s_1t_4, s_1s_2, s_1\} = H_4$ and is of order 96. There can not be more than $3 \cdot 16/4$ substitutions of degree 12 and order 2 in the transitive constituent of order 96. The 12 substitutions $s_1, s_2, \cdots, t_8$ are conjugate in H_4. To extend this H_4 we have only to consider the single type of substitution $(a_1f_1)\cdots$, on account of t_3 and t_4, which fix a_1. Nor need we go outside the cases (a), (b), $\cdots$ of § 14. Since $\tau S_{4\,6}\tau = S_{4\,1}$ it is sufficient to use $S_{4\,1}$ for s_5. It will be convenient however to put

$$s_5 = t_6 S_{4\,1}\; t_6 = a_1a_2\cdot b_1c_1\cdot b_2c_2\cdot d_1f_1\cdot d_2f_2\cdot d_3f_3\cdot f_4g_4.$$

The group $\{H_3, s_5\}$ has a symmetric-5 constituent in the letters $g_1g_2g_3g_4f_4$. In the other constituent of degree 15 there is an octic subgroup $\{s_1, s_2t_3\}$ which fixes f_3. This includes $s_2t_3s_1s_2t_3 = s_3$, and $\{s_1, s_2t_3, s_5\}$ includes $(s_5s_2t_3)^3 = s_2^3(s_5t_3)^3 = s_2$, so that $\{s_1, s_2, s_3, s_5\}$ coincides with $\{s_1, s_2t_3, s_5\}$. If now the symmetric group of order 120 is represented as the group according to which its 15 octic subgroups are permuted, we have the above constituent of degree 15. Let the 15 octic groups in (12345) *all* be characterized by an operator of order 4:

$$\begin{aligned} a_1 &= 1254, & a_2 &= 1245, & b_1 &= 2345, & b_2 &= 1345, & c_1 &= 2354,\\ c_2 &= 1354, & d_1 &= 1253, & d_2 &= 1235, & d_3 &= 1325, & e_1 &= 2435,\\ e_2 &= 1435, & e_3 &= 1425, & f_1 &= 1243, & f_2 &= 1432, & f_3 &= 1324; \end{aligned}$$

then 12, 13·24, and 45 give rise to

$$a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2,$$
$$a_1b_1\cdot a_2e_1\cdot b_2d_1\cdot c_1e_3\cdot c_2d_3\cdot d_2e_2,$$

and

$$a_1a_2\cdot b_1c_1\cdot b_2c_2\cdot d_1f_1\cdot d_2f_2\cdot d_3f_3,$$

respectively. Hence $\{s_1, s_2, s_3, s_5\}$ is exactly of order 120 and is a simple isomorphism between its two constituents. We next note that H_5 has an invariant subgroup of order 16 that leaves every system of imprimitivity of four letters fixed. It is generated by

$$\begin{aligned} s_1t_4 &= a_1a_2\cdot b_1e_1\cdot b_2e_2\cdot c_1g_1\cdot c_2g_2\cdot d_1d_2\cdot e_3g_3\cdot d_3g_4,\\ s_2s_4 &= a_1g_3\cdot a_2e_3\cdot b_1g_1\cdot b_2c_2\cdot c_1e_1\cdot d_1d_3\cdot d_2g_4\cdot e_2g_2,\\ s_5s_1t_4s_5 &= a_1a_2\cdot b_1g_1\cdot b_2g_2\cdot c_1e_1\cdot c_2e_2\cdot e_3g_3\cdot f_1f_2\cdot f_3f_4,\\ s_5s_2s_4s_5 &= a_1e_3\cdot a_2g_3\cdot b_1e_1\cdot b_2c_2\cdot c_1g_1\cdot e_2g_2\cdot f_1f_3\cdot f_2f_4. \end{aligned}$$

The 15 substitutions of order 2 are similar and with the identity constitute an Abelian group. This group has no substitution other than the identity in common with $\{s_1, s_2t_3, s_5\}$ and therefore the two groups together generate a group of order 1920. This clearly includes s_2, s_3, and s_4. H_4, of order 96, is its subgroup that leaves one letter fixed. The only systems of imprimitivity permitted by H_5 are $a_1a_2e_3g_3$, $d_1d_2d_3g_4$, $b_1c_1e_1g_1$, $b_2c_2e_2g_2$, and $f_1f_2f_3f_4$, so that if H_5 is a subgroup of a primitive group of higher degree, it is in a multiply transitive group of degree 21. It is known that there is no primitive group of degree 20 and class 14.

17. There should now be sought a substitution $s_6 = (\alpha f_4) \cdots$ which with H_5 will generate a doubly transitive group of degree 21. Any one of the substitutions $s_1, s_2, \cdots, t_8$ generates with s_6 an Abelian or an octic group. Since s_6 certainly displaces one of the 16 letters $a_1, a_2, \cdots, g_4$, it may be assumed that g_4 is in s_6. Because $\{s_5, s_6 = (\alpha f_4)(g_4 —) \cdots\}$ is octic, the letter that follows g_4 in s_6 is e_1, e_2, e_3, g_1 g_2, or g_3, and after transformation, if necessary, by substitutions of H_2 we may make it e_3 or g_3. But H_5 is transformed into itself by $a_1a_2 \cdot b_1c_1 \cdot b_2c_2 \cdot e_1g_1 \cdot e_2g_2 \cdot e_3g_3$, and therefore $s_6 = (\alpha f_4)(g_4e_3) \cdots$ if we like. Suppose then that s_4s_6 is of order 4; s_6 is commutative with s_1 and t_6, and in consequence fixes the 8 letters $c_1c_2d_1d_2a_1a_2g_1g_2$. Then s_4s_6 is of order 2 and

$$s_6 = a_1g_1 \cdot a_2g_2 \cdot c_1d_2 \cdot c_2d_1 \cdot e_1e_2 \cdot e_3g_4 \cdot f_4\alpha.$$

It will presently be shown that H_6 exists; that is, that H_5 is the subgroup of H_6 that leaves one letter fixed. We proceed on the assumption that H_6 exists and is of order 21·20·96.

Each subgroup of H_6 that leaves two letters fixed, as H_4, has one and only one regular subgroup of degree and order 16. Then there are just 21 of these subgroups of order 16 in H_6 and each is invariant in a subgroup of order 1920, one constituent of which is the symmetric group of degree 5. Since we have laid down the condition that G contains no substitution of degree 15 and no substitution of degree 14 and order 7, the largest subgroup of G in which the regular subgroup of degree 16, Abelian of type (1, 1, 1, 1), is invariant has no substitutions other than the 16 of the regular group on the 16 letters only of the latter group. Since the constituent in the other five letters is already a symmetric group, and since the subgroup of G that leaves two letters fixed can have but one regular elementary subgroup of order 16, G and H_6 are identical.

18. Now we seek to find a triply transitive group of degree 22 of which H_6 is a subgroup. Obviously s_7 is not $(\beta\alpha)(f_4g_4) \cdots$, because of s_6. Hence H_6 is not a subgroup of a quadruply transitive group with the present limitations. Since H_4 has a transitive constituent of degree 16, we can, after proper transformation, put $s_7 = (\beta\alpha)(f_4 —)(g_4 —) \cdots$. Since $\{s_6, s_7\}$ is octic, the letter by which s_7 replaces f_4 is fixed by s_6 and since we may use for s_7 any one of its transforms by substitutions of H_2, it is enough to consider $(\beta\alpha)(f_4f_3) \cdots$ only. We find

$$s_7 = \beta\alpha \cdot f_4f_3 \cdot g_4d_3 \cdot e_3g_3 \cdot f_1f_2 \cdot d_1d_2 \cdot a_1a_2.$$

There exists a quintuply transitive group of degree 24 and of class 16, discovered by Mathieu* and our groups H_6 and H_7 are concealed in it as transitive constituents of certain intransitive subgroups. In fact H_7 is the triply transitive constituent of that largest subgroup of Mathieu's group in which the subgroup that leaves two letters fixed is invariant. The generating substitutions of the quintuply transitive group are

$$\begin{aligned}
Z_1 &= (4, 19)(15, 12)(16, 7)(10, 8)(18, 9)(6, 13)(11, 20)(21, 22),\\
Z_2 &= (4, 16)(19, 7)(15, 10)(12, 8)(18, 11)(9, 20)(6, 21)(13, 22),\\
Z_3 &= (4, 18)(19, 9)(15, 6)(12, 13)(16, 11)(7, 20)(10, 21)(8, 22),\\
Z_4 &= (4, 12)(19, 15)(16, 8)(7, 10)(18, 13)(9, 6)(11, 22)(20, 21),\\
Y &= (3, 14, 17)\ (7, 19, 21)\ (13, 15, 12)\ (4, 10, 11)\ (9, 8, 16)\\
&\qquad (20, 18, 22),\\
U &= (2, 16, 9, 6, 8)\ (4, 3, 12, 13, 18)\ (10, 11, 22, 7, 17)\\
&\qquad (20, 15, 14, 19, 21),\\
B &= (1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12)\ (5, 10, 20, 17, 11, 22, 21, 19,\\
&\qquad 15, 7, 14),\\
A &= (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,\\
&\qquad 20, 21, 22),\\
X &= (0, \infty)\ (1, 22)\ (2, 11)\ (3, 15)\ (4, 17)\ (5, 9)\ (6, 19)\ (7, 13)\\
&\qquad (8, 20)\ (10, 16)\ (12, 21)\ (14, 18).
\end{aligned}$$

If now the substitutions $A^{-2}Z_1A^2$, $Y^{-1}A^{-2}Z_1A^2Y$, $U^{-1}Y^{-1}A^{-2}Z_1A^2YU$, $U^{-1}Z_1UY^{-1}A^{-2}Z_1A^2YU^{-1}Z_1U$, $A^{-3}Z_4A^3$, $Z_1Z_4A^{-19}BZ_1B^{-1}A^{19}Z_4Z_1$, $(B^{-2}XA^{-2}Z_1A^2XB^2Z_3)^{-1}U^{-1}Z_1Z_2Z_3Z_4A^{-4}Z_2Z_3A^4Z_1Z_2Z_3Z_4U(B^{-2}XA^{-2}Z_1A^2XB^2Z_3)$, are transformed by $2g_4\cdot 3d_3\cdot 4e_3\cdot 5\alpha\cdot 6g_1\cdot 7g_3\cdot 8a_1\cdot 9e_2\cdot 10f_2\cdot 11c_2\cdot 12f_1\cdot 13b_2\cdot 14d_2\cdot 15a_2\cdot 16f_3\cdot 17d_1\cdot 18e_1\cdot 19f_4\cdot 20c_1\cdot 21g_2\cdot 22b_1\cdot \infty\beta$, they go into s_1, s_2, s_3, s_4, s_5, s_6, s_7, respectively, multiplied by the transposition (0, 1).

19. With H_3 as before, let $S_{4\ 2}$ be our s_4:

$$\begin{aligned}
s_4 &= a_1f_2\cdot a_2f_1\cdot c_1e_1\cdot c_2e_2\cdot d_1d_2\cdot e_3f_3\cdot g_3g_5,\\
s_3s_4s_3 = t_4 &= a_1a_2\cdot b_1c_1\cdot b_2c_2\cdot d_1f_1\cdot d_2f_2\cdot d_3f_3\cdot g_4g_5,\\
s_1t_4 &= b_1c_2\cdot b_2c_1\cdot d_1f_2\cdot d_2f_1\cdot d_3f_3\cdot e_1e_2\cdot g_1g_2\cdot g_4g_5.
\end{aligned}$$

For s_5 we have to try only $(a_1g_1)\cdots$ and $(a_1g_3)\cdots$. If $s_3 = (a_1g_3)(b_1)(g_1)\cdots$, $s_5s_2s_5 = (a_1g_1)\cdots$. If $s_5 = (a_1g_1)(a_2)(g_2)\cdots$, $s_1t_4s_5s_1s_5s_1t_4 = (a_1g_1)(a_2g_2)\cdots$. If $s_5 = (a_1g_1)(b_1)(g_3)\cdots$, $s_5s_2s_5 = (a_1g_3)(b_1g_1)\cdots$. Then s_5 is $S_{4\ 3}$, $S_{4\ 9}$, $(a_1g_3)(b_1g_1)\cdots$, or $(a_1g_3)(b_1 —)(g_1 —)\cdots$. But $S_{4\ 3}$ and $S_{4\ 9}$ depend (§ 14) on $S_{4\ 6}$. If $s_5 = (a_1g_3)(b_1g_1)\cdots$, we find at once the unique substitution

$$s_5 = a_1g_3\cdot b_1g_1\cdot b_2d_1\cdot c_2d_3\cdot d_2g_2\cdot e_2g_4\cdot f_1f_3 = s_1s_3S_{4\ 6}\,s_3s_1.$$

Unless g_1 in $s_5 = (a_1g_3)(b_1 —)(g_1 —)\cdots$ is followed by c_1 or e_1, s_3 can be transformed by substitutions of H_4 into $(a_1g_1)\cdots$. Both letters are impossible.

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* Mathieu, *Liouville's Journal*, Ser. 2, Vol. 18 (1873), p. 25.

20. Let $S_{4\,5}$ be s_4:

$$\begin{aligned} s_4 &= a_1f_1\cdot b_1f_3\cdot a_2d_1\cdot c_1d_3\cdot d_2f_2\cdot e_2f_4\cdot g_1g_3,\\ s_4t_1s_4 = t_4 &= b_1d_2\cdot b_2d_1\cdot c_1f_2\cdot c_2f_1\cdot d_3f_3\cdot e_3f_4\cdot g_1g_2,\\ t_2s_4t_2 = t_5 &= a_2e_1\cdot b_1d_2\cdot c_1f_3\cdot c_2f_4\cdot d_3f_2\cdot e_3f_1\cdot g_3g_4. \end{aligned}$$

Only $(a_1g_1)\cdots$ need be used for s_5. If $s_5=(a_1g_1)(b_1)(g_3)\cdots$, $t_5s_1s_3t_4s_5s_2s_5t_4s_1s_3t_5=(a_1g_1)\ (b_1g_3)\cdots$. If $s_5=(a_1g_1)\ (a_2)\ (g_2)\cdots$, $t_1s_5s_1s_5t_4=(a_1g_1)(a_2g_2)\cdots$. Thus we are restricted to the five cases of § 11 but none of those substitutions give us anything new.

21. Let $S_{4\,13}$ be our s_4.

$$\begin{aligned} s_4 &= a_1g_2\cdot a_2e_1\cdot b_1g_4\cdot c_2d_3\cdot d_2g_3\cdot e_2g_1\cdot f_2f_4,\\ s_3s_4s_3 = t_4 &= a_2g_4\cdot b_1d_2\cdot b_2g_1\cdot c_2e_3\cdot d_1g_2\cdot e_1g_3\cdot f_1f_4. \end{aligned}$$

Therefore we have only to try two types $(a_1\ f_1)\cdots$ and $(a_1f_2)\cdots$ for s_5. All the substitutions s_5 come under cases (a), (b), $\cdots$ of § 14, and they have all been discussed.

22. Referring to § 13, we take up the study of the other group H_3, of which the substitutions of order 2 and degree 14 are s_1, s_2, t_1,

$$\begin{aligned} s_3 &= a_1d_1\cdot a_2d_2\cdot b_1e_1\cdot b_2e_2\cdot d_3e_3\cdot f_3f_4\cdot g_3g_4,\\ t_2 &= a_1e_3\cdot b_1d_3\cdot c_1d_2\cdot c_2e_2\cdot d_1e_1\cdot f_1f_4\cdot g_1g_4,\\ t_3 &= a_2e_3\cdot b_2d_3\cdot c_1e_1\cdot c_2d_1\cdot d_2e_2\cdot f_2f_4\cdot g_2g_4. \end{aligned}$$

This group H_3 is invariant under $\sigma=a_2c_2\cdot b_2c_1\cdot d_1e_3\cdot d_2e_2\cdot d_3e_1\cdot f_1f_3\cdot g_1g_3$ and $f_1g_1\cdot f_2g_2\cdot f_3g_3\cdot f_4g_4$. Then s_4 is either $(a_1g_1)\cdots$ or $(a_1g_2)\cdots$. If we use the transformation by σ as in § 14, we have the same five cases as there and by the same brief process find

$$s_4=a_1g_1\cdot a_2g_2\cdot b_1f_1\cdot b_2f_2\cdot d_3f_4\cdot e_3g_4\cdot f_3g_3.$$

It is true that two other substitutions seem possible with the present H_3; they are

$$\begin{aligned} s_4' &= a_1g_1\cdot a_2e_1\cdot b_1g_3\cdot c_1e_3\cdot d_2g_4\cdot e_2g_2\cdot f_1f_3,\\ s_4'' &= a_1g_1\cdot a_2e_1\cdot b_1g_3\cdot c_1e_3\cdot d_2g_4\cdot e_2g_2\cdot f_2f_4. \end{aligned}$$

But $a_2b_1\cdot b_2c_1\cdot d_1g_1\cdot d_2g_3\cdot d_3g_2\cdot e_2e_3\cdot f_2f_3$ transforms $\{s_2,\ s_1,\ s_4'\}$ into $\{s_1,\ s_2,\ a_1d_1\cdot a_2d_2\cdot b_1e_1\cdot b_2e_2\cdot d_3e_3\cdot f_1f_2\cdot g_3g_4\}$, the H_3 of § 14, while the latter group is transformed into $\{t_3,\ t_2,\ s_4''\}$ by $a_1c_1d_1e_3g_1d_3\cdot a_2e_1g_4g_3b_1d_2\cdot b_2e_2g_2\cdot f_1f_4f_3$.

Now H_4 is a transitive group of degree 20. If it is a subgroup of a primitive group, that group is multiply transitive, and in it there is a substitution

$(a_1g_2)\cdots$ of order 2 and degree 14. We have just seen that there is not such a substitution under our five cases. Then if $s=(a_1g_2)\cdots$ exists, $\{s, s_1\}$ is octic and $ss_1s=s_4$. But ss_1s and s_1 have no cycle in common, so that $\{s, s_1\}$ can not be octic. The substitution $s=(a_1g_2)\cdots$ therefore does not exist.

This completes the discussion of D_5, *the diedral group of order* 6 *with one regular constituent.*

23. In the group D_0 there are four substitutions of degree 14:

$$\begin{aligned} s_1 &= a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2\cdot g_1g_2,\\ s_2 &= a_1b_1\cdot c_1\alpha_1\cdot c_2\alpha_2\cdot d_1\beta_1\cdot d_2\beta_2\cdot e_1\gamma_1\cdot e_2\gamma_2,\\ t_1 &= a_1b_2\cdot a_2b_1\cdot \alpha_1\alpha_2\cdot \beta_1\beta_2\cdot \gamma_1\gamma_2\cdot f_1f_2\cdot g_1g_2,\\ t_2 &= a_2b_2\cdot c_1\alpha_2\cdot c_2\alpha_1\cdot d_1\beta_2\cdot d_2\beta_1\cdot e_1\gamma_2\cdot e_2\gamma_1. \end{aligned}$$

All the substitutions of $\{f_1f_2,\ g_1g_2,\ f_1g_1\cdot f_2g_2,\ c_1c_2\cdot\alpha_1\alpha_2,\ d_1d_2\cdot\beta_1\beta_2,\ e_1e_2\cdot\gamma_1\gamma_2,\ c_1d_1\cdot c_2d_2\cdot\alpha_1\beta_1\cdot\alpha_2\beta_2,\ c_1e_1\cdot c_2e_2\cdot\alpha_1\gamma_1\cdot\alpha_2\gamma_2\}$ transform D_0 into itself. Then s_3 is either $(a_1c_1)\cdots$ or $(a_1f_1)\cdots$.

Let $s_3=(a_1c_1)\cdots$. If $s_1s_3=s_3s_1$, $\{s_2, s_3\}$ is octic, $s_3=(a_1c_1)(a_2c_2)(b_1)(b_2)\cdots$, and $(t_1s_3)^3=1$. Therefore $s_3=(a_1c_1)(f_1—)(f_2)\cdots$ is impossible. If $s_2s_3=s_3s_2$, $s_3=(a_1c_1)(b_1\alpha_1)(a_2)(c_2)(\alpha_2)\cdots$ and $\{t_2, s_3\}$ is of order 6. Then $s_3=(b_2\delta_1)\cdots$ and replaces d_2 or β_1 by a letter new to t_2 and in s_2. This is impossible.

Now $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are both octic; $\{t_2, s_3\}$ is octic because s_3 fixes c_2 and α_1. Hence s_3 should replace α_2 by a letter fixed by t_2 and displaced by s_2, and this is impossible because s_3 fixes b_1.

Let $s_3=(a_1f_1)\cdots$. Neither s_1 nor t_1 can be commutative with s_3. The groups $\{s_1, s_3\}$ and $\{t_1, s_3\}$ are octic. Let $(s_2s_3)^3=1$. Now s_3 can be transformed into $(a_1f_1)(c_1x)(d_1y)(e_1z)(a_2)(b_1)(b_2)(f_2)(\alpha_1)(\beta_1)(\gamma_1)\cdots$, where x, y, z are new to s_2, so that we may put $x=g_1$ or δ_1. But $(a_1f_1)(c_1g_1)(g_2)\cdots$ is inconsistent with t_1. So too is $(a_1f_1)(c_1\delta_1)(d_1\delta_2)(e_1\delta_3)\cdots$. Let $(s_2s_3)^4=1$. Immediately, $s_3=(a_1f_1)(b_1g_1)(a_2)(b_2)(f_2)(g_2)\cdots$. If $\{s_1, s_3\}$ is D_2, s_3 fixes both letters of one of the three cycles (c_1c_2), (d_1d_2), (e_1e_2), and by proper transformation it can be made (e_1e_2). Thus $s_3=(a_1f_1)(b_1g_1)(c_1x_1)(c_2x_2)(d_1y_1)(d_2y_2)(z_1z_2)(a_2)(b_2)(e_1)(e_2)(f_2)(g_2)\cdots$, where $x_1, x_2, \cdots$ are new to s_1. If $x_1=\alpha_2$, β_1, or β_2, s_3 fixes c_2, d_1, or d_2. If $x_1=\gamma_1$, γ_2 is replaced by d_1, d_2, or c_2; that is, by d_1 or c_2. If by c_2, $s_3=(d_1\beta_1)(d_2\beta_2)\cdots$ or $(d_1\beta_2)(d_2\beta_1)\cdots$. And $s_3=(a_1f_1)(b_1g_1)(c_1\gamma_1)(d_1\gamma_2)(a_2)(b_2)(e_1)(e_2)(f_2)(g_2)(\alpha_1)(\alpha_2)(\beta_1)(\beta_2)\cdots$ is impossible. Then finally $s_3=(a_1f_1)(b_1g_1)(c_1\delta_1)(c_2\delta_2)(d_1\delta_3)(d_2\delta_4)\cdots$. But $\{s_2, s_3\}$ is octic. If $\{s_1, s_3\}$ is D_0', $s_3=(a_1f_1)(b_1g_1)(a_2)(b_2)(f_2)(g_2)\cdots$ unites two of the three cycles (c_1c_2), (d_1d_2), (e_1e_2). The two fixed letters can be made c_2 and d_2.

Therefore $s_3 = (a_1f_1)(b_1g_1)(c_1d_1)(e_1x_1)(e_2x_2)(a_2)(b_2)(f_2)(g_2)(\alpha_1)(\beta_1)\cdots$ and is commutative with t_2, replacing α_2 by β_2. The letters x_1 and x_2 must be new to s_1 and in t_2, so that $s_3 = (e_1\gamma_1)(e_2\gamma_2)\cdots$ or $(e_1\gamma_2)(e_2\gamma_1)\cdots$, which, however, would make it commutative with s_2.

The proof is complete that G does not include D_0 or D_0'

24. The group D_2 has the four substitutions:

$$\begin{aligned} s_1 &= a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2, \\ s_2 &= a_1b_1 \cdot c_1d_1 \cdot e_1\alpha_1 \cdot e_2\alpha_2 \cdot f_1\beta_1 \cdot f_2\beta_2 \cdot \gamma_1\gamma_2, \\ t_1 &= a_1b_2 \cdot a_2b_1 \cdot c_1d_2 \cdot c_2d_1 \cdot \alpha_1\alpha_2 \cdot \beta_1\beta_2 \cdot g_1g_2, \\ t_2 &= a_2b_2 \cdot c_2d_2 \cdot e_1\alpha_2 \cdot e_2\alpha_1 \cdot f_1\beta_2 \cdot f_2\beta_1 \cdot \gamma_1\gamma_2. \end{aligned}$$

It is transformed into itself by $\{s_1(g_1g_2),\ s_2(\gamma_1\gamma_2),\ a_1b_1\cdot a_2b_2,\ c_1d_1\cdot c_2d_2,\ e_1e_2\cdot\alpha_1\alpha_2,\ f_1f_2\cdot\beta_1\beta_2,\ a_1c_1\cdot a_2c_2\cdot b_1d_1\cdot b_2d_2,\ e_1f_1\cdot e_2f_2\cdot\alpha_1\beta_1\cdot\alpha_2\beta_2,\ a_1e_1\cdot a_2\alpha_1\cdot b_1e_2\cdot b_2\alpha_2\ c_1f_1\cdot c_2\beta_1\cdot d_1f_2\cdot d_2\beta_2\cdot g_1\gamma_1\cdot g_2\gamma_2\}$. There are therefore but the three following types to try for s_3:

$$(g_1a_1)\cdots, \qquad (g_1\gamma_1)\cdots, \qquad (g_1e_1)\cdots.$$

Let $s_3 = (g_1a_1)\cdots$. It is commutative with neither s_1 nor t_1. Then it fixes b_2. If b_1 is also fixed, $s_3 = (a_1g_1)(\gamma_1x)(\gamma_2)\cdots$, where x is new to s_2, and is a letter of t_2. After transformation by $c_1d_1\cdot c_2d_2$, $s_3 = (a_1g_1)(\gamma_1c_2)\cdots$. Because of s_1 and s_2, $s_3 = (a_1g_1)(\gamma_1c_2)(d_1)\cdots$, which is impossible with t_2. Then $s_3 = (a_1g_1)(b_1x)(a_2)(g_2)(b_2)\cdots$, where x is new to s_2 and in s_1, and therefore can be made c_2. Because of t_1, d_1 is fixed, and then because of s_1, d_2 is also fixed. Thus $\{t_2, s_3\}$ is neither of our two possible types of non-Abelian diedral groups. The primitive group G contains no substitution of degree 14 that replaces $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ by g_1 or g_2; nor $e_1, e_2, f_1, f_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ by γ_1 or γ_2.

25. Let $s_3 = (g_1\gamma_1)\cdots$. If s_3 fixes g_2 or γ_2, we can transform it into $(g_1\gamma_1)(g_2)\cdots$. Now $(s_1s_3)^3 = 1$, and after another transformation $s_3 = (g_1\gamma_1)(a_1 —)(g_2)(a_2)(b_2)\cdots$ and this means that $\{t_2, s_3\}$ is octic; therefore $s_3 = (g_1\gamma_1)(\gamma_2x)(g_2)(a_1 —)(a_2)(b_2)\cdots$. Here x is new to s_2 and t_2, and is a letter of s_1, an absurdity. If $s_3 = (g_1\gamma_1)(g_2x)(\gamma_2y)\cdots$, x is new to s_1 and t_1, and y is new to s_2 and t_2; that is, $x = \delta_1$ and $y = \delta_2$. Then s_3 is either $(g_1\gamma_1)(g_2\gamma_2)\cdots$ or $(g_1\gamma_1)(g_2\delta_1)(\gamma_2\delta_2)\cdots$, and the four diedral groups generated by s_3 and s_1, s_2, t_1, and t_2 are octic. One of the four letters a_1, a_2, b_1, and b_2 is certainly displaced by s_3, and by appropriate transformation it can be made a_1. If $s_3 = (a_1\delta_3)\cdots$ it displaces a_2, b_1, and b_2. Not all these letters can be followed by letters new to D_2. Then we can say that x in $s_3 = (a_1x)\cdots$ is a letter of D_2. Moreover it is not b_1, b_2, or a_2; but after

transformation can be c_1, c_2, or e_1. Now $s_3 = (g_1\gamma_1)(a_1c_1)\cdots$ fixes a_2, c_2, b_1, d_1, b_2 and d_2 and is impossible with s_2. Let $s_3 = (a_1c_2)(a_2)(c_1)(b_2)(d_1)\cdots$. It must replace b_1 by a letter new to s_2 and in s_1; that is, by d_2. It must omit both e_1 and e_2 or else both f_1 and f_2. If e_1 and e_2 are fixed by s_3, we may transform it by $e_1f_1\cdot e_2f_2\cdot\alpha_1\beta_1\cdot\alpha_2\beta_2$. Now $s_3 = (a_1c_2)(b_1d_2)(a_2)(c_1)(b_2)(d_1)(f_1)(f_2)(e_1x)(e_2y)\cdots$, where x and y are new to s_1 and are in s_2. Neither e is followed by an α. We can transform by $f_1f_2\cdot\beta_1\beta_2$ if necessary, and write $s_3 = (a_1c_2)(b_1d_2)(e_1\beta_1)(e_2\beta_2)(a_2)(c_1)(b_2)(d_1)(f_1)(f_2)(\alpha_1)(\alpha_2)\cdots$. Therefore two substitutions s_3 are determined:

$$s_{3\,1} = g_1\gamma_1 \cdot g_2\gamma_2 \cdot a_1c_2 \cdot b_1d_2 \cdot e_1\beta_1 \cdot e_2\beta_2 \cdot \delta_1\delta_2,$$
$$s_{3\,2} = g_1\gamma_1 \cdot g_2\delta_1 \cdot \gamma_2\delta_2 \cdot a_1c_2 \cdot b_1d_2 \cdot e_1\beta_1 \cdot e_2\beta_2.$$

Now let $s_3 = (g_1\gamma_1)(a_1e_1)(a_2)(e_2)(b_1)(\alpha_1)(b_2x)(\alpha_2y)\cdots$. Here x is new to t_1 and is in t_2, while y is new to t_2 and is in t_1. Hence y is c_1 or d_1 and we can make it c_1. It follows that s_3 replaces c_2 by a letter new to s_1 and fixes d_2. Then x is f_1 or f_2. Since transformation by $f_1f_2\cdot\beta_1\beta_2$ is possible, $x = f_1$. Now s_3 must replace c_2 by a letter new to s_1 and in t_2, but not by β_2. Therefore

$$s_{3\,3} = g_1\gamma_1 \cdot g_2\gamma_2 \cdot a_1e_1 \cdot b_2f_1 \cdot c_1\alpha_2 \cdot c_2\beta_1 \cdot \delta_1\delta_2,$$

or

$$s_{3\,4} = g_1\gamma_1 \cdot g_2\delta_1 \cdot \gamma_2\delta_2 \cdot a_1e_1 \cdot b_2f_1 \cdot c_1\alpha_2 \cdot c_2\beta_1.$$

26. Let $s_3 = (g_1e_1)\cdots$. First let s_1s_3 be of order 4. Then $s_3 = (e_1g_1)(e_2)(g_2)(\beta_1\delta_1)(\alpha x)\cdots$, and x is in s_1 and new to t_1, that is, x is f_1 or f_2. But not both $\{s_2, s_3\}$ and $\{t_2, s_3\}$ are octic. Now we have $s_3 = (g_1e_1)(g_2e_2)\cdots$ and since $\{t_1, s_3\}$ is octic it displaces an α or a β. In case $s_3 = (g_1e_1)(g_2e_2)(\alpha_1x)(\alpha_2y)\cdots$, x and y are new to s_2 and to t_2. There must be a cycle in s_3 new to s_2. If $s_3 = (g_1e_1)(g_2e_2)(\alpha_1\delta_1)(\alpha_2\delta_2)(\delta_3\delta_4)\cdots$, it displaces no other letter new to s_2, or to t_2, and therefore is impossible. Then the new cycle in s_3 displaces two of the four letters a_2, b_2, c_2, and d_2. Of these six cycles, a_2b_2 and c_2d_2 are in t_2 and the other four are conjugate. Then

$$s_3 = (g_1e_1)(g_2e_2)(\alpha_1\delta_1)(\alpha_2\delta_2)(a_2c_2)(a_1c_1)(b_1)(b_2)(d_1)(d_2)(\beta_1)(\beta_2)\cdots,$$

or finally,

$$s_{3\,5} = g_1e_1\cdot g_2e_2\cdot a_1c_1\cdot a_2c_2\cdot f_1f_2\cdot\alpha_1\delta_1\cdot\alpha_2\delta_2.$$

Now let $s_3 = (g_1e_1)(g_2e_2)(\beta_1 —)(\beta_2 —)(\gamma_1 —)(\alpha_1)(\alpha_2)(\gamma_2)\cdots$. The letters following β_1, β_2, and γ_1 are new to s_2 and t_2, and both the letters of the cycle (f_1f_2) are fixed. Because of s_2, s_3 must displace a_1 or b_1 and also c_1 or d_1. Since s_3 can be transformed by $a_1b_1\cdot a_2b_2$ and by $c_1d_1\cdot c_2d_2$,

$$s_{3\,6} = g_1e_1 \cdot g_2e_2 \cdot \beta_1\delta_1 \cdot \beta_2\delta_2 \cdot \gamma_1\delta_3 \cdot a_1a_2 \cdot c_1c_2,$$

or

$$s_{3\,7} = g_1e_1 \cdot g_2e_2 \cdot \beta_1\delta_1 \cdot \beta_2\delta_2 \cdot \gamma_1\delta_3 \cdot a_1c_2 \cdot a_2c_1.$$

27. It is now necessary to consider the groups generated by $s_{3\,i}$ and D_2, $i = 1, 2, \cdots, 7$. We recall that G contains no substitution of degree 14 that replaces g_1 or g_2 by a_1, a_2, b_1, b_2, c_1, c_2, d_1, or d_2; nor γ_1 or γ_2 by e_1, e_2, f_1, f_2, α_1, α_2, β_1, or β_2.

Consider $s_{3\,1}$. A substitution $s_4 = (g_1x) \cdots$, where x is $e_1, e_2, \cdots, \beta_2$, is transformed by $s_{3\,1}$ into $(\gamma_1y) \cdots$, where y is one of the same letters. Then uniquely $s_4 = (g_1\delta_1) \cdots$. Because of $s_{3\,1}$ and $t_3 = a_2c_1 \cdot b_2d_1 \cdot e_1\beta_2 \cdot e_2\beta_1 \cdot g_1\gamma_2 \cdot g_2\gamma_1 \cdot \delta_1\delta_2$, neither $(g_1\delta_1)(\gamma_1\delta_2) \cdots$ nor $(g_1\delta_1)(\gamma_2\delta_2) \cdots$ is possible. Then s_4 fixes γ_1, γ_2 and δ_2. If $s_4 = (g_1\delta_1)(g_2x) \cdots$, x is new to s_1 and to t_1, but is in $s_{3\,1}$, impossible. Now $s_4 = (g_1\delta_1)(g_2)(\gamma_1)(\gamma_2) \cdots$ and $(s_1s_4)^3 = 1$. We seek a substitution $s_5 = (g_1\epsilon) \cdots$, where ϵ, new to H_3, replaces in s_4 a letter of s_1. Now $t = s_{3\,1}\ s_4s_{3\,1} = (\gamma_1\delta_2)(x\epsilon)(g_1)(g_2) \cdots$ and $ts_5t = (g_1x) \cdots$, where x is a letter of D_2, not g_2, γ_1, or γ_2. This disposes of $s_{3\,1}$, and a fortiori of $s_{3\,2}$.

For $s_{3\,3}$ and $s_{3\,4}$ the eight substitutions s_4 which may replace g_1 by $e_1, e_2, \cdots, \beta_2$ reduce on transformation by substitutions of D_2 to $(g_1e_2) \cdots$ and $(g_1f_2) \cdots$. Since $s_{3\,3}s_4s_{3\,3}$ (and $s_{3\,4}s_4s_{3\,4}$ likewise) is $(\gamma_1e_2) \cdots$ or $(\gamma_1f_2) \cdots$, $s_{3\,4}$ goes out at once and s_4 in the case of $s_{3\,3}$ can only be $(g_1\delta_1) \cdots$. Just as in the preceding paragraph this, too, is impossible.

We next take up $s_{3\,5}$. There are four transitive constituents in H_3. But s_4 can replace g_1 only by the letters of the constituent $f_1f_2\beta_1\beta_2$. Since s_2 and $f_1f_2 \cdot \beta_1\beta_2$ transform $\{D_2, s_{3\,5}\}$ into itself, there is but one substitution $s_4 = (g_1f_1) \cdots$ to be considered. Now $s_1s_4 \neq s_4s_1$, because of $s_{3\,5}$. Since it was proved that $(g_1e_1) \cdots$ must be commutative with s_1, $(g_1f_1) \cdots$ must also be commutative with s_1, contradicting the last remark.

28. Only $s_{3\,6}$ and $s_{3\,7}$ remain. The arguments leading to the rejection of $s_{3\,6}$ will be seen to be equally valid for $s_{3\,7}$.

$$\begin{aligned} s_{3\,6} &= g_1e_1 \cdot g_2e_2 \cdot \beta_1\delta_1 \cdot \beta_2\delta_2 \cdot \gamma_1\delta_3 \cdot a_1a_2 \cdot c_1c_2, \\ t_3 &= \alpha_1g_1 \cdot \alpha_2g_2 \cdot f_1\delta_1 \cdot f_2\delta_2 \cdot \gamma_2\delta_3 \cdot b_1a_2 \cdot d_1c_2, \\ t_4 &= e_1g_2 \cdot e_2g_1 \cdot \beta_2\delta_1 \cdot \beta_1\delta_2 \cdot \gamma_1\delta_3 \cdot b_2b_1 \cdot d_2d_1, \\ t_5 &= \alpha_2g_1 \cdot \alpha_1g_2 \cdot f_2\delta_1 \cdot f_1\delta_2 \cdot \gamma_2\delta_3 \cdot a_1b_2 \cdot c_1d_2. \end{aligned}$$

The substitution $f_1f_2 \cdot \beta_1\beta_2 \cdot \delta_1\delta_2$ transforms H_3 into itself. There are then but the three substitutions, $(g_1\delta_3) \cdots$, $(g_1f_1) \cdots$, $(g_1\delta_1) \cdots$, to try for s_4. Let $s_4 = (g_1\delta_3) \cdots$: $s_{3\,6}\,s_4s_{3\,6} = (\gamma_1e_1) \cdots$, which has been shown to be impossible. Let $s_4 = (g_1f_1) \cdots$. It is commutative with s_1. Then $s_4 = (g_1f_1)(g_2f_2)(\alpha_1\delta_1)(\alpha_2\delta_2) \cdots$ if it is commutative with t_3 also. Since $s_{3\,6}\,s_4$ is of order 3,

s_4 displaces γ_1 or δ_3, not both. Like $s_{3\,6}$ itself two cycles of s_4 are on four of the eight letters $a_1, a_2, \cdots, d_2$. Therefore one cycle of s_4 is $(\gamma_1\epsilon)$, and s_5 can be nothing but $(g_1\epsilon)\cdots$. But $s_4s_5s_4=(f_1\gamma_1)\cdots$. Suppose that s_4 is not commutative with t_3. If $s_4=(g_1f_1)(g_2f_2)(e_1x)(e_2y)\cdots$, x and y are in s_1 and not in $s_{3\,6}$ or t_4, an impossibility. If $s_4=(g_1f_1)(g_2f_2)(\alpha_1)(\alpha_2)(e_1)(e_2)\cdots$, it fixes the letters of two cycles of s_2 and is not commutative with s_2. Let $s_4=(g_1\delta_1)\cdots$. It may be that $s_{3\,6}$ and s_4 are commutative. If so, $s_4=(g_1\delta_1)(e_1\beta\ )\cdots$. If s_4 displaces g_2, it does not replace it by e_2 (because of s_1) nor by a new letter ϵ. But $(g_1\delta_1)(g_2\delta_2)(e_1\beta_1)(e_2\beta_2)(\alpha_1)(\alpha_2)\cdots$ is impossible with s_2. If s_4 fixes g_2, it fixes also e_2, β_2, and δ_2, and s_1s_4 is of order 3. Now $s_4=(g_1\delta_1)(\alpha_1f_1)(e_1\beta_1)(a—)(b—)(c—)(d—)(g_2)(e_2)(f_2)(\alpha_2)(\beta_2)(\delta_2)\cdots$ if $t_3s_4=s_4t_3$, and

$$s_4=(g_1\delta_1)(e_1\beta_1)(\alpha_2f_2)(e_2)(g_2)(f_1)(\alpha_1)(\beta_2)(\delta_2)\cdots \text{ if } t_3s_4\neq s_4t_3.$$

All the letters of the two sets $g_1g_2e_1e_2\alpha_1\alpha_2$ and $f_1f_2\beta_1\beta_2\delta_1\delta_2$ of H_3 are accounted for, so that H_4 is certainly intransitive and s_5 can replace g_1 only by a letter ϵ of $s_4=(x\epsilon)\cdots$, x being one of the eight letters $a_1a_2b_1b_2c_1c_2d_1d_2$. A substitution of D_2 (s' say) transforms $(\delta_1x)\cdots$ into $(\delta_1x')\cdots$, where x' is fixed by s_4. Then $s_4s's_4s_5s_4s's_4=(g_1x')\cdots$, known to be impossible. Thus s_4 is not commutative with $s_{3\,6}$, nor with its transforms by s_2 or $t_2 : t_3$ and t_5. Hence finally $s_4=(g_1\delta_1)(e_2\beta_2)(e_1)(f_1)(f_2)\cdots$ or $(g_1\delta_1)(e_1)(e_2)(f_1)(f_2)\cdots$ and these too are impossible.

Hence no G contains D_2 without also containing D_5. *In what follows the only non-Abelian diedral group is D_4.*

29. There are 10 Abelian groups generated by two substitutions of degree 14.

Consider the group generated by

$$s_1 \text{ and } s_2=a_1b_1\cdot a_2b_2\cdot c_1d_1\cdot c_2d_2\cdot e_1f_1\cdot e_2f_2\cdot\alpha_1\alpha_2.$$

We may transform this group by $a_1a_2\cdot b_1b_2$ and by $a_1b_1\cdot a_2b_2$. Then only $(g_1a_1)\cdots$ and $(g_1\alpha_1)\cdots$ need be considered for s_3. Let $s_3=(g_1a_1)\cdots$. We have immediately $s_3=(g_1a_1)(g_2a_2)(b_1)(b_2)\cdots$, and since we may transform by $c_1d_1\cdot c_2d_2$ and by $e_1f_1\cdot e_2f_2$ if necessary, $s_3=(g_1a_1)(g_2a_2)(c_1x)(c_2y)(e_1z)(e_2w)(\alpha_1—)(b_1)(b_2)(\alpha_2)(d_1)(d_2)(f_1)(f_2)$, where x, y, z and w should be letters of s_1. Let $s_3=(g_1\alpha_1)\cdots$. Uniquely $s_3=g_1\alpha_1\cdot a_1\delta_1\cdot b_2\delta_2\cdot c_1\delta_3\cdot d_2\delta_4\cdot e_1\delta_5\cdot f_2\delta_6$. Next $s_4=(g_1\delta_i)\cdots$ and $s_3s_4s_3=(\alpha_1x)\cdots$, where x is a letter of s_1, not g_1 or g_2. But this, as shown above, is not possible.

Consider s_1 and $s_2=a_1b_1\cdot a_2b_2\cdot c_1d_1\cdot c_2d_2\cdot e_1e_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2$. Now s_3 is

$(e_1a_1)\cdots$ or $(e_1f_1)\cdots$. If we try the first, $s_3=(e_1a_1)(e_2a_2)\cdots$ by s_1 and $(e_1a_1)(e_2b_1)\cdots$ by s_2. If $s_3=(e_1f_1)(e_2f_2)\cdots$, $\{s_2, s_3\}$ is octic.

Consider s_1 and $s_2=a_1b_1\cdot a_2b_2\cdot c_1c_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\cdot\delta_1\delta_2$,

$$s_1 \text{ and } s_2=a_1a_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\cdot\delta_1\delta_2\cdot\epsilon_1\epsilon_2\cdot\zeta_1\zeta_2.$$

For s_3 both $(c_1a_1)\cdots$ and $(c_1d_1)\cdots$ are impossible.

30. Consider s_1 and $s_2=a_1b_1\cdot a_2b_2\cdot c_1c_2\cdot d_1d_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2$. Both $(c_1a_1)\cdots$ and $(c_1e_1)\cdots$ are impossible as before. Let $s_3=(c_1d_1)(c_2d_2)\cdots$. No s_4 replaces c_1 by one of the other 16 letters of our group D. Then s_3 has at least one new cycle $(\delta_1\delta_2)$. In fact s_4 is uniquely determined as $s_4=c_1\delta_1\cdot d_1\delta_2\cdot a_1\epsilon_1\cdot b_2\epsilon_2\cdot e_1\alpha_1\cdot f_1\beta_1\cdot g_1\gamma_1$. Since transformation by $a_1b_2\cdot a_2b_1\cdot\epsilon_1\epsilon_2$ is possible, s_5 is $(c_1\epsilon_1)\cdots$ or $(c_1\zeta_1)\cdots$, $(\zeta_1\zeta_2)$ being another cycle of s_3. If $s_5=(c_1\epsilon_1)\cdots$, $s_1s_5s_1=(c_2\epsilon_1)(c_1)\cdots$, and $s_4s_1s_5s_1s_4=(c_2a_1)\cdots$. We now have $s_5=(c_1\zeta_1)(d_1\zeta_2)(c_2)(d_2)\cdots$. Because the transitive constituent $c_1,\cdots$ is increased only by the addition of the two letters of a cycle of s_3 at this and each succeeding step, the substitutions s_1, s_2, $\cdots$ cannot in this case generate a transitive group.

Consider s_1 and $s_2=a_1a_2\cdot b_1b_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\cdot\delta_1\delta_2\cdot\epsilon_1\epsilon_2$. The reasoning of the preceding paragraph is again applicable.

31. Consider s_1 and $s_2=a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\cdot\delta_1\delta_2$. This is the last of the Abelian D groups in which s_1 and s_2 have a cycle in common. Therefore $s_3=(a_1b_1)(a_2b_2)(c_1)(c_2)(\epsilon_1\epsilon_2)\cdots$. To follow this we have $s_4=(a_1c_1)(a_2c_2)\cdots$ or $s_4=(a_1\epsilon_1)(b_1\epsilon_2)(c_1\kappa)\cdots$. If $s_4=(a_1c_1)(a_2c_2)\cdots$, $s_5=(a_1\epsilon_1)(b_1\epsilon_2)(c_1\epsilon_3)(a_2)(b_2)(c_2)\cdots$ or $(a_1\epsilon_2)(b_1\epsilon_1)(c_2\lambda)(\epsilon_3)(a_2)(b_2c_1)\cdots$. In either case the letters of the last four cycles of s_3 are letters of s_1 and s_2. Like s_5, s_6 will replace a_1 by a letter of a transitive constituent of $\{s_3, s_4\}$ of degree 3, and so on. But these steps will not lead us to a transitive group. If $s_4=(a_1\epsilon_1)(b_1\epsilon_2)(c_1\kappa)\cdots$, it can be completed as follows:

$$s_4=a_1\epsilon_1\cdot b_1\epsilon_2\cdot c_1\kappa\cdot d_1\alpha_1\cdot e_1\beta_1\cdot e_2\gamma_1\cdot f_1\delta_1. \text{ If } s_5=(a_1\kappa)\cdots,\ s_4s_1s_5s_1s_4=(a_2c_1)\cdots,$$

already discussed. If $s_5=(a_1\zeta_1)(b_1\zeta_2)\cdots$, the last four cycles are again built up from the last four cycles of s_1 and s_2. The third cycle is $(c_1\lambda)$ or $(c_2\kappa)$. It is evident that we are not going to build up a transitive group by continuing with s_6, $s_7,\cdots$.

32. Consider s_1 and $s_2=a_1b_1\cdot a_2b_2\cdot c_1d_1\cdot c_2d_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2$. For s_3 we have $(e_1a_1)\cdots$, $(e_1f_1)\cdots$, or $(e_1\alpha_1)\cdots$. It is to be remembered that *no two substitutions of degree* 14 *have a common cycle.* In the first case, immediately and uniquely, $s_3=e_1a_1\cdot e_2a_2\cdot c_1f_1\cdot c_2f_2\cdot\alpha_1\alpha_3\cdot\beta_1\beta_3\cdot\gamma_1\gamma_3$. This case can

be continued by either $(g_1a_1)\cdots$ or $(g_1\gamma_1)\cdots$. Since H_3 is transformed into itself by $a_1a_2\cdot b_1b_2\cdot e_1e_2\cdot$ and $c_1c_2\cdot d_1d_2\cdot f_1f_2$, we get

$$\begin{aligned} s_{4\,1} &= g_1a_1\cdot g_2a_2\cdot d_1f_2\cdot d_2f_1\cdot \alpha_1\alpha_4\cdot \beta_1\beta_4\cdot \gamma_1\gamma_4,\\ s_{4\,2} &= g_1a_1\cdot g_2a_2\cdot d_1f_2\cdot d_2f_1\cdot \alpha_1\alpha_4\cdot \beta_2\gamma_3\cdot \gamma_2\beta_3,\\ s_{4\,3} &= g_1\gamma_1\cdot e_1\alpha_2\cdot b_1\alpha_3\cdot f_1\beta_2\cdot d_1\beta_3\cdot a_2\alpha_4\cdot c_2\beta_4. \end{aligned}$$

Let us take up the three corresponding groups $H_{4\,1}$, $H_{4\,2}$ and $H_{4\,3}$ in turn. Because $c_1c_2\cdot d_1d_2\cdot f_1f_2$ transforms $H_{4\,1}$ into itself, only $(g_1c_1)\cdots$, $(g_1\alpha_1)\cdots$, $(g_1\alpha_4)\cdots$ need be tried for s_5. A short calculation shows that all three are impossible. With $H_{4\,2}$ there are four possibilities for s_5 : $(g_1c_1)\cdots$, $(g_1\alpha_1)\cdots$, $(g_1\alpha_4)\cdots$ and $(g_1\beta_1)\cdots$. The substitution s_5 can be completed only in the last case, $s_5 = g_1\beta_1\cdot a_2\delta\cdot b_1\gamma_3\cdot e_1\gamma_2\cdot c_1\alpha_4\cdot d_2\alpha_3\cdot f_2\alpha_2$. The transitive sets of H_5 are $c_1c_2d_1d_2f_1f_2\alpha_1\alpha_2\alpha_3\alpha_4$ and one other, $\delta a_1\cdots$. Only two forms of s_6 are to be tried, $(\delta c_1)\cdots$ and $(\delta\alpha_1)\cdots$; or, if we prefer, their transforms, $(g_1\alpha_1)\cdots$ and $(g_1\alpha_4)\cdots$. These two substitutions were inadmissible when proposed for s_5. Let us consider $H_{4\,3}$. To extend this group we have $(g_1\alpha_1)\cdots$, $(g_1\alpha_4)\cdots$, and $(g_1a_1)\cdots$. Calculation shows that there are but two substitutions s_5, one of which is $s_5 = g_1\alpha_1\cdot\gamma_2e_1\cdot b_1\gamma_3\cdot a_2\delta_1\cdot c_1\beta_4\cdot f_2\beta_2\cdot d_2\beta_3$. With this group H_5 only one substitution s_6 need be considered because H_5 has just two transitive sets, one of which is $c_1c_2d_1d_2f_1f_2\beta_1\beta_2\beta_3\beta_4$. We may put $s_6 = (\delta_1b_4)\cdots$ and this is at once seen to be impossible. The other $s_5 = g_1a_1\cdot g_2a_2\cdot\gamma_2\alpha_3\cdot\gamma_3\alpha_2\cdot f_1d_2\cdot f_2d_1\cdot\beta_1\beta_4$ goes out with $s_{4\,2}$.

33. The second possibility for s_3, $(e_1\alpha_1)\cdots$, is to be studied. Uniquely, $s_3 = e_1\alpha_1\cdot f_1\beta_1\cdot g_1\gamma_1\cdot a_1\delta_1\cdot b_2\delta_2\cdot c_1\epsilon_1\cdot d_2\epsilon_2$. The group H_3 is on the five sets of letters: $a_1a_2b_1b_2\delta_1\delta_2$, $c_1c_2d_1d_2\epsilon_1\epsilon_2$, $e_1e_2\alpha_1\alpha_2$, $f_1f_2\beta_1\beta_2$, $g_1g_2\gamma_1\gamma_2$. For s_4 we can have only $(e_1f_1)\cdots$ or $(e_1f_2)\cdots$. Now $(e_1f_2)\cdots$ is impossible because it would have to replace g_1 or γ_1 by one of the letters a_1, $a_2, \cdots, d_2$. Then $s_4 = e_1f_1\cdot e_2f_2\cdot\alpha_1\beta_1\cdot\alpha_2\beta_2\cdot a_1b_2\cdot a_2b_1\cdot\delta_1\delta_2$ or $s_4 = e_1f_1\cdot e_2f_2\cdot\alpha_1\beta_1\cdot\alpha_2\beta_2\cdot\zeta_1\zeta_2\cdot\eta_1\eta_2\cdot\theta_1\theta_2$. The sets of H_4 are $e_1e_2f_1f_2\alpha_1\alpha_2\beta_1\beta_2$, $a_1a_2b_1b_2\delta_1\delta_2$, $g_1g_2\gamma_1\gamma_2, \cdots$. Then $s_5 = (e_1g_1)\cdots$, $(e_1g_2)\cdots$, or, in case $s_4 = (\zeta_1\zeta_2)\cdots$, $(e_1\zeta_1)\cdots$. We need not have written down $(e_1g_2)\cdots$ for since $(e_1f_2)\cdots$ cannot be s_4, it cannot be s_5. Now $s_5 = (e_1g_1)(e_2g_2)(\alpha_1\gamma_1)(\alpha_2\gamma_2)(f_1)\cdots$. Considering only the first form of s_4, it is seen that s_5 must displace a_1 or b_2, and by means of s_1s_2 this letter can be made a_1. But $s_5 = (e_1g_1)(e_2g_2)(\alpha_1\gamma_1)(\alpha_2\gamma_2)(a_1c_1)\cdots$ or $(e_1g_1)(e_2g_2)(\alpha_1\gamma_1)(\alpha_2\gamma_2)(a_1d_2)\cdots$ requires eight cycles. There remains only $s_4 = e_1f_1\cdot e_2f_2\cdot\alpha_1\beta_1\cdot\alpha_2\beta_2\cdot\zeta_1\zeta_2\cdot\eta_1\eta_2\cdot\theta_1\theta_2$. Try $s_5 = e_1g_1\cdot e_2g_2\cdot\alpha_1\gamma_1\cdot\alpha_2\gamma_2\cdot\zeta_1\zeta_3\cdot\eta_1\eta_3\cdot\theta_1\theta_3$. For s_6 we have only $(e_1\zeta_1)\cdots$ and $(e_1\zeta_2)\cdots$. If $s_6 = (e_1\zeta_1)(f_1\zeta_2)(g_1\zeta_3)(ax)\cdots$, it fixes α_1, α_2, β_1, β_2, γ_1 and γ_2, while x should be in s_2 and not in s_1, an impossibility. If $s_6 = (e_1\zeta_2)(f_1\zeta_1)\cdots$, it fixes

g_2, α_1, α_2, β_1 and β_2; therefore it must replace γ_1 by a letter of one of the first four cycles of s_2, which takes us back to the case of $s_3 = (e_1a_1) \cdots$. Finally try $s_5 = (e_1\zeta_1)(f_1\zeta_2) \cdots$. This is impossible with s_1 and s_2.

The third and last possibility for s_3 is to be studied: $s_3 = (e_1f_1)(e_2f_2) \cdots$. The substitutions $s_3, s_4, \cdots$ can replace e_1 only by g_1, g_2, or a letter δ new to H_2, and any one of these substitutions is commutative with s_2. Hence no one of them adds a new letter to $\alpha_1\alpha_2$, $\beta_1\beta_2$, or $\gamma_1\gamma_2$, and therefore the group $\{s_1, s_2, s_3, \cdots\}$ is not transitive.

This disposes of the group

$$\{s_1,\ s_2 = a_1b_1\cdot a_2b_2\cdot c_1d_1\cdot c_2d_2\cdot \alpha_1\alpha_2\cdot \beta_1\beta_2\cdot \gamma_1\gamma_2\}.$$

Only two Abelian D groups, one of degree 24 and one of degree 28, and only one non-Abelian D group, of order 3 and degree 21, are left.

34. In the group D_4 there are the following three substitutions:

$$\begin{aligned} s_1 &= a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2\cdot g_1g_2,\\ s_2 &= a_1a_3\cdot b_1b_3\cdot c_1c_3\cdot d_1d_3\cdot e_1e_3\cdot f_1f_3\cdot g_1g_3,\\ t &= a_2a_3\cdot b_2b_3\cdot c_2c_3\cdot d_2d_3\cdot e_2e_3\cdot f_2f_3\cdot g_2g_3. \end{aligned}$$

There is at least one substitution s_3, similar to s_1, non-commutative with at least two of the above three substitutions. We assume without loss of generality that s_3 is not commutative with s_1 and s_2. Since this substitution s_3 may or may not connect the transitive sets of D_4, two cases arise, as $\{t, s_3\}$ is of degree 24 or 28.

35. Case I:

$$s_3 = a_2b_3\cdot a_3b_2\cdot c_1c_4\cdot d_1d_4\cdot e_1e_4\cdot f_1f_4\cdot g_1g_4.$$

In H_3 are also
$$\begin{aligned} t_1 &= a_1b_3\cdot a_3b_1\cdot c_2c_4\cdot d_2d_4\cdot e_2e_4\cdot f_2f_4\cdot g_2g_4,\\ t_2 &= a_1b_2\cdot a_2b_1\cdot c_3c_4\cdot d_3d_4\cdot e_3e_4\cdot f_3f_4\cdot g_3g_4. \end{aligned}$$

We may extend H_3 by $s_4 = a_1c_1\cdot a_2c_2\cdot a_3c_3\cdot d_4d_5\cdot e_4e_5\cdot f_4f_5\cdot g_4g_5$ or by $a_1c_3\cdot a_3c_1\cdot b_2c_4\cdot d_2d_5\cdot e_2e_5\cdot f_2f_5\cdot g_2g_5$. These two substitutions are conjugate under $a_2b_2\cdot c_1c_3\cdot c_2c_4\cdot d_1d_3\cdot d_2d_4\cdot e_1e_3\cdot e_2e_4\cdot f_1f_3\cdot f_2f_4\cdot g_1g_3\cdot g_2g_4$, which transforms s_1, s_2, s_3 into t_2, s_2, s_3, respectively. Then we use the first only. In H_4 there are also the substitutions

$$\begin{aligned} t_3 &= a_1c_4\cdot b_3c_2\cdot b_2c_3\cdot d_1d_5\cdot e_1e_5\cdot f_1f_5\cdot g_1g_5,\\ t_4 &= b_3c_1\cdot a_2c_4\cdot b_1c_3\cdot d_2d_5\cdot e_2e_5\cdot f_2f_5\cdot g_2g_5,\\ t_5 &= b_2c_1\cdot b_1c_2\cdot a_3c_4\cdot d_3d_5\cdot e_3e_5\cdot f_3f_5\cdot g_3g_5. \end{aligned}$$

If s_5 replaces a_1 by d_3, $s_5 = a_1d_3\cdot a_3d_1\cdot b_2d_4\cdot c_2x\cdot e_2y\cdot f_2z\cdot g_2w$, where x is new to H_4, y, z and w are new to H_3, but displaced by s_4. Then $s_5 = a_1d_3\cdot a_3d_1\cdot b_2d_4\cdot c_2\alpha\cdot e_2f_5\cdot f_2e_5\cdot g_2$ —, an impossibility. Then

$$s_5 = a_1d_1 \cdot a_2d_2 \cdot a_3d_3 \cdot c_4d_5 \cdot e_4e_6 \cdot f_4f_6 \cdot g_4g_6.$$

This s_5 is unique because $(a_1d_4)\cdots$ and $(a_1d_2)\cdots$ can be transformed into $(a_1d_1)\cdots$ or $(a_1d_3)\cdots$ by a substitution of H_4 that fixes a_1. Next,

$$\begin{aligned} s_6 &= a_1e_1 \cdot a_2e_2 \cdot a_3e_3 \cdot c_4e_5 \cdot d_4e_6 \cdot f_4f_7 \cdot g_4g_7, \\ s_7 &= a_1f_1 \cdot a_2f_2 \cdot a_3f_3 \cdot c_4f_5 \cdot d_4f_6 \cdot e_4f_7 \cdot g_4g_8, \\ s_8 &= a_1g_1 \cdot a_2g_2 \cdot a_3g_3 \cdot c_4g_5 \cdot d_4g_6 \cdot e_4g_7 \cdot f_4g_8. \end{aligned}$$

The groups H_6, H_7, H_8 are unique. H_8 is simply isomorphic to the symmetric group of degree 9. In fact it is the group according to which that symmetric group permutes its 36 transpositions. Because of the impossibility of a substitution $(a_1d_3)\cdots$ of degree 14, H_8 is not a subgroup of a double transitive group of the same or higher degree. H_8 contains all the permissible substitutions similar to s_1 on the letters of H_8 only, except perhaps $(a_1b_1)\cdots$, $(a_1c_2)\cdots$ and $(a_1c_3)\cdots$. Now t_4 and t_5 reduce $(a_1c_2)\cdots$ and $(a_1c_3)\cdots$ to $(a_1b_1)\cdots$. This last substitution $(a_1b_1)(a_2b_2)(a_3b_3)\cdots$ fixes all the other letters of H_3, and therefore displaces d_5, e_5, f_5 and g_5, because of s_4. In the same way s_5 demands e_6, f_6, g_6, s_6 calls for f_7, g_7, and s_7 for g_8. Hence H_8 is invariant in any larger primitive group G, if one exists. But H_8 is a complete group.

36. Case II.

$$s_3 = a_1a_4 \cdot b_1b_4 \cdot c_1c_4 \cdot d_1d_4 \cdot e_1e_4 \cdot f_1f_4 \cdot g_1g_4.$$

The case in which s_i $(i = 3, 4, \cdots)$ is non-commutative with the substitutions of such a group as D_4 and connects two of its sets of transitivity has been fully considered. Then since we know that there is a substitution similar to s_1 which replaces a_1 by a letter of another constituent of H_i $(i = 3, 4 \cdots)$, we write down immediately and uniquely:

$$\begin{aligned} s_4 &= a_1b_1 \cdot a_2b_2 \cdot a_3b_3 \cdot a_4b_4 \cdot a_5b_5 \cdot a_6b_6 \cdot a_7b_7, \\ s_5 &= a_1c_1 \cdot a_2c_2 \cdot a_3c_3 \cdot a_4c_4 \cdot a_5c_5 \cdot a_6c_6 \cdot a_7c_7, \\ s_6 &= a_1d_1 \cdot a_2d_2 \cdot a_3d_3 \cdot a_4d_4 \cdot a_5d_5 \cdot a_6d_6 \cdot a_7d_7, \\ s_7 &= a_1e_1 \cdot a_2e_2 \cdot a_3e_3 \cdot a_4e_4 \cdot a_5e_5 \cdot a_6e_6 \cdot a_7e_7, \\ s_8 &= a_1f_1 \cdot a_2f_2 \cdot a_3f_3 \cdot a_4f_4 \cdot a_5f_5 \cdot a_6f_6 \cdot a_7f_7, \\ s_9 &= a_1g_1 \cdot a_2g_2 \cdot a_3g_3 \cdot a_4g_4 \cdot a_5g_5 \cdot a_6g_6 \cdot a_7g_7, \\ s_{10} &= a_1a_5 \cdot b_1b_5 \cdot c_1c_5 \cdot d_1d_5 \cdot e_1e_5 \cdot f_1f_5 \cdot g_1g_5, \\ s_{11} &= a_1a_6 \cdot b_1b_6 \cdot c_1c_6 \cdot d_1d_6 \cdot e_1e_6 \cdot f_1f_6 \cdot g_1g_6, \\ s_{12} &= a_1a_7 \cdot b_1b_7 \cdot c_1c_7 \cdot d_1d_7 \cdot e_1e_7 \cdot f_1f_7 \cdot g_1g_7. \end{aligned}$$

This imprimitive group H_{12} of degree 49 and order $(7!)^2$ is a subgroup of one and only one primitive group G, and G is of degree 49 and order $2(7!)^2$. A substitution of G that transforms H_{12} into itself is

$a_2b_1 \cdot a_3c_1 \cdot a_4d_1 \cdot a_5e_1 \cdot a_6f_1 \cdot a_7g_1 \cdot b_3c_2 \cdot b_4d_2 \cdot b_5e_2 \cdot b_6f_2 \cdot b_7g_2 \cdot c_4d_3 \cdot c_5e_3 \cdot c_6f_3 \cdot$
$c_7g_3 \cdot d_5e_4 \cdot d_6f_4 \cdot d_7g_4 \cdot e_6f_5 \cdot e_7g_5 \cdot f_7g_6$.

37. At this point we remove the restriction that the positive subgroup of G is of class > 15. But G contains no substitution of degree 15 and order 5, for the primitive groups of class 15 containing such a substitution are of degree 16 or 17. The substitutions of degree 15 of G are all of order 3 and generate an invariant, and therefore transitive, subgroup H of G of degree > 20. For the determination of H we have precisely the conditions of our former study of the primitive groups of class 15.* The transitive groups H of class 15 (with no substitution of degree 15 and order 5), generated by substitutions of degree 15 and order 3 were found to be:

(1) A simply transitive primitive group of degree 21 simply isomorphic to the alternating group of degree 7.

(2) An imprimitive group of degree 25 simply isomorphic to the direct product of two alternating groups of degree 5.

But there was a serious oversight in that study of the groups of class 15 (as will be shown in § 39) because of which another such group is to be taken into account:

(3) The positive subgroup of the larger transitive constituent of that subgroup of Mathieu's quintuply transitive group of degree 24 in which the subgroup that fixes three letters is invariant.

38. It will now be shown that no G of class 14 contains either (1) or (2).

Suppose the group H of degree 21 is a subgroup of G. Now G contains a substitution of degree 14 and order 2 which transforms H into itself and therefore with H generates a primitive group of degree 21 and order 7!. It is easy to show that this G is of class < 14. Let G_1 and H_1 be the subgroups of G and H, respectively, which fix one letter. The subgroup H_1 is known. It is isomorphic to the symmetric group of degree 5, a complete group. Then G_1 is a direct product of H_1 and a subgroup of G_1 of order 2. Of the two transitive constituents of degree 10 of H_1, one is primitive and the other is imprimitive, so that G_1 is also intransitive, with one constituent, at least, primitive. This primitive constituent of G_1 can have no invariant substitution except the identity, and therefore remains of order 120 in G_1, as in H_1. The invariant substitution of G_1 is therefore of degree 10 on the letters of the other transitive constituent.

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* *American Journal of Mathematics*, Vol. 39 (1917), p. 281.

As to (2), the imprimitive group of degree 25 and order 3600, it has been proved[*] that the only primitive groups of degree 25 in which it is invariant are of class 10 or 15.

39. In order to conclude this investigation of the primitive groups of class 14, it is necessary to correct the error made by the author in line 20, page 284, of the paper on primitive groups of class 15. The statement "nor is H an imprimitive group of degree 16, for then its order is of necessity 48, and this case has been examined" is false. The conclusion referred to is in the second and third lines from the bottom of page 283, "the primitive groups under consideration do not contain a transitive group of degree 16 and order 48", quite true because the groups in question are generated by two substitutions of degree 15 and order 3. But the reasoning is not applicable to the groups of the next page, which have three, instead of two, generators of degree 15 and order 3. This paper was written up completely as the research progressed. It was laid aside for some weeks at a point between these two statements and on resuming the task the last quoted result was misinterpreted.

The lines quoted from page 284 are to be replaced by §§ 40-48 below.

40. The group D is tetraedral, with one transitive constituent of degree 12 and one of degree 4:

$$D = \{s_1,\ s_2 = a_1b_2c_3 \cdot b_1a_2d_3 \cdot c_1d_2a_3 \cdot d_1c_2b_3 \cdot e_1e_3x\}.$$

For the larger constituent, its conjoin might be used, but it is conjugate to the given regular group.

The group $H = \{D, s_3\}$ is transitive. Suppose it to be of degree 16. It is imprimitive. It is not in a primitive group of lower degree than 21, and then only by virtue of having 5 systems of imprimitivity of four letters each with a letter (x, say) in common. Each of the systems to which x belongs is left fixed by s_1, and these 5 systems can only be $xe_1e_2e_3$, $xa_1a_2a_3$, $xb_1b_2b_3$, $xc_1c_2c_3$, $xd_1d_2d_3$. The group $\{s_1, s_3\}$ is of order 12 and like D has transitive constituents of degrees 12 and 4. The letter following x in s_3 is one of the 12 letters of the large constituent of D, and by transformation we can make it d_2. Then $s_3 = (d_1d_3x)(d_2)(e_1 —)(e_2 —)(e_3 —)\cdots$. The 20 systems of imprimitivity of H are completely determined by D. They are

$$\begin{array}{lllll} xa_1a_2a_3, & xb_1b_2b_3, & xc_1c_2c_3, & xd_1d_2d_3, & xe_1e_2e_3, \\ e_1b_2d_3c_1, & e_1a_2c_3d_1, & e_1d_2b_3a_1, & e_1c_2a_3b_1, & d_1a_1b_1c_1, \\ e_3c_3b_1d_2, & e_3d_3a_1c_2, & e_3a_3d_1b_2, & e_3b_3c_1a_2, & d_2a_2b_2c_2, \\ e_2d_1c_2b_3, & e_2c_1d_2a_3, & e_2b_1a_2d_3, & e_2a_1b_2c_3, & d_3a_3b_3c_3. \end{array}$$

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[*] *American Journal of Mathematics*, Vol. 39 (1917), pp. 308, 309.

It may be well to note that after the first 17 systems are written down from D, the remaining system $d_1, \cdots$ has no letter except d_1 in common with any of the preceding systems, and therefore is $d_1a_1b_1c_1$. Likewise we have the systems $d_2a_2b_2c_2$ and $d_3a_3b_3c_3$. Now s_3 replaces the system $xa_1a_2a_3$ by $e_2d_1c_2b_3$, the latter by $e_1b_2d_3c_1$, and similar statements can be made for the systems $xb_1b_2b_3, \cdots, xc_1c_2c_3, \cdots$. Then

$$s_3 = (d_1d_3x)(a_1b_3e_1)(c_1a_3e_2)(b_1c_3e_3)(a_2c_2b_2),$$

completely determined by the 20 systems of imprimitivity of H.

41. This transitive group H has an elementary subgroup E of order 16 generated by

$$\begin{aligned}
e_1 &= s_1{}^2s_2 = a_1c_1\cdot a_2b_2\cdot b_1d_1\cdot c_2d_2\cdot a_3d_3\cdot b_3c_3\cdot e_1x\cdot e_2e_3,\\
d_1 &= s_1{}^2s_3 = a_1e_2\cdot a_2b_3\cdot b_1e_1\cdot b_2c_3\cdot c_1e_3\cdot c_2a_3\cdot d_1x\cdot d_2d_3,\\
e_3 &= s_2s_1{}^2 = a_1b_1\cdot a_2d_2\cdot b_2c_2\cdot c_1d_1\cdot a_3c_3\cdot b_3d_3\cdot e_3x\cdot e_1e_2,\\
d_3 &= s_3s_1{}^2 = a_1b_2\cdot a_2c_1\cdot b_1c_2\cdot b_3e_3\cdot c_3e_2\cdot e_1a_3\cdot d_1d_2\cdot d_3x.
\end{aligned}$$

As a notation for the 15 substitutions of the regular group E we use the letter which follows x in that substitution. Now s_1 transforms $\{e_1, e_3\}$ and $\{d_1, d_3\}$ each into itself. Moreover, $\{E, s_1\} = \{s_1, s_2, s_3\} = H$. Then E is invariant in H and H is of order 48. H has 16 conjugate subgroups of order 3.

That subgroup of the group of isomorphisms of the elementary group E, in which s_1 is invariant, is the transitive group

$$\begin{aligned}
I_s = \{s_1,\ & a_1b_1\cdot a_2b_2\cdot a_3b_3\cdot c_1d_1\cdot c_2d_2\cdot c_3d_3,\\
& a_1c_1\cdot a_2c_2\cdot a_3c_3\cdot b_1d_1\cdot b_2d_2\cdot b_3d_3,\\
& a_1e_1\cdot a_2e_2\cdot a_3e_3\cdot b_1d_3\cdot b_2d_1\cdot b_3d_2,\\
& a_1b_1\cdot a_2b_3\cdot a_3b_2\cdot c_2c_3\cdot d_2d_3\cdot e_1e_2\},
\end{aligned}$$

of order 360. These generators transform $\{E, s_1\}$ into itself. The group of isomorphisms of the elementary group of order 16 is isomorphic to the alternating-8 group, in which $\{(abc)\}$ is invariant in a subgroup of order 360, and $\{(abc)(def)\}$ is invariant in a subgroup of order 36, so that the order of I_s is right.

42. In the imprimitive group of degree 20 towards which we are working there are 5 systems of four letters each permuted according to the alternating-5 group. We can take for the substitution which with H generates this imprimitive group (call it H_1) a substitution of E and transform it into a substitution t which fixes an arbitrary one of the 5 systems and transforms the new system $y_1y_2y_3y_4$ into the system $x \cdots$. Transformation of H and t by substitutions of I_s permits us to write

$$t = (y_4x)(y_1e_1)(y_2e_2)(y_3e_3)(a_1—)(b_1—)(c_1—)(d_1—)(a_3)(b_3)(c_3)(d_3).$$

The four blanks in t are to be filled by a_2, b_2, c_2 and d_2 in some order as yet unknown. Now $ts_1t = (y_1y_2y_3)(x)(e_1)(e_2)(e_3)(y_4)\cdots$ is commutative with s_1, and therefore one of its cycles is the inverse of one of the cycles of s_1. The same is true if for s_1 we put $s_2 = (a_1b_2c_3)(b_1a_2d_3)(d_1c_2b_3)(c_1d_2a_3)(e_1e_3x)$, $(a_1c_2d_3)(b_1d_2c_3)(c_1a_2b_3)(d_1b_2a_3)(e_1xe_2)$, or $(a_1d_2b_3)(b_1c_2a_3)(c_1b_2d_3)(d_1a_2c_3)(e_2xe_3)$, all of D. Then one cycle of t is a_1a_2, b_1b_2, c_1c_2, or d_1d_2. Transformation by the substitution $a_1b_1\cdot a_2b_2\cdot a_3b_3\cdot c_1d_1\cdot c_2d_2\cdot c_3d_3$, $a_1d_1\cdot a_2d_2\cdot a_3d_3\cdot b_1c_1\cdot b_2c_2\cdot b_3c_3$, or $a_1c_1\cdot a_2c_2\cdot a_3c_3\cdot b_1d_1\cdot b_2d_2\cdot b_3d_3$ of I_s enable us to fix upon b_1b_2 as this cycle. The reason for this choice is that it seems to be the most convenient when we come later to connect our groups with subgroups of the quintuply transitive group of degree 24. From s_2 we see that t has either (d_1c_2) or (c_1d_2); from e_1s_1, either (a_1c_2) or (c_1a_2); from b_3s_1, either (d_1a_2) or (a_1d_2). If $t = (a_1c_2)\cdots$, the other cycles are determined by these considerations. Thus t may be

$$b_1b_2\cdot a_1c_2\cdot c_1d_2\cdot d_1a_2\cdot y_1e_1\cdot y_2e_2\cdot y_3e_3\cdot y_4x.$$

But if this t is multiplied by the substitution b_8 of E the product is of degree 12. Then

$$t = a_1d_2\cdot b_1b_2\cdot c_1a_2\cdot d_1c_2\cdot e_1y_1\cdot e_2y_2\cdot e_3y_3\cdot xy_4,$$

uniquely.

43. Consider the imprimitive group $\{E, t\}$. It has the same 5 systems of four letters each as H_1, and permuted according to the alternating group. There is a subgroup $\Delta = \{e_1, e_3, te_1t, te_3t\}$ in $\{E, t\}$, an elementary group of order 16 with 5 regular axial constituents, obviously invariant in $\{E, t\}$. Another subgroup of $\{E, t\}$ is $\Gamma = \{a_2, a_3, t\}$, of which one transitive constituent is the simple group $\{(b_3e_3)(d_1c_2), (b_3d_1)(c_2e_2), (d_1c_2)(e_2y_2)\}$ of order 60. The other transitive constituent is of degree 15 and order 60, and because of the systems of imprimitivity of $\{E, t\}$, Γ is not a direct product. Since Δ and Γ are permutable groups and have only the identity in common, $\{\Delta, \Gamma\}$ is of order 960. Now $\{\Delta, \Gamma\} = \{e_1, e_3, te_1t, te_3t, a_2, a_3, t\} = \{E, t\}$.

Because $s_1^2Es_1 = E$ and $s_1^2ts_1 = c_1ta_2tc_1$, $\{E, t\}$ is an invariant subgroup of $H_1 = \{E, s_1, t\}$, and the latter is therefore of order 2880. The head of H_1 is the intransitive subgroup $\Delta_8 = \{\Delta, V\}$ of order 48, where $V = ts_1ts_1 = a_1d_1c_1\cdot a_2d_2c_2\cdot a_3d_3c_3\cdot e_1e_2e_3\cdot y_1y_2y_3$.

The order of the subgroup F of H_1 that fixes y_4 is 144. F has two transitive constituents, one on $y_1y_2y_3$ and the other on the remaining 16 letters.

44. If H_1 is the subgroup that fixes one letter of the doubly transitive group H_2, there exists a substitution $T = (y_i z) \cdots$ which transforms H into H. T can be assumed to be a transform of one of the substitutions of E. If T displaces x, its transform by one of the substitutions of E will fix x. Since $THT = H$, T replaces no letter of H by a y_1, y_2, y_3, or y_4. Then $T = (y_i z)(x) \cdots$ transforms s_1 into s_1 or s_1^2, and 6 of its 8 cycles come from I_8; the only cycle unaccounted for must be (yy), with subscripts to be determined. The substitutions of order 2 of I_8 which permute four cycles of s_1 fall into two sets of conjugates under the substitutions of Δ_8 that fix x (the head of H_1 was called Δ_8), with the three that fix the cycle $(e_1e_2e_3)$ of s_1 in one set, and the 12 that displace $(e_1e_2e_3)$ in the other set. For example, one substitution of Δ_8 is $te_3s_1ts_1 = V_2 = b_1c_1d_1 \cdot b_2c_2d_2 \cdot b_3c_3d_3 \cdot c_1c_2c_3 \cdot y_1y_3y_4$. Another is $V_1 = a_1b_1d_1 \cdot a_2b_2d_2 \cdot a_3b_3d_3 \cdot e_1e_2e_3 \cdot y_2y_4y_3$, the transform of s_1ts_1t by te_1t. Since all the substitutions of Δ_8 that fix x are commutative with s_1, and transform E into E, we need try for T only two of these 15 possible substitutions of I_8. Similarly, of the 10 substitutions of I_8 which permute only two cycles of s_1, the 6 which fix $(e_1e_2e_3)$ are conjugate under the substitutions of Δ_8 which fix x, and the four which displace $(e_1e_2e_3)$ are conjugate under the same subgroup of the head. There are then four possibilities for T:—

$$\begin{aligned}
T_0 &= a_1e_1 \cdot a_2e_2 \cdot a_3e_3 \cdot b_1d_3 \cdot b_2d_1 \cdot b_3d_2 \cdot yy \cdot yz,\\
T_1 &= a_1b_1 \cdot a_2b_2 \cdot a_3b_3 \cdot c_1d_1 \cdot c_2d_2 \cdot c_3d_3 \cdot yy \cdot yz,\\
T_2 &= a_1e_3 \cdot a_2e_2 \cdot a_3e_1 \cdot b_2b_3 \cdot c_1c_3 \cdot d_1d_2 \cdot yy \cdot yz,\\
T_3 &= a_1b_1 \cdot a_2b_3 \cdot a_3b_2 \cdot c_2c_3 \cdot d_2d_3 \cdot e_1e_2 \cdot yy \cdot yz.
\end{aligned}$$

If $te_2t = a_1b_1 \cdot c_1d_1 \cdot a_2b_2 \cdot c_2d_2 \cdot a_3b_3 \cdot c_3d_3 \cdot y_1y_3 \cdot y_2y_4$ of Δ is multiplied by T_1, the product is of degree $\leqq 5$ and is not the identity. If te_2t is multiplied by T_3, the product is $(c_1d_1) \cdots (z \cdots) \cdots$, with a square not the identity and yet of degree $\leqq 5$. Multiply T_2 by the substitution V_1 of Δ_8; the product is

$$(a_1e_1b_3d_2)(a_2e_3b_1d_1)(a_3e_2b_2d_3)(c_2c_3)(zy)(yy)(y_2y_4y_3).$$

The product of $(zy)(yy)$ by $(y_2y_4y_3)$, where the undetermined subscripts are 1, 2, 3, or 4, is a substitution of the alternating-5 group and therefore not a cycle of four letters. It is of order 2, 3, or 5, and the degree of the second or fourth power of the partial product is $\leqq 12$, and is not the identity.

In T_0 the subscripts of the y's are to be determined. Consider $V_1T_0 = (a_1d_3e_3)(a_2d_1e_1)(a_3d_2e_2)(b_1b_2b_3)(y_2y_4y_3)(yy)(yz)$. The partial product of $(y_2y_4y_3)$ by $(yy)(yz)$ must be of order 3 and therefore T_0 fixes y_1. $V_2T_0 = (a_1e_1a_2e_2a_3e_3)(b_1c_1b_2c_2b_3c_3)(d_1d_2d_3) \cdots$. The product of $(y_1y_3y_4)$ by $(yy)(yz)(y_1)$ must be of order 2 and degree 4, hence is $(y_1y_4)(z\text{-})$, so that

$$T_0 = y_3y_4 \cdot y_2z \cdot a_1e_1 \cdot a_2e_2 \cdot a_3e_3 \cdot b_1d_3 \cdot b_2d_1 \cdot b_3d_2.$$

After transformation by te_2t, we have finally the unique substitution

$$T = y_4z \cdot y_1y_2 \cdot b_1e_1 \cdot b_2e_2 \cdot b_3e_3 \cdot a_1c_3 \cdot a_2c_1 \cdot a_3c_2.$$

45. It must next be proved that H_1 is indeed the subgroup of $H_2 = \{H_1, T\}$ that fixes z. Now

$$H_1 = F_s + F_stF_s + F_ste_2tF_s,$$

where F_s is the subgroup of H_1 that fixes y_4. We have to show that $TF_sT = F_s$, and that $(tT)^3$ and $(te_2tT)^3$ are substitutions of F_s. Now $F_s = \{H, V = s_1ts_1t\}$. But $THT = H$, as we know, and $TVT = s_1V^2$. Also,

$$tT = xzy_4 \cdot b_1e_2y_1 \cdot b_2e_1y_2 \cdot b_3e_3y_3 \cdot a_1d_2c_3 \cdot a_3c_2d_1$$

and

$$te_2tT = y_4y_3z \cdot a_1a_2a_3 \cdot b_1d_1e_1 \cdot b_2d_2e_2 \cdot b_3d_3e_3 \cdot c_1c_3c_2,$$

so that $(tT)^3 = 1$ and $(te_2tT)^3 = 1$. Then H_2 exists. Since there is no positive doubly transitive group of order $21 \cdot 20 \cdot 144$ of class < 15, H_2 is certainly of class 15, an essential point not previously proved.

Since $s_1T = Ts_1$, the doubly transitive group $\{E, t, T\}$ is invariant in H_2. Now $F = \{E, Y\}$ where

$$Y = tc_1ta_2te_2 = a_1a_2e_2 \cdot b_1b_2x \cdot c_1c_2e_1 \cdot d_1d_2e_3 \cdot a_3d_3c_3 \cdot y_1y_3y_2,$$

is the subgroup of $\{E, t\}$ that fixes y_4, and

$$\{E, t\} = F + FtF + Fte_2tF;$$

while $TET = E$, and $TYT = c_1Y^2$. We have as above $(tT)^3 = (te_2tT)^3 = 1$, so that $\{E, t, T\}$ is of order $21 \cdot 20 \cdot 48$.

46. In this connection it is easy to prove the existence of Mathieu's quintuply transitive group of degree 24.* We shall show that the group $\{E, t, T\}$ whose existence has just been proved is the subgroup that leaves three letters of Mathieu's group fixed, and that H_2 is a subgroup of the transitive constituent of the largest subgroup of Mathieu's G^{24} in which $\{E, t, T\}$ (or say G^{21}) is invariant.

The generating substitutions of G^{24} given by Mathieu are in § 18 above. Since the connection with the immediately preceding work must be made clear, we now replace Mathieu's letters x_i, or rather the subscripts i ($i = \infty$,

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* Mathieu, *Liouville's Journal*, Vol. 18 (1873), p. 25.

Miller, *Messenger of Mathematics*, Vol. 27 (1897), p. 187; *Bulletin de la Société Mathématique de France*, Vol. 28 (1900), p. 206.

De Séguier, *Groupes de substitutions* (1912), pp. 156-163.

0, 1, · · ·, 22), by ∞, 0, w, y_4, y_3, a_1, z, b_3, b_1, c_1, e_1, a_2, e_2, d_2, e_3, y_2, d_1, c_2, y_1, a_3, b_2, c_3, x, d_3, respectively. Then Mathieu's $\{Z_1, Z_2, Z_3, Z_4\}$ is precisely our group E, for $Z_1 = d_3$, $Z_2 = b_3$, $Z_3 = a_2$, $Z_4 = c_3$, and his Y is our Y. Moreover,

$$\begin{aligned} U &= y_4c_2e_1b_3c_1 \cdot y_3d_2e_3a_3a_1 \cdot y_1a_2e_2d_3b_1 \cdot y_2b_2xc_3d_1 \\ &= (b_2Yb_2te_1)^3, \end{aligned}$$

a substitution of $\{E, t\}$. And t is in $\{E, Y, U\}$, for $t = b_2Y^2b_2U^2e_1$. Thus it is proved that $\{E, t\} = \{E, Y, U\}$.

Mathieu gives no substitution that with G^{20} generates G^{21}, but the substitution $YBa_3B^{-1}Y^{-1}$ fixes ∞, 0 and w, and is our T. From Mathieu's set of generators we also get

$$\begin{aligned} T' &= Y^{-1}B^{-7}c_2B^7Y \\ &= zw \cdot y_1y_2 \cdot a_2d_3 \cdot a_3b_2 \cdot b_1d_1 \cdot b_3c_2 \cdot c_3d_2 \cdot e_2e_3, \\ T'' &= A^{-2}d_3A^2b_3 \\ &= wo \cdot y_1y_2 \cdot a_1c_2 \cdot a_2c_1 \cdot a_3c_3 \cdot b_1b_2 \cdot d_1d_2 \cdot e_1e_2, \\ T''' &= Ye_2TA^{-1}XT''XATe_2tY^{-1}e_2 \\ &= 0\,\infty \cdot y_1y_2 \cdot a_1c_1 \cdot a_2c_3 \cdot a_3c_2 \cdot b_2b_3 \cdot d_2d_3 \cdot e_2e_3. \end{aligned}$$

Since

$$\begin{aligned} T'''T'' &= wo\,.\,\infty \cdot a_1a_2a_3 \cdot b_1b_2b_3 \cdot c_1c_2c_3\,.\,d_1d_2d_3 \cdot e_1e_2e_3, \\ T''T' &= zwo \cdot a_1b_3c_2 \cdot a_2c_1d_3 \cdot a_3d_2b_1 \cdot b_2d_1c_3 \cdot e_1e_3e_2, \\ T'T &= y_4zw \cdot a_1c_3d_2 \cdot a_2d_3c_1 \cdot a_3e_2b_3 \cdot b_1d_1e_1 \cdot b_2c_2e_3, \end{aligned}$$

the existence of G^{24} will be proved as soon as it is shown that

$$\begin{aligned} &T'\{E, t\}T' = \{E, t\}, \\ &T''\{E, t, T\}T'' = \{E, t, T\}, \\ &T'''\{E, t, T, T'\}T''' = \{E, t, T, T'\}. \end{aligned}$$

This is established as follows:

$$\begin{aligned} &T'ET' = T''ET'' = T'''ET''' = E, \\ &T'tT' = T'''tT''' = tb_3Y^2b_3,\ T''tT'' = t, \\ &T''TT'' = T'''TT''' = T, \\ &T'''T'T''' = T'. \end{aligned}$$

In the course of the above proof of the existence of the quintuply transitive group G^{24} we found a substitution $T'''T''$ of order 3 which transforms $\{E, t, T\}$ into itself, and which except for the cycle $(wo\,\infty)$ is our s_1. The transitive constituent of degree 21 of $\{G^{21}, T'''T''\}$ is H_2.

It should be noted that G^{24} is of class 16.

47. It remains to be seen if H_2 is a subgroup of a larger primitive group of class 15. Suppose it a subgroup of a group G of the same degree. Because of H, G is not triply transitive, and G_1 (the subgroup that fixes z) has the same 5 systems of imprimitivity as H_1. The transitive constituents

of the head of G_1 cannot be symmetric groups, for then the class of G would be $\leqq 10$. If the 5 alternating-4 groups of the head are combined to make its order > 48, there must be more substitutions of order 2 and of degree 16 or 20, and this is impossible because such a substitution would have at least four cycles in common with a substitution of Δ and the resulting product would be of degree $\leqq 12$. The head of G_1 is Δ_s of order 48. Then the group in the systems is the symmetric group of order 120.

In H_1 there are 96 subgroups of order 5, and in H_1, $\{U\}$, for example, is invariant in the subgroup $\{U, c_3, e_2V_1e_2\}$ of order 30. According to Sylow's Theorem, G_1 has 96 subgroups of order 5 and each is invariant in a subgroup I_u of order 60. Suppose that I_u has substitutions which transform U into U^2 without permutation of cycles. One such substitution (Q) fixes y_4 and therefore as a substitution of the imprimitive group G_1 permutes y_1, y_2 and y_3 among themselves, that is, fixes y_1, y_2 and y_3. Therefore

$$Q = c_2e_1c_1b_3 \cdot d_2e_3a_1a_3 \cdot a_2e_2b_1d_3 \cdot b_2xd_1c_3.$$

Now $b_2Q = a_1a_2c_1d_2b_1c_2 \cdot a_3b_3d_3 \cdot e_1e_2e_3 \cdot b_2d_1$, and $(b_2Q)^3$ is of degree 8. Then all the substitutions of I_u not in the invariant subgroup $\{U, c_3\}$ permute cycles of U according to a group of order 6, the symmetric-3 group. A substitution of I_u which permutes two cycles of U cannot be commutative with U, for then it or its fifth power would be of degree 10. The known subgroup $\{U, e_2V_1e_2\}$ of I_u shows that the cycle $(y_1 \cdot \cdot \cdot)$ of U is fixed by I_u. Then a substitution which permutes two cycles of U is $W = (y_2y_3)(y_4)(y_1) \cdot \cdot \cdot$, with the property of transforming U into U^4 or U^2. No one of the 30 substitutions of I_u which permute two cycles of U is commutative with U. Now c_3 transforms U into its inverse. If W does the same, Wc_3 is commutative with U, and therefore W is in $\{U, c_3, e_2V_1e_2\}$. Then $W^{-1}UW = U^2$. The subgroup $\{U, W\}$ has one transitive constituent of degree 10 and two of degree 5 each. The transitive constituent of degree 10 is completely determined by U and the given cycle (y_2y_3) of W, and W is completely determined as to the other two constituents because y_1 and y_4 are fixed by it. Thus

$$W = c_2e_1c_1b_3 \cdot a_2e_2b_1d_3 \cdot y_2y_3 \cdot d_2xa_1c_3 \cdot b_2e_3d_1a_3.$$

Since $W^2 = c_3$, $W^3EW = E$, $W^3s_1W = a_2s_1$, $W^3tW = a_1ta_1$, $W^3TW = VTV^2$, $\{H_2, W\}$ is a doubly transitive group of order $21 \cdot 20 \cdot 288$. There is no doubly transitive group of this order among the groups of class < 14. Now

$$Vd_2WV^2 = a_1c_2 \cdot a_2c_1 \cdot a_3c_3 \cdot b_1b_2 \cdot c_1c_2 \cdot e_1e_2 \cdot y_1y_2,$$

so that $\{H_2, W\}$ is actually of class 14. There is therefore no group of class 15 and degree 21 of which our H_2 is a subgroup.

The substitution Vd_2WV^2 above is $(wo)T''$, so that $\{H_2, W\}$ is the transitive constituent of degree 21 of the largest subgroup $\{E, t, T, T'', T'''\}$ of G^{24} in which $\{E, t, T\}$ is invariant. We have shown that $\{H_2, W\}$ is the only primitive group of class 14 in which H_2 is invariant.

48. Can H_2 be the subgroup that leaves one letter fixed of a triply transitive group H_3 of degree 22? If H_3 exists, there is a substitution $T_1 = (zw)(y_4) \cdots$ of order 2 and degree 16 in H_3 which transforms $H_1 = \{E, S_1, t\}$ into itself. Now H_1 has only one system of imprimitivity of four letters including y_4, $y_1y_2y_3y_4$, and therefore T_1 permutes the three letters y_1, y_2 and y_3 among themselves. T_1 can be chosen to fix x. This T_1 transforms E into E and $\{s_1\}$ into $\{s_1\}$. Then T_1 has 6 cycles from I_3 and may now be written $(zw)(y_1y_2)(y_4)(y_3)(x) \cdots$. Since T_1 respects the systems of four letters each of H_1, only three substitutions of order 2 of I_3 are available for T_1: $a_1b_1 \cdot c_1d_1 \cdot a_2b_2 \cdot c_2d_2 \cdot a_3b_3 \cdot c_3d_3$, $a_1c_1 \cdot b_1d_1 \cdot a_2c_2 \cdot b_2d_2 \cdot a_3c_3 \cdot b_3d_3$ and $a_1d_1 \cdot b_1c_1 \cdot a_2d_2 \cdot b_2c_2 \cdot a_3d_3 \cdot b_3c_3$. But these three substitutions occur in substitutions of Δ, which, multiplied by T_1, would lower the class of H_3 to 4 or 6. There is no group H_3 of degree 22 with H_2 as a subgroup of index 22.

49. At line 32 of page 288 of the paper on Class 15 all the partitions of the degree of the subgroup F have been rejected except two: 12, 9 and 9, 12. There is a slip in the discussion of these two cases which may be corrected by noting that the conditions of the following theorem are fulfilled. By the notation $G(x)$ is meant the subgroup of G that leaves fixed the letter x, and by $G(x)(a)$ the subgroup of $G(x)$ that fixes a.

Let G be a transitive group whose $G(x)$ has just two transitive constituents, A on the $m+h$ letters $a, a', \ldots$ and B of a lower degree m. Let $G(x)(a)$, of degree $\geqq 2m$, have no constituent of degree $< h$, but at least one transitive constituent of degree m. Then G is imprimitive.

The theorem is true if $m < 3$.* Then it is assumed that $m \geqq 3$. Let $b_1, b_2, \cdots, b_m$ be the letters of B.

Suppose the theorem false. Then $G(x)$ is a maximal subgroup of G. Since A is the only constituent of $G(x)$ of its degree, it is paired with itself and G contains a substitution $S = (xa) \cdots$† which transposes $G(x)$ and $G(a)$ and transforms $G(x)(a)$ into itself. S cannot permute the m letters of B among themselves for then $G(a)$ and $G(x)$ would generate an intransitive group, a subgroup of G, contrary to the assumption that $G(x)$ is maximal. If S transforms the given transitive constituent of degree m of $G(x)(a)$ into itself, its letters are certainly $a_1, a_2, \cdots, a_m$ of the constituent A. All the other $h-1$ letters $a', \cdots$ of A are fixed by $G(x)(a)$, for if displaced they would occur in transitive constituents of degree $< h$, contrary to hypothesis. Then because the degree of $G(x)(a)$ is $\geqq 2m$, it displaces all the letters of B and permutes them among themselves, which is impossible.

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* Miller, *Proceedings of the London Mathematical Society*, Vol. 38 (1897), p. 533.

† Burnside, *Proceedings of the London Mathematical Society*, Vol. 33 (1901), p. 162.

But $G(x)(a)$ may have two transitive constituents of degree m, transposed by S. Then one is on $b_1, b_2, \cdots, b_m$ and the other is on $a_1, a_2, \cdots, a_m$. Now $h \geqq 2$, for if $h = 1$, A is doubly transitive, which is impossible when G is primitive.[*] The other letters of $G(x)$ are as before all fixed by $G(x)(a)$, so that $G(x)(a) = G(x)(a')$. In this case both $G(a)$ and $G(a')$ have transitive constituents of degree m on $a_1, a_2, \cdots, a_m$. The group generated by $G(a)$ and $G(a')$ is intransitive. But $G(a)$, a conjugate of $G(x)$, is a maximal subgroup of G.

Hence $G(x)$ cannot be a maximal subgroup of G, and G is therefore imprimitive.

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[*] *Transactions of the American Mathematical Society*, Vol. 20 (1919), p. 66, Th. 16.

Summability of Infinite Products.

By George M. Robison.

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1. *Introduction.* Among the definitions giving a value to a divergent series are those which are expressed by means of a linear transformation. Let $s_1, s_2, \cdots, s_n, \cdots$ be the sequence corresponding to the infinite series $\sum_{n=1}^{\infty} u_n$. Form a new sequence t_n, where $t_n = \sum_{k=1}^{n} a_{n,k} s_k$. If the limit of t_n, as n becomes infinite, exists, we say that this limit is the generalized value of the series $\sum_{n=1}^{\infty} u_n$ by the transformation $A:(a_{n,k})$. For such a definition to be useful, it must include convergence; that is, if we apply it to a convergent series, the value assigned to the series must be the same as the convergent value. If a transformation A satisfies this condition, we say that it is regular. The conditions, which the terms $(a_{n,k})$ of a transformation must satisfy in order that it be regular, are stated in the Silverman-Toeplitz Theorem.*

The theory of infinite products furnishes us an analogous problem. From the given infinite product $\prod_{n=1}^{\infty} (1 + u_n)$, form a sequence P'_n as follows: $P'_n = \prod_{k=1}^{n} (1 + a_{n,k} \cdot u_k)$, where $a_{n,k}$ is the element in the nth row and kth column of a triangular matrix of numbers defining the transformation A. If $\lim_{n\to\infty} P'_n$ exists, we define this limit to be the value assigned to the given infinite product by the transformation $A:(a_{n,k})$. Let us consider the following illustration:

To the infinite product $\prod_{n=1}^{\infty} [1 + (-2)]$ let us apply the transformation $A:(a_{n,k} = n - k + 1/n)$. The resulting sequence is as follows:

$$
\begin{aligned}
P'_n &= 0, && n \text{ even,} \\
P'_n &= (-1)^{(n+1)/2} 1^2 \cdot 3^2 \cdot 5^2 \cdots (n-1)^2/n^{n-1} && n \text{ odd,}
\end{aligned}
$$

where $\lim_{n\to\infty} P'_n = 0$. Thus, zero is the value assigned to the above divergent infinite product by the transformation $A:(a_{n,k} = n - k + 1/n)$.

2. *Regularity of transformations.* A transformation $A:(a_{n,k})$ is said to

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* L. L. Silvermann, Missouri Dissertation, (1910); Toeplitz, *Prace Matematyczno-fizyczne*, Vol. 22 (1911), p. 113.

be regular, if it gives to every convergent infinite product the value to which it converges. With regard to the regularity of these transformations, we have the following theorems:

THEOREM I. *A necessary condition that a transformation* $A:(a_{n,k})$ *be regular is*

$$1)\quad \lim_{n\to\infty} a_{n,k} = 1 \qquad \textit{for each } k;$$

$$2)\quad | a_{n,k} | < M,$$

where M *is some constant.*

Proof. Let us consider the infinite product $\prod_{n=1}^{\infty}(1+u_n)$, where $u_n = u_k$ for $n = k$; $u_n = 0$, $n \neq k$. Thus, $P_n = \prod_{k=1}^{n}(1+u_n) = 1 + u_k$, if $n > k$, otherwise $P_n = 1$. Applying transformation $A:(a_{n,k})$, we have

$$P'_n = 1 + a_{n,k}\cdot u_k, \qquad n > k.$$

From definition, $P = 1 + u_k$. If $\lim_{n\to\infty} P'_n = P$, it is necessary that $\lim_{n\to\infty} a_{n,k} = 1$.

To show that condition (2) is necessary:

Under the assumption that the transformation $A:(a_{n,k})$ is not bounded, we shall transform a convergent product into a divergent sequence. We shall assume that the elements of the transformation satisfy condition (1) of the theorem. Hence, we can find a set of integers

$$r_0 < n_0 < r_1 < n_1 < r_2 < n_2 < \cdots$$

such that

$$\begin{array}{lll} | a_{n_0,r_0} | > 4, & & \\ | a_{n_1,r_1} | > 4^3, & | a_{n_1,r_0} | < 2, & \\ | a_{n_2,r_2} | > 4^5, & | a_{n_2,r_1} | < 2, & | a_{n_2,r_0} | < 2, \end{array}$$

and in general,

$$| a_{n_i,r_i} | > 4^{2i+1}, \quad | a_{n_i,r_l} | < 2, \qquad l = 0, 1, 2, \cdots, i-1.$$

We shall transform, by the above transformation, the convergent infinite product $\prod_{n=1}^{\infty}(1+u_n)$, where

$$u_n = 0,\quad n \neq r_0,\quad n \leqq n_0;\quad u_{r_0} = (1/4)\operatorname{sgn} a_{n_0,r_0}$$

$$u_n = 0,\quad n \neq r,\quad n_0 < n \leqq n_1;\quad u_{r_1} = (1/8)\operatorname{sgn} a_{n_1 r_1}.$$

In general

$$u_n = 0,\quad n \neq r_i,\quad n_{i-1} < n \leqq n_i;\quad u_{r_i} = (\tfrac{1}{2}{}^{i+2})\operatorname{sgn} a_{n_i,r_i}.$$

We shall show that a certain sub-sequence of the sequence P'_n has no limit.

$$P'_{n_0} = \prod_{r=1}^{n_0} (1 + a_{n_0,r} \cdot u_r) = 1 + a_{n_0,r_0} u_{r_0} = 1 + (1/4) \cdot | a_{n_0,r_0} | > 2,$$

$$P'_{n_1} = \prod_{r=1}^{n_1} (1 + a_{n_1 r} \cdot u_r) = (1 + a_{n_1} u_{r_0})(1 + a_{n_1,r_1} \cdot u_{r_1}),$$

$$| P'_{n_1} | \geqq (1/2) \cdot (1 + 8) > 4,$$

$$P'_{n_2} = \prod_{r=1}^{n_2} (1 + a_{n_2,r} \cdot u_r)(1 + a_{n_2,r_0} \cdot u_{r_0})(1 + a_{n_2,r_1} \cdot u_{r_1}) \cdot (1 + a_{n_2,r_2} \cdot u_{r_2}),$$

$$| P'_{n_2} | \geqq (1/2) \cdot (3/4)[1 + (4^5/2^4)],$$

and in general

$$P'_{n_i} = (1 + a_{n_i,r_0} \cdot u_{r_0})(1 + a_{n_i,r_1} \cdot u_{r_1})(1 + a_{n_i,r_2} \cdot u_{r_2}) \cdots (1 + a_{n_i,r_i} \cdot u_{r_i}),$$

$$| P'_{n_i} | \geqq \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdot \,\cdot\,\cdot \frac{2^i - 1}{2^i}\left(1 + \frac{4^{2i+1}}{2^{i+2}}\right) \geqq \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdot \,\cdot\,\cdot \frac{2^{i-1}}{2^i} \cdot \frac{4^{2i+1}}{2^{i+2}}$$

$$\geqq \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdot \,\cdot\,\cdot \frac{2^{i-1}}{2^i}\, 2^{2i+1} \cdot 2^{i-1} \geqq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \,\cdot\,\cdot \frac{1}{2} \cdot 2^{2i+1} \cdot 2^{i-1} \geqq 2^{2i},$$

$$| P'_{n_i} | \geqq 2^{2i}.$$

$\therefore \lim\limits_{i \to \infty} | P'_{n_i} | = \infty$, thus showing the necessity of condition (2).

THEOREM II. *A sufficient condition that* $\lim\limits_{n \to \infty} P_n'$ *shall exist and be equal to* $\lim\limits_{n \to \infty} P_n$, *where* $P_n = \prod_{k=1}^{n} (1 + u_k)$, *and where* $\sum_{k=1}^{\infty} | u_k |$ *converges is that* 1) $\lim\limits_{n \to \infty} a_{n,k} = 1$, *for each* k; 2) $| a_{n,k} | < M$, M some constant.

Proof. Let us write

$$P_n' = \prod_{k=1}^{n} (1 + a_{n,k} \cdot u_k) = \prod_{k=1}^{n} (1 + u_k + \delta_{n,k} \cdot u_k), \qquad \delta_{n,k} = a_{n,k} - 1,$$

$$P_n' = P_n + P_n \cdot \sum_{k=1}^{n} \cdot \delta_{n,k} \cdot [u_k/(1 + u_k)]$$

$$+ P_n \cdot \sum_{\substack{k=1,\, l=1 \\ k \neq l}}^{m,n} \delta_{n,k} \cdot \delta_{n,l}\, [u_k \cdot u_l/(1 + u_k)(1 + u_l)]$$

$$+ \cdots + \prod_{k=1}^{n} \delta_{n,k} \cdot u_k.$$

us call

$$\sum_{k=1}^{n} \delta_{n,k} \cdot u_k/(1 + u_k) = Q_{n,1}, \text{ etc., and}$$

$$V_n = \sum_{k=1}^{n} | \delta_{n,k} \cdot u_k/(1 + u_k) | = \sum_{k=1}^{n} | \delta_{n,k} \cdot u_k | \cdot 1/| 1 + u_k |$$

$_{n,1} | \leqq V_n, \qquad | Q_{n,2} | \leqq V_n^2/2, \cdots, \qquad | Q_{n,p} | \leqq V_n^p/p!, \cdots.$

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Substituting in the above, we have

$$|P_n' - P_n| \leqq |P_n| \{V_n + (V_n^2/2) + \cdots + (V_n^n/n!)\}$$
$$|P_n' - P_n| \leqq |P_n| \cdot [e^{V_n} - 1].$$

A sufficient condition that $\lim_{n\to\infty} P_n' = \lim_{n\to\infty} P_n$, provided the latter limit exists, is that $\lim_{n\to\infty} V_n = 0$. If $\lim_{n\to\infty} P_n$ exists $\neq 0$, $\lim_{n\to\infty} u_n = 0$.

Hence, at most, only a finite number of u_n can be found such that $R(u_n) \leqq -1$.

(a) We shall consider only infinite products, where $R(u_n) > -1$

$$\therefore \quad |1 + u_k| > 1 + L, \text{ where } R(u_n) > L > -1$$

$$\therefore \quad V_n = \sum_{k=1}^{n} |\delta_{n,k} \cdot u_k| \, 1/|1 + u_k| \leqq 1/1 + L \sum_{k=1}^{n} |\delta_{n,k} \cdot u_k|$$

If $\lim_{n\to\infty} \sum_{k=1}^{n} |\delta_{n,k} \cdot u_k| = 0$, $\lim_{n\to\infty} V_n = 0$.

It can be shown that $\lim_{n\to\infty} \sum_{k=1}^{n} |\delta_{n,k} \cdot u_k|$ is zero, provided $|\delta_{n,k}| < M$ and $\lim_{n\to\infty} \delta_{n,k} = 0$ for each k.

This proof can be modified to hold in the case of a finite number of the $R(u_n)$ being < -1.

Hence, the sufficiency of the above condition.

3. *Summability of Double Infinite Products.* The definitions of the first section can be extended to double infinite products as follows:

Let $P_{m,n} = \prod_{k=1,\, l=1}^{m,n} (1 + u_{k,l})$ represent the sequence corresponding to the infinite product $\prod_{k=1,\, l=1}^{\infty\infty} (1 + u_{k,l})$. A new sequence $P'_{m,n}$ is defined by the relation

$$P'_{m,n} = \prod_{k=1,\, l=1}^{m,n} (1 + a_{m,n,k,l} \cdot u_{k,l}),$$

where $A:(a_{m,n,k,l})$ is a given set of constants. If $\lim_{m,n\to\infty} P'_{m,n}$ exists we define this limit to be the value assigned to the infinite product by the transfor tion $A:(a_{m,n,k,l})$.

A transformation is said to be regular if, when applied to a con infinite product, the value assigned is the same as its convergent val shall now prove certain theorems with regard to the regularity transformations.

THEOREM III. *A necessary condition that a transformation* $A:(a_{m,n,k,l})$ *be regular is that*

1) $\lim\limits_{m,n\to\infty} a_{m,n,k,l} = 1$ *for each* k *and* l;

2) $|a_{m,n,k,l}| < M$ *where* M *is some fixed constant.*

Proof. Let us consider the infinite product

$$\prod_{m=1,n=1}^{\infty\infty} (1 + u_{mn}) \text{ where } u_{p,r} = u_{p,r} \neq 0,$$

p,r being two fixed integers, all other $u_{m,n} = 0$.

Applying the transformation, we have

$$P'_{m,n} = 1 + a_{m,n,p,r} \cdot u_{p,r}.$$

From definition, $$P = \lim_{m,n\to\infty} P_{m,n} = 1 + u_{p,r}.$$

Therefore, if $\lim\limits_{m,n\to\infty} P'_{m,n} = P$, it is necessary that

$$\lim_{m,n\to\infty} a_{m,n,p,r} = 1.$$

To show that condition (2) is necessary: Under the assumption that the terms of transformation $A:(a_{m,n,k,l})$ are not bounded, we shall transform a convergent product into a divergent sequence. We shall assume that the transformation satisfies condition (1) of the theorem.

Since the condition

$$|a_{m,n,k,l}| < M$$

is not satisfied, we can find a set of integers

$$(m_0, n_0, p_0, r_0), (m_1, n_1, p_1, r_1) \cdots (m_r, n_r, p_r, r_r) \text{ where}$$
$$p_0 < m_0 < p_1 < m_1 < p_2 < m_2 \cdots$$
$$r_0 < n_0 < r_1 < n_1 < r_2 < n_2 \cdots$$

such that

$$|a_{m_0,n_0,p_0,r_0}| > 4,$$
$$|a_{m_1,n_1,p_1,r_1}| > 4^3, \qquad |a_{m_1,n_1,p_0,r_0}| < 2,$$
$$|a_{m_2,n_2,p_2,r_2}| > 4^5, \qquad |a_{m_2,n_2,p_1,r_1}| < 2, \qquad |a_{m_2,n_2,p_0,r_0}| < 2,$$

and in general,

$$|a_{m_i,n_i,p_i,r_i}| > 4^{2i+1}, \qquad |a_{m_i,n_i,p_i,r_i}| < 2, \qquad j = 0, 1, 2, \cdots, i-1.$$

We shall define the sequence of $u_{m,n}$ as follows:

$$\begin{array}{lllll}
u_{m,n} = 0, & m \neq p_0, & n \neq r_0, & m \leqq m_0, & n \leqq n_0, \\
u_{m,n} = 0, & m \neq p_0, & n = r_0, & m \leqq m_0, & \\
u_{m,n} = 0, & m = p_0, & n \neq r_0, & n \leqq n_0, & \\
u_{mn} = \tfrac{1}{4} \operatorname{sgn} a_{m_0, n_0, p_0, r_0}, & & & m = p_0, \quad n = r_0, & \\
u_{m,n} = 0, & m \neq p_1, & n \neq r_1, & m_0 < m \leqq m_1, & n_0 < n \leqq n_1, \\
u_{m,n} = 0, & m \neq p_1, & n = r_1, & m_0 < m \leqq m_1, & \\
u_{m,n} = 0, & m = p_1, & n \neq r_1, & n_0 < n \leqq n_1, & \\
u_{mn} = \tfrac{1}{8} \cdot \operatorname{sgn} a_{m_1, n_1, p_1, r_1} & & & m = p_1, \quad n = r_1. &
\end{array}$$

In general

$$\begin{array}{lllll}
u_{m,n} = 0, & m \neq p_i, & n \neq r_i, & m_{i-1} < m \leqq m_i, & n_{i-1} < n \leqq n_i; \\
u_{mn} = 0, & m = p_i, & n \neq r_i, & n_{i-1} < n \leqq n_i, & \\
u_{m,n} = 0, & m \neq p_i, & n = r_i, & m_{i-1} < m \leqq m_i, & \\
u_{m,n} = \tfrac{1}{2}i + 2 \cdot \operatorname{sgn} a_{m_i, n_i, p_i, r_i}. & & & &
\end{array}$$

From the above, we have

$$\lim_{m,n \to \infty} P_{mn} = \lim_{m,n \to \infty} \prod_{p=1, r=1}^{m,n} (1 + u_{p,r}) \text{ exists, for } \sum_{p=1, r=1}^{\infty \infty} | u_{p,r} | \text{ converges.}$$

We shall show that a certain sub-sequence of the sequence $P'_{m,n}$ has no limit as m and n become infinite.

$$P'_{m_0, n_0} = \prod_{p=1, r=1}^{m_0, n_0} (1 + a_{m_0, n_0, p, r} \cdot u_{p,r}) = 1 + a_{m_0, n_0, p_0, r_0} \cdot u_{p_0, r_0} > 2,$$

$$P_{m_1, n_1} = (1 + a_{m_1, n_1, p_0, r_0} \cdot u_{p_0, r_0}) (1 + a_{m_1, n_1, p_1, r_1} \cdot u_{p_1, r_1}),$$

$$| P'_{m_1, n_1} | \geqq \tfrac{1}{2} \cdot (1 + 8) \geqq 4,$$

$$P'_{m_2, n_2} = (1 + a_{m_2, n_2, p_0, r_0} \cdot u_{p_0, r_0}) (1 + a_{m_2, n_2, p_1, r_1} \cdot u_{p_1, r_1}) (1 + a_{m_2, n_2, p_2, r_2} \cdot u_{p_2, r_2}),$$

$$| P'_{m_2, n_2} | \geqq (1/2) \cdot (3/4) [1 + (4^5/2^4)].$$

In general

$$| P'_{m_i, n_i} | \geqq \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdots \frac{2^i - 1}{2^i} \left(1 + \frac{4^{2i+1}}{2^{i+2}} \right) \geqq \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdots \frac{2^i - 1}{2^i} \; 4$$

$$\geqq \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdots \frac{2^i - 1}{2^i} 2^{2i+1} \cdot 2^{i-1} \geqq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} \cdot 2^{2i+1} \cdot 2^{i-1} = 2^{2i+1}$$

$$\therefore \lim_{i \to \infty} | P'_{m_i, n_i} | = \infty.$$

This shows the necessity of condition (4).

THEOREM IV. *A sufficient condition that* $\lim\limits_{m,n\to\infty} P'_{m,n}$ *shall exist and equal* $\lim\limits_{m,n\to\infty} P_{m,n}$, *where* $P_{m,n} = \prod\limits_{k=1,l=1}^{m,n}(1+u_{k,l})$ *and where* $\sum\limits_{k=1,l=1}^{\infty\infty}|u_{k,l}|$ *shall converge,* *d* $R|u_{k,l}| > -1$ *is that*

1) $\lim\limits_{m,n\to\infty} a_{m,n,k,l} = 1$ *for each* k *anl* l;

2) $\lim\limits_{m,n\to\infty} \sum\limits_{k=1}^{m} |a_{m,n,k,l}-1| = 0$ *for each* l;

3) $\lim\limits_{m,n\to\infty} \sum\limits_{l=1}^{n} |a_{m,n,k,l}-1| = 0$ *for each* k;

4) $|a_{m,n,k,l}| < K$ *where* K *is some constant.*

Proof. We have

$$P'_{m,n} = \prod_{k=1,l=1}^{n,n}(1+a_{m,n,k,l}\cdot u_{k,l}) = \prod_{k=1,l=1}^{m,n}(1+u_{k,l}+\delta_{m,n,k,l}\cdot u_{k,l}).$$

$$a_{m,n,k,l} = 1+\delta_{m,n,k,l}.$$

$$P'_{mn} = P_{m,n} + P_{m,n}\cdot\sum_{k=1,l=1}^{m,n}\frac{\delta_{m,n,k,l}\cdot u_{k,l}}{1+u_{k,l}} + P_{m,n}\sum_{k_1=1,l_1=1}^{m,n}\ \sum_{\substack{k_2=1,l_2=1\\(k_1l_1)\neq(k_2l_2)}}^{m,n}$$

$$\frac{\delta_{m,n,k_1,l_1}\cdot\delta_{m,n,k_2,l_2}u_{k_1,l_1}\cdot u_{k_2,l_2}}{(1+u_{k_1l_1})(1+u_{k_2l_2})} + \cdots + \prod_{k=1,l=1}^{n,n}\delta_{m,n,k,l},\cdot u_{k,l},. \tag{1}$$

Let us put

$$\sum_{k=1,l=1}^{m,n}\frac{\delta_{m,n,k,l}\cdot u_{k,l}}{1+u_{k,l}} = Q_{m,n,1}, \text{ etc.}$$

We define

$$V_{m,n} = \sum_{k=1,l=1}^{m,n}\left|\delta_{m,n,k,l}\cdot\frac{u_{k,l}}{1+u_{k,l}}\right|.$$

Thus

$$|Q_{m,n,1}| \leq V_{m,n};\ |Q_{m,n,2}| \leq \frac{(V_{m,n})^2}{2!}:\cdots|Q_{m,nmn}| \leq \frac{(V_{m,n})^{mn}}{m\cdot n!}$$

Substituting in (1), we have

$$|P'_{m,n}-P_{m,n}| \leq |P_{m,n}|\left\{V_{m,n}+\frac{(V_{mn})^2}{2}+\frac{(V_{mn})^3}{3!}+\cdots+\frac{(V_{mn})^{mn}}{(mn)!}\right.$$

(2)

$$|P'_{m,n}-P_{mn}| \leq |P_{m,n}|\cdot\{e^{V_{m,n}}-1\}.$$

From (2), we can see that a sufficient condition for $\lim\limits_{m,n\to\infty} P'_{m,n} = \lim\limits_{m,n\to\infty} P_{m,n}$, provided the latter limit exists, is that $\lim\limits_{m,n\to\infty} V_{m,n} = 0$. Let us assume $\lim\limits_{m,n\to\infty} P_{mn}$ exists, and is different from zero; then $\lim\limits_{m,n\to\infty} u_{m,n} = 0$. Further, assume that $R(u_{m,n}) \geq L > -1$. Then

$$V_{mn} = \sum_{k=1,l=1}^{m,n}\left|\delta_{m,n,k,l}\cdot\frac{u_{k,l}}{1+u_{k,l}}\right| \leq 1/(1+L)\sum_{k=1,l=1}^{m,n}|\delta_{m,n,k,l}\cdot u_{k,l}| = 1/(1+L)\cdot\sigma_{m,n}.$$

If we assume $\sum_{k=1,\, l=1}^{\infty\,\infty} | u_{k,l} |$ converges, then a sufficient condition that $\lim_{m,n\to\infty} V_{m,n}=0$ is that

1) $\lim_{m,n\to\infty} \delta_{m,n,k,l} = 0$ for every k and l;

2) $\lim_{m,n\to\infty} \sum_{k=1}^{m} | \delta_{m,n,k,l} | = 0$ for each l;

3) $\lim_{m,n\to\infty} \sum_{l=1}^{n} | \delta_{m,n,k,l} | = 0$ for each k;

4) $| \delta_{m,n,k,l} | < M$ where M is some constant.

$$\sigma_{m,n} = \sum_{k=1,\, l=1}^{p,q} | \delta_{m,n,k,l} \cdot u_{k,l} | + \sum_{k=1,\, l=q+1}^{p,n} | \delta_{m,n,k,l} \cdot u_{k,l} |$$
$$+ \sum_{k=p+1,\, l=1}^{m,q} | \delta_{m,n,k,l} \cdot u_{k,l} | + \sum_{k=p+1,\, l=q+1}^{m,n} | \delta_{m,n,k,l} \cdot u_{k,l} |.$$

From condition (4), $| \delta_{m,n,k,l} | < K$. Let us choose p and q such that

$$\sum_{k=p+1,\, l=q+1}^{\infty\,\infty} | u_{k,l} | < (\epsilon/4k).$$

Choose M_1 and N_1 so that whenever $m > M_1$, and $n > N_1$,

$$| \delta_{m,n,k,l} | < \epsilon/4{\cdot}p{\cdot}qA, \qquad \begin{pmatrix} k = 1, 2, \cdots, p \\ l = 1, 2, \cdots, q \end{pmatrix};$$

where $| u_{k,l} | < A$. Choose M_2 and N_2 so that whenever $m > M_2$, and $n > N$

$$\sum_{k=p+1}^{m} | \delta_{m,n,k,l} | < \epsilon/4qA, \qquad (l = 1, 2, \cdots, q).$$

Choose M_3 and N_3 so that whenever $m > M_3$, and $n > N_3$,

$$\sum_{l=q+1}^{n} | \delta_{m,n,k,l} | < \epsilon/4pA, \qquad (k = 1, 2, \cdots, p).$$

Let M be the largest of M_1, M_2, and M_3, and let N be the largest of N_1 N_2, and N_3. For any $m > M$, $n > N$,

$$\sigma_{m,n} \leq (\epsilon/4.p.q.A) \cdot p.q.A$$
$$+ (\epsilon/4pA) \cdot p.A + (\epsilon/4.q.A) q.A + (\epsilon/4k) \cdot K = \epsilon.$$

Therefore $$\lim_{m,n\to\infty} \sigma_{m,n} = 0.$$

Expressing these conditions in terms of $a_{m,n,k,l}$, we have the conditions state in the theorem.

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CORRECTION, VOLUME 51.

Walter H. Gage, *Representations in the Form* $xy + yz + zx$.

Pages 347-8, equations (11) and (12), change "$(\alpha + 1/2)$" to "$\frac{1}{2}(\alpha + 1)$."